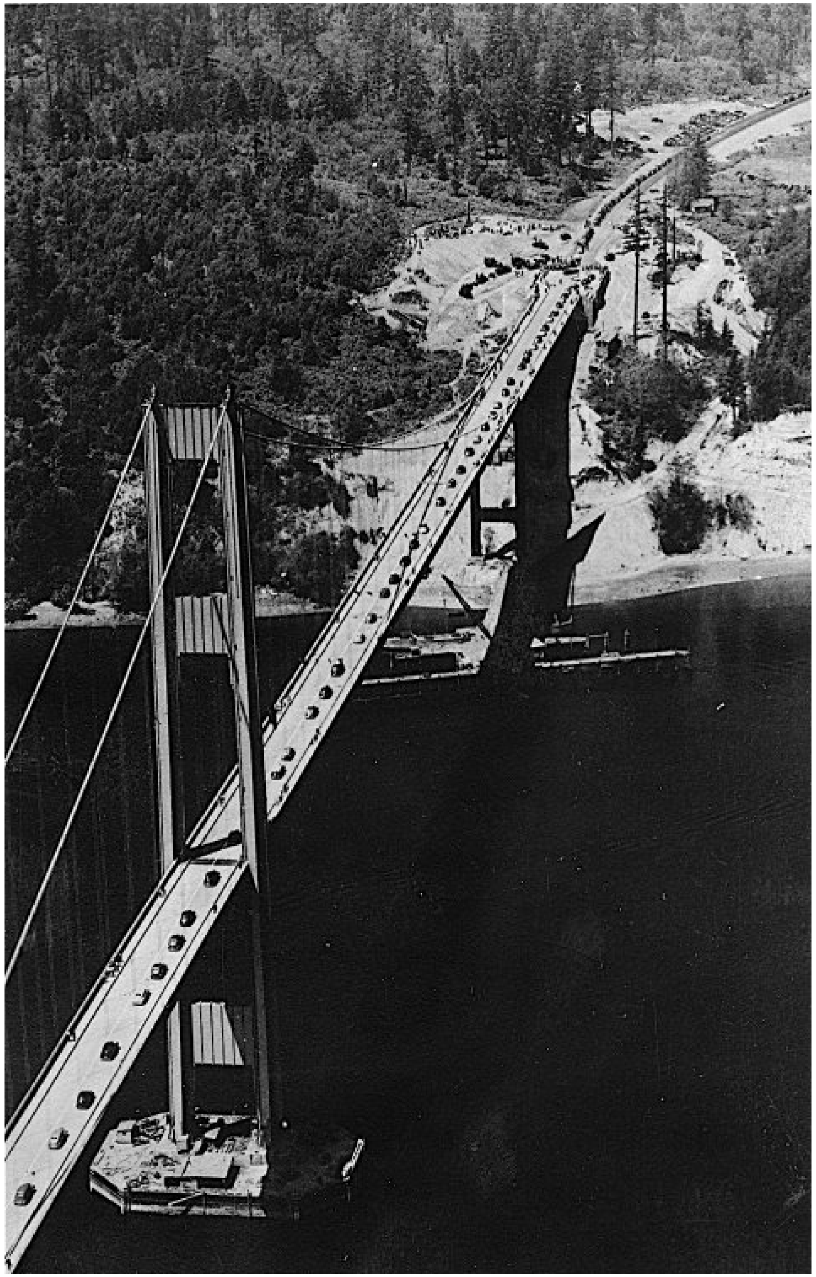
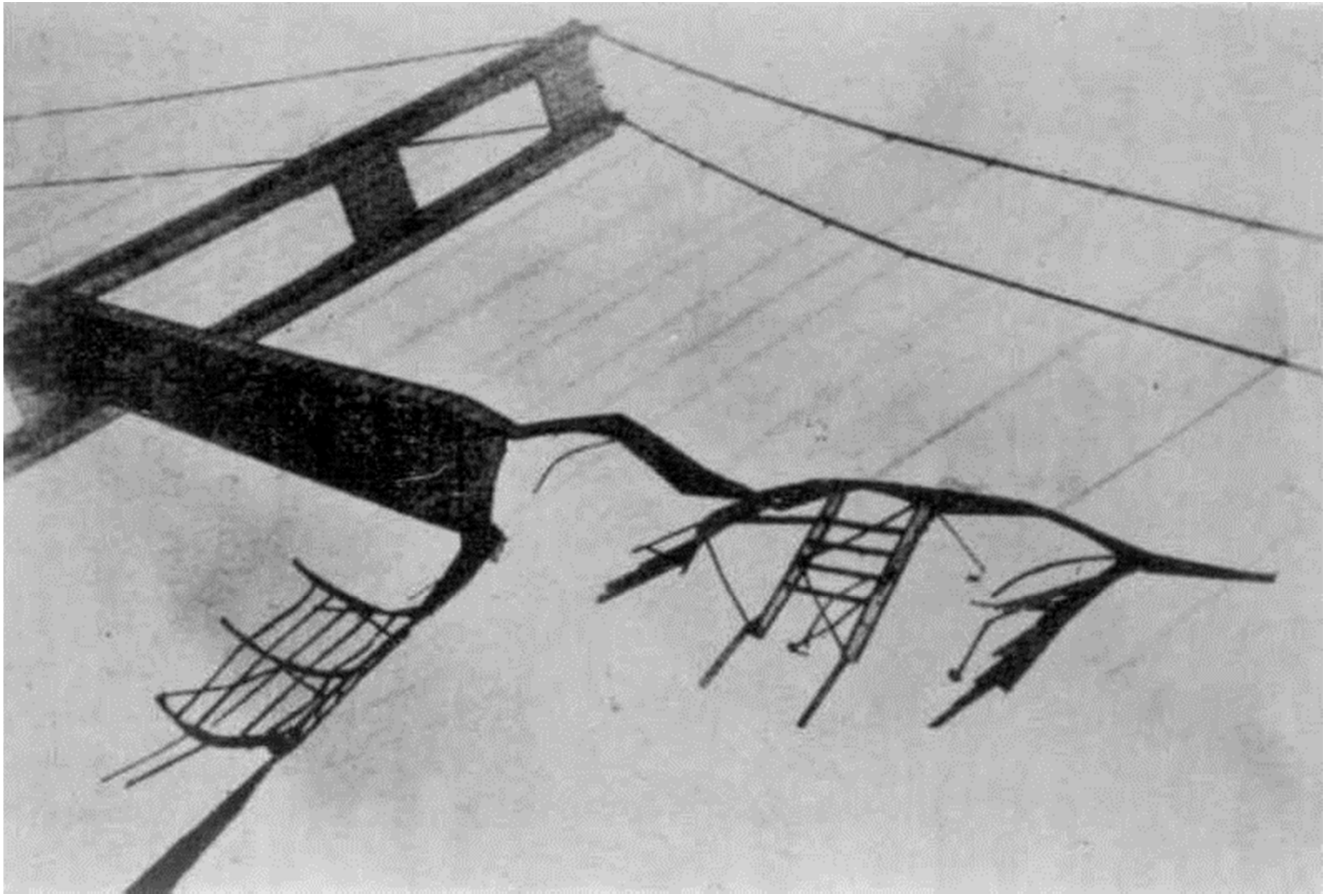


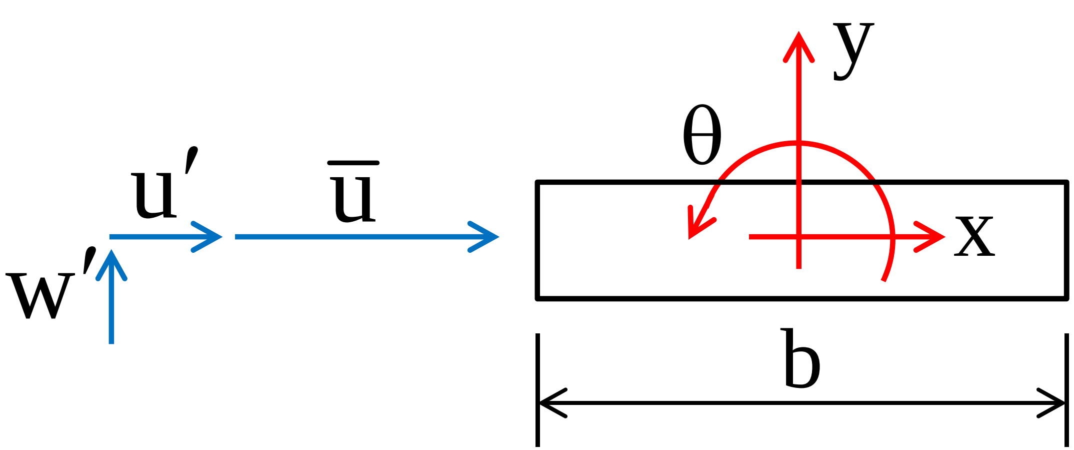
Tacoma Narrows Bridge, 1940

Tacoma Narrows Bridge, 1940

Tacoma Narrows Bridge, November 7, 1940



Tacoma Narrows Bridge, November 7, 1940



Symplifying hypotheses : x negligible

y and θ structurally independent

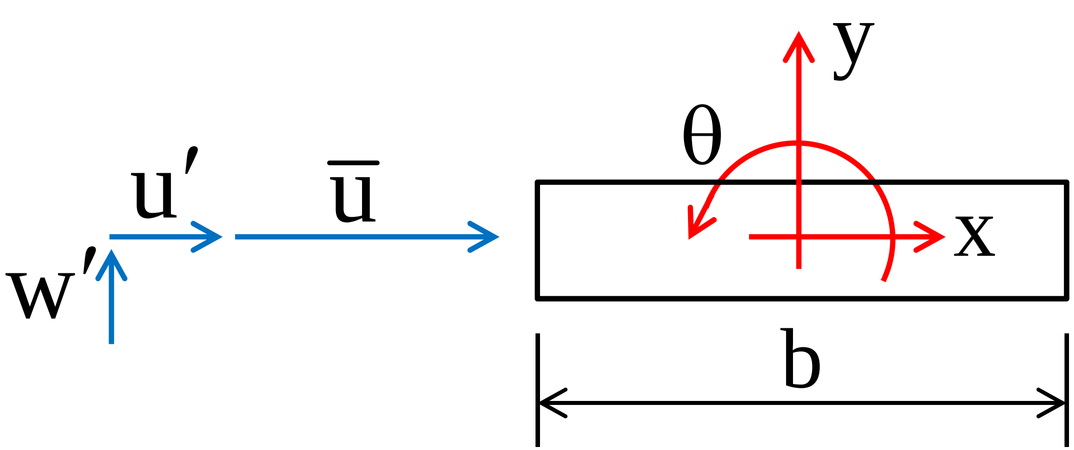
Equations of motion

$$\begin{cases} m\left[\ddot{y}(t)+2\xi_y\omega_y\dot{y}(t)+\omega_y^2 y(t)\right]=\bar{F}_y+F'_y(t)+L_a(t) \\ I\left[\ddot{\theta}(t)+2\xi_\theta\omega_\theta\dot{\theta}(t)+\omega_\theta^2\theta(t)\right]=\bar{F}_\theta+F'_\theta(t)+M_a(t) \end{cases}$$

$\bar{F}_y, \bar{M}_\theta$ = mean wind actions

F'_y, M'_θ = fluctuating wind actions

L_a, M_a = aeroelastic (motion – induced) actions



Equations of motion

$$\begin{cases} m\left[\ddot{y}(t)+2\xi_y\omega_y\dot{y}(t)+\omega_y^2 y(t)\right]=\bar{F}_y+F'_y(t)+L_a(t) \\ I\left[\ddot{\theta}(t)+2\xi_\theta\omega_\theta\dot{\theta}(t)+\omega_\theta^2\theta(t)\right]=\bar{F}_\theta+F'_\theta(t)+M_a(t) \end{cases}$$

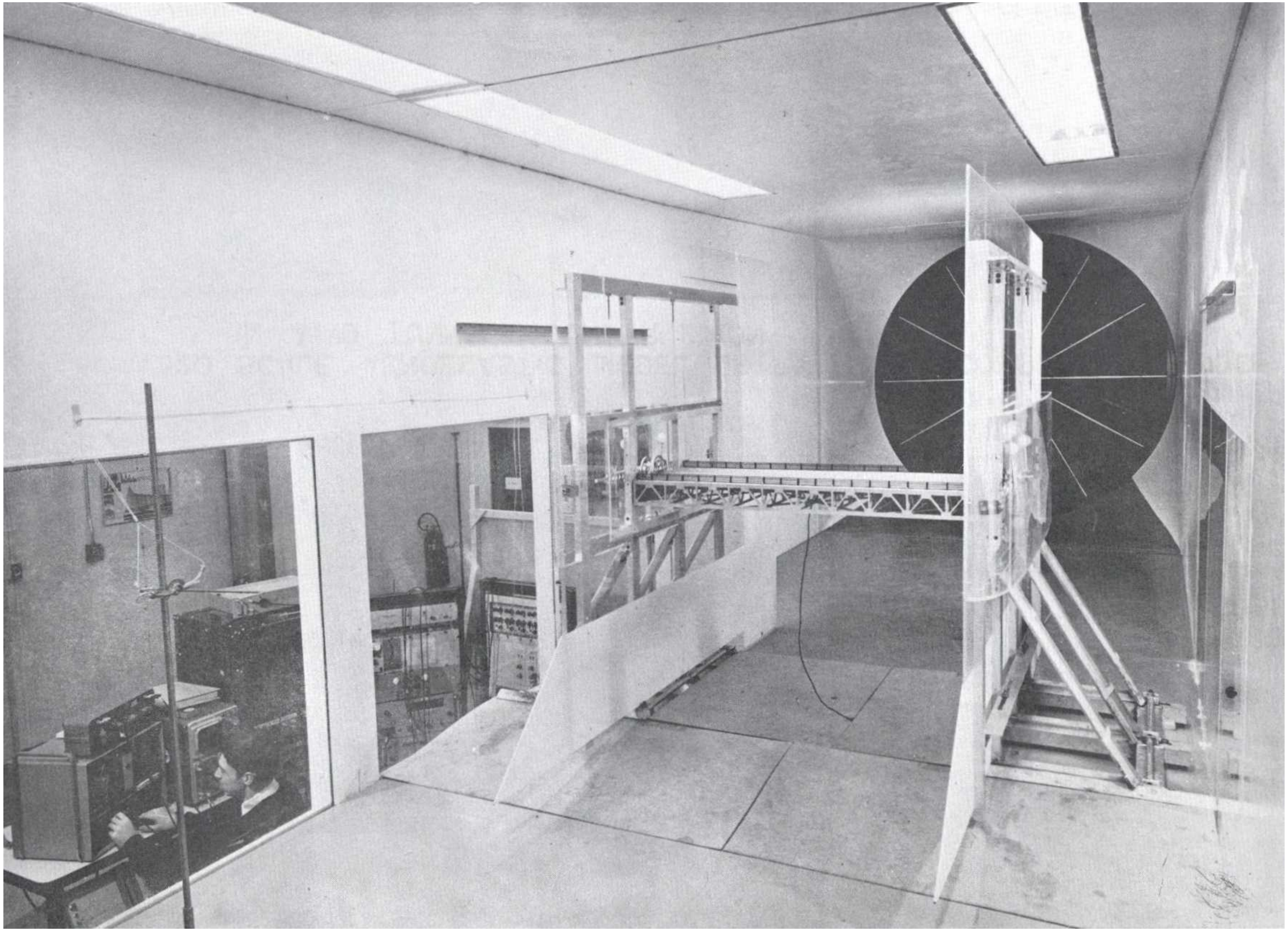
Aeroelastic actions

$$L_a(t)=\frac{1}{2}\rho\bar{u}^2 b\left[KH_1^*(K)\frac{\dot{y}(t)}{\bar{u}}+KH_2^*(K)\frac{b\dot{\theta}(t)}{\bar{u}}+K^2H_3^*(K)\theta(t)+K^2H_4^*(K)\frac{y(t)}{b}\right]$$

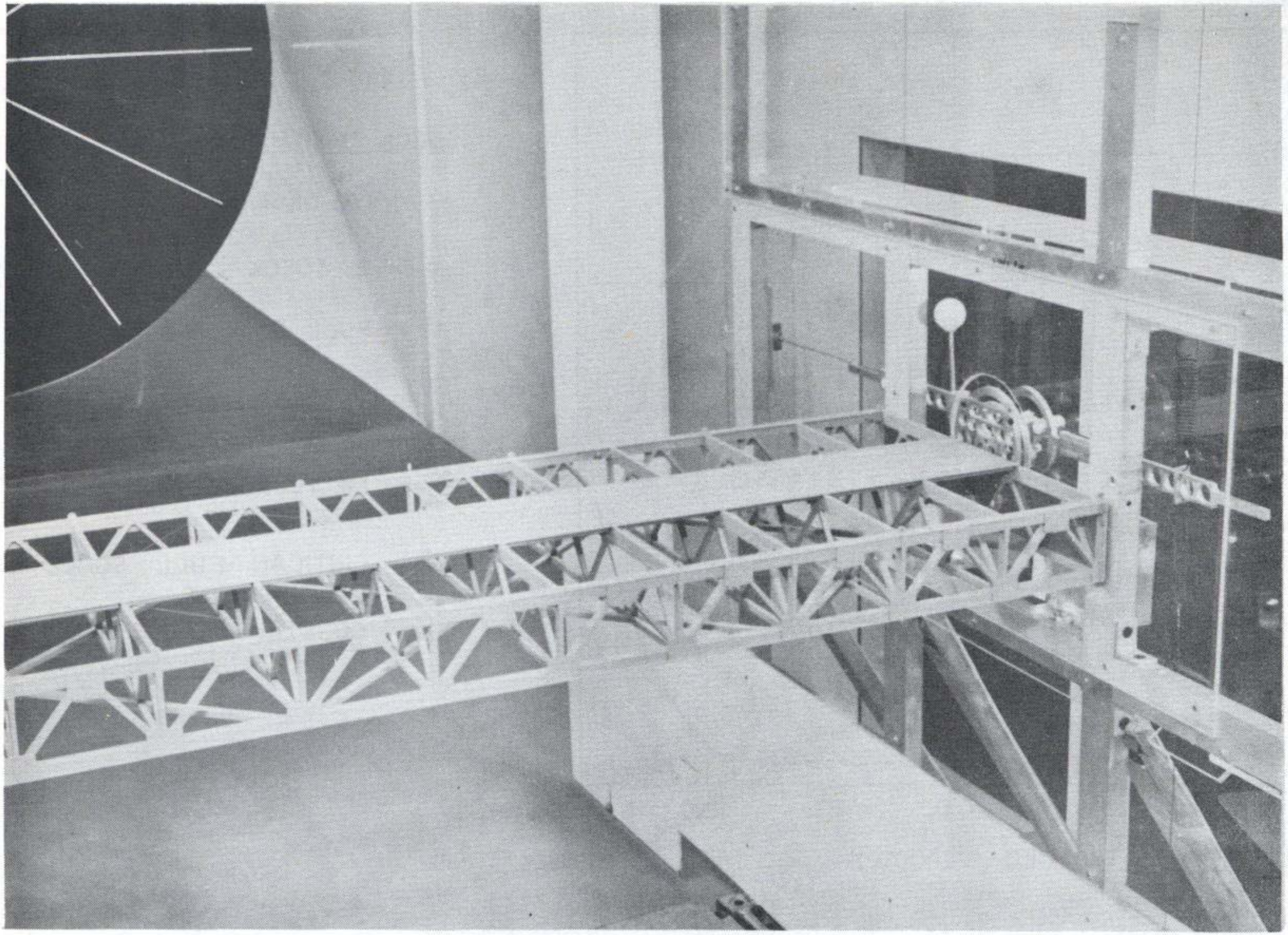
$$M_a(t)=\frac{1}{2}\rho\bar{u}^2 b^2\left[KA_1^*(K)\frac{\dot{y}(t)}{\bar{u}}+KA_2^*(K)\frac{b\dot{\theta}(t)}{\bar{u}}+K^2A_3^*(K)\theta(t)+K^2A_4^*(K)\frac{y(t)}{b}\right]$$

$H_i^*(K), A_i^*(K)\ (i=1,2,3,4)$ = aerodynamic derivatives

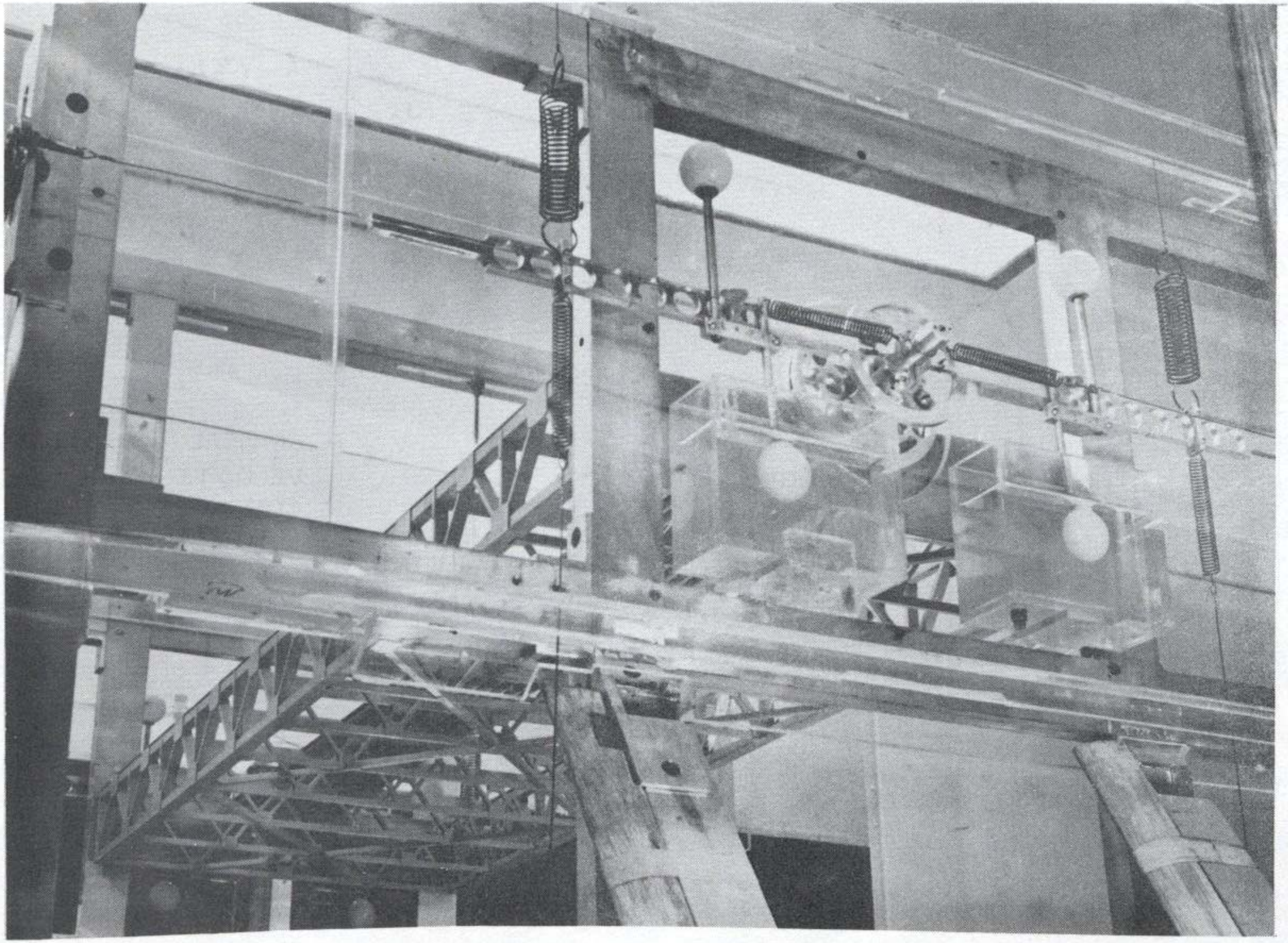
$K=\omega b/\bar{u}$ = reduced circular frequency



Free-vibration wind-tunnel tests



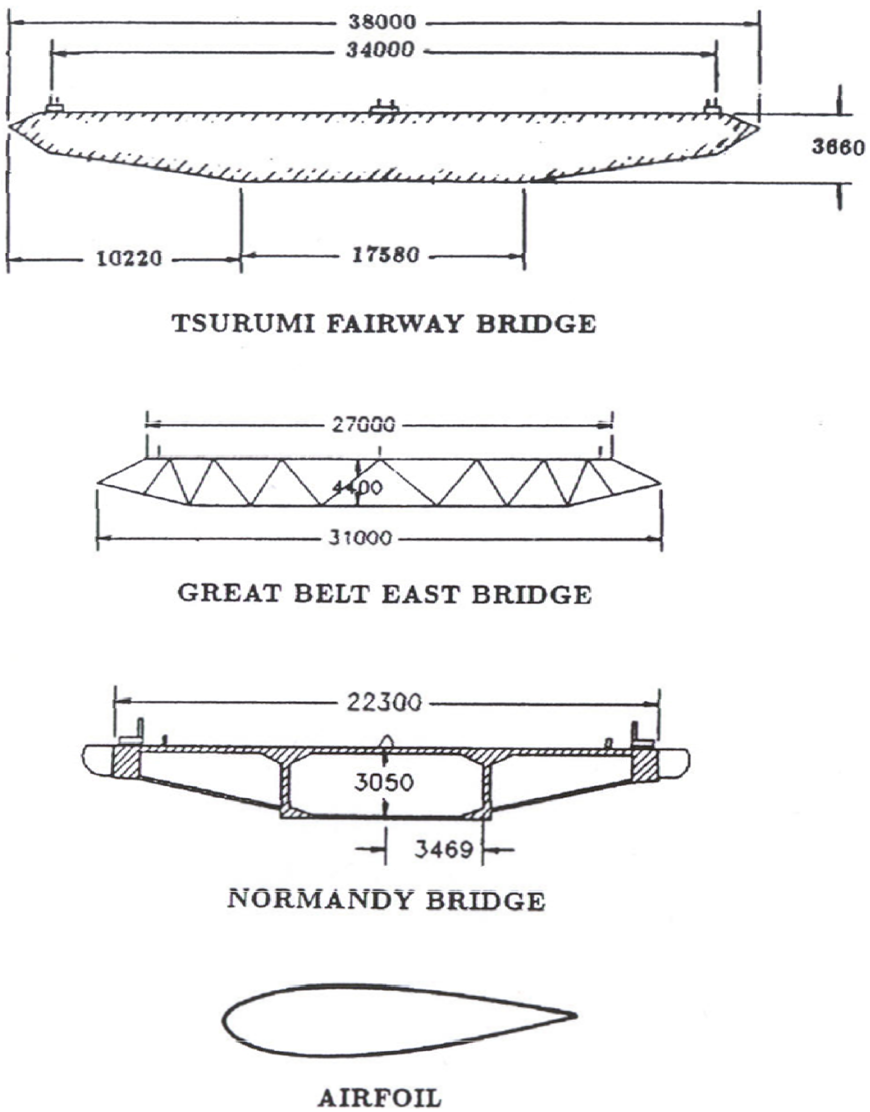
Free-vibration wind-tunnel tests

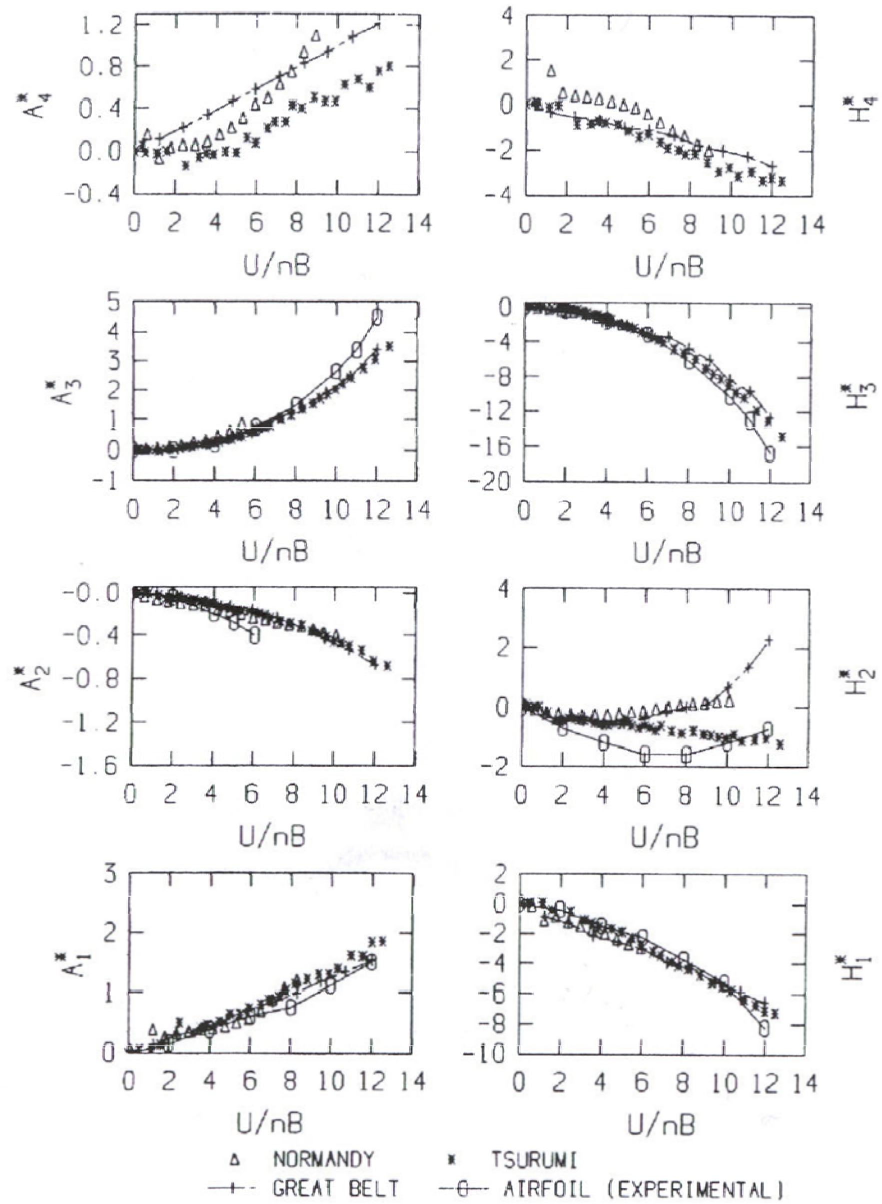


Free-vibration wind-tunnel tests

Free-vibration wind-tunnel tests

Forced-vibration wind-tunnel tests





Aerodynamic derivatives

Equations of motion

$$\begin{cases} m\left[\ddot{y}+2\xi_y\omega_y\dot{y}+\omega_y^2 y\right]=\overline{F}_y+F'_y+L_a \\ I\left[\ddot{\theta}+2\xi_\theta\omega_\theta\dot{\theta}+\omega_\theta^2\theta\right]=\overline{F}_\theta+F'_\theta+M_a \end{cases}$$

Aeroelastic actions

$$L_a=\frac{1}{2}\rho\overline{u}^2 b\left[KH_1^*\frac{\dot{y}}{\overline{u}}+KH_2^*\frac{b\dot{\theta}}{\overline{u}}+K^2H_3^*\theta+K^2H_4^*\frac{y}{b}\right]$$

$$M_a=\frac{1}{2}\rho\overline{u}^2 b^2\left[KA_1^*\frac{\dot{y}}{\overline{u}}+KA_2^*\frac{b\dot{\theta}}{\overline{u}}+K^2A_3^*\theta+K^2A_4^*\frac{y}{b}\right]$$

Instability analysis $\Rightarrow \overline{F}_\alpha=F'_\alpha=0\left(\alpha=y,\theta\right)$

$$\begin{cases} m\left[\ddot{y}+2\xi_y\omega_y\dot{y}+\omega_y^2 y\right]=\frac{1}{2}\rho\overline{u}^2 b\left[KH_1^*\frac{\dot{y}}{\overline{u}}+KH_2^*\frac{b\dot{\theta}}{\overline{u}}+K^2H_3^*\theta+K^2H_4^*\frac{y}{b}\right] \\ I\left[\ddot{\theta}+2\xi_\theta\omega_\theta\dot{\theta}+\omega_\theta^2\theta\right]=\frac{1}{2}\rho\overline{u}^2 b^2\left[KA_1^*\frac{\dot{y}}{\overline{u}}+KA_2^*\frac{b\dot{\theta}}{\overline{u}}+K^2A_3^*\theta+K^2A_4^*\frac{y}{b}\right] \end{cases}$$

Equations of motion

$$\begin{cases} m\left[\ddot{y}+2\xi_y\omega_y\dot{y}+\omega_y^2 y\right]=\dfrac{1}{2}\rho\bar{u}^2 b\left[KH_1^*\dfrac{\dot{y}}{\bar{u}}+KH_2^*\dfrac{b\dot{\theta}}{\bar{u}}+K^2H_3^*\theta+K^2H_4^*\dfrac{y}{b}\right] \\ I\left[\ddot{\theta}+2\xi_\theta\omega_\theta\dot{\theta}+\omega_\theta^2\theta\right]=\dfrac{1}{2}\rho\bar{u}^2 b^2\left[KA_1^*\dfrac{\dot{y}}{\bar{u}}+KA_2^*\dfrac{b\dot{\theta}}{\bar{u}}+K^2A_3^*\theta+K^2A_4^*\dfrac{y}{b}\right] \end{cases}$$

$$s=\frac{\bar{u}t}{b}\Rightarrow\left(\dot{}\right)=\frac{d(\ )}{dt}=\frac{d(\ )}{ds}\frac{ds}{dt}=(\ )'\frac{\bar{u}}{b}$$

$$K_y=\frac{\omega_y b}{\bar{u}};\ K_\theta=\frac{\omega_\theta b}{\bar{u}}$$

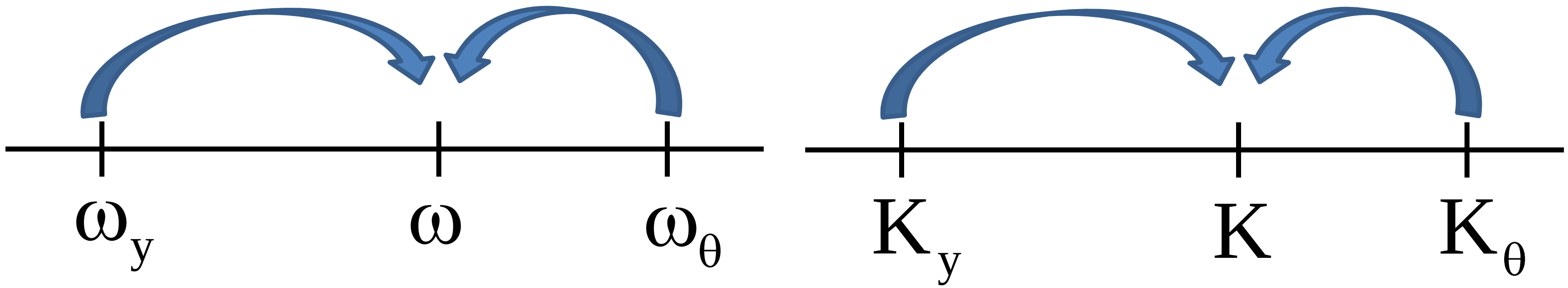
$$\begin{cases} \dfrac{y''}{b}+2\xi_y K_y\dfrac{y'}{b}K_y^2\dfrac{y}{b}=\dfrac{\rho b^2}{2m}\left[KH_1^*\dfrac{y'}{b}+KH_2^*\theta'+K^2H_3^*\theta+K^2H_4^*\dfrac{y}{b}\right] \\ \theta''+2\xi_\theta K_\theta\theta'+K_\theta^2\theta=\dfrac{\rho b^4}{2I}\left[KA_1^*\dfrac{h'}{b}+KA_2^*\theta'+K^2A_3^*\theta+K^2A_4^*\dfrac{y}{b}\right] \end{cases}$$

Equations of motion

$$\begin{cases} \dfrac{y''}{b} + 2\xi_y K_y \dfrac{y'}{b} K_y^2 \dfrac{y}{b} = \dfrac{\rho b^2}{2m}\left[K H_1^* \dfrac{y'}{b} + K H_2^* \theta' + K^2 H_3^* \theta + K^2 H_4^* \dfrac{y}{b} \right] \\ \theta'' + 2\xi_\theta K_\theta \theta' + K_\theta^2 \theta = \dfrac{\rho b^4}{2I}\left[K A_1^* \dfrac{h'}{b} + K A_2^* \theta' + K^2 A_3^* \theta + K^2 A_4^* \dfrac{y}{b} \right] \end{cases}$$

$$y(t) = y_0 e^{i\omega t} \; ; \theta(t) = \theta_0 e^{i\omega t}$$

$$\frac{y(s)}{b} = \frac{y_0}{b} e^{iKs} \; ; \theta(s) = \theta_0 e^{iKs}$$



$$y(s) = y_0 e^{iKs} \; ; \; \theta(s) = \theta_0 e^{iKs}$$

Equations of motion

$$\begin{cases} \left[-K^2 + 2i\xi_y K_y K + K_y^2 - \dfrac{\rho b^2}{2m}\left(iK^2 H_1^* + K^2 H_4^* \right) \right] \dfrac{y_0}{b} - \left[\dfrac{\rho b^2}{2m}\left(iK^2 H_2^* + K^2 H_3^* \right) \right] \theta_0 = 0 \\ \left[-\dfrac{\rho b^4}{2I}\left(iK^2 A_1^* + K^2 A_4^* \right) \right] \dfrac{y_0}{b} + \left[-K^2 + 2i\xi_\theta K K_\theta + K_\theta^2 - \dfrac{\rho b^4}{2I}\left(iK^2 A_2^* + K^2 A_3^* \right) \right] \theta_0 = 0 \end{cases}$$

$$\frac{y_0}{b} = \theta_0 = 0 \Rightarrow \text{trivial solution}$$

A non – trivial solution exist if and only if the matrix of the coefficients has determinant $\Delta = 0$

$$\begin{aligned} \Delta = & \left[-K^2 + 2i\xi_y K_y K + K_y^2 - \frac{\rho b^2}{2m}\left(iK^2 H_1^* + K^2 H_4^* \right) \right] \cdot \\ & \cdot \left[-K^2 + 2i\xi_\theta K K_\theta + K_\theta^2 - \frac{\rho b^4}{2I}\left(iK^2 A_2^* + K^2 A_3^* \right) \right] + \\ & + \left[-\frac{\rho b^4}{2I}\left(iK^2 A_1^* + K^2 A_4^* \right) \right] \left[-\frac{\rho b^2}{2m}\left(iK^2 H_2^* + K^2 H_3^* \right) \right] = 0 \end{aligned}$$

$$\Delta = \left[-K^2 + 2i\xi_y K_y K + K_y^2 - \frac{\rho b^2}{2m} \left(iK^2 H_1^* + K^2 H_4^* \right) \right] \cdot$$

$$\cdot \left[-K^2 + 2i\xi_\theta K K_\theta + K_\theta^2 - \frac{\rho b^4}{2I} \left(iK^2 A_2^* + K^2 A_3^* \right) \right] +$$

$$+ \left[-\frac{\rho b^4}{2I} \left(iK^2 A_1^* + K^2 A_4^* \right) \right] \left[-\frac{\rho b^2}{2m} \left(iK^2 H_2^* + K^2 H_3^* \right) \right] = 0$$

$$X = \frac{K}{K_y} = \frac{\omega}{\omega_y} \Rightarrow$$

$$\Delta = \left[-X^2 + 2i\xi_y X + 1 - \frac{\rho b^2}{2m} \left(iX^2 H_1^* + X^2 H_4^* \right) \right] \cdot$$

$$\cdot \left[-X^2 + 2i\xi_\theta X \frac{\omega_\theta}{\omega_y} + \left(\frac{\omega_\theta}{\omega_y} \right)^2 - \frac{\rho b^4}{2I} \left(iX^2 A_1^* + iX^2 A_3^* \right) \right] +$$

$$+ \left[-\frac{\rho b^4}{2I} \left(iX^2 A_1^* + X^2 A_4^* \right) \right] \left[\frac{\rho b^2}{2m} \left(iX^2 H_2^* + X^2 H_3^* \right) \right] = 0$$

$$\Delta = \left[-X^2 + 2i\xi_y X + 1 - \frac{\rho b^2}{2m}\left(iX^2 H_1^* + X^2 H_4^* \right) \right] \cdot$$

$$\cdot \left[-X^2 + 2i\xi_\theta X \frac{\omega_\theta}{\omega_y} + \left(\frac{\omega_\theta}{\omega_y} \right)^2 - \frac{\rho b^4}{2I}\left(iX^2 A_1^* + iX^2 A_3^* \right) \right] +$$

$$+ \left[-\frac{\rho b^4}{2I}\left(iX^2 A_1^* + X^2 A_4^* \right) \right]\left[\frac{\rho b^2}{2m}\left(iX^2 H_2^* + X^2 H_3^* \right) \right] = 0 \Rightarrow$$

Complex equation of the 4th degree in the complex unknown X

$$a_4(K)X^4 + a_3(K)X^3 + a_2(K)X^2 + a_1(K)X + a_0(K) = 0$$

$$X = \frac{\omega}{\omega_y} = X_r + iX_i \Rightarrow \omega = \omega_r + i\omega_i \Rightarrow$$

$$\begin{cases} y(t) = y_0 e^{i\omega t} = y_0 e^{i\omega_r t} e^{-\omega_i t} \\ \theta(t) = \theta_0 e^{i\omega t} = \theta_0 e^{i\omega_r t} e^{-\omega_i t} \end{cases}$$

$\omega_i > 0 \Rightarrow$ asymptotically stable system

$\omega_i = 0 \Rightarrow$ stable system

$\omega_i < 0 \Rightarrow$ unstable system

$\omega_i = 0 \Rightarrow$ bifurcation condition $\Rightarrow$ flutter occurrence $\Rightarrow \omega = \omega_r$

Complex equation of the 4th degree in the complex unknown X as a function of K

$$a_4(K)X^4 + a_3(K)X^3 + a_2(K)X^2 + a_1(K)X + a_0(K) = 0$$

Search for the solution of the complex equation of the 4th degree under the condition that the unknown X is real

$$\left[b_4(K)X^4 + b_3(K)X^3 + b_2(K)X^2 + b_1(K)X + b_0(K)\right] +$$

$$+i\left[c_4(K)X^4 + c_3(K)X^3 + c_2(K)X^2 + c_1(K)X + c_0(K)\right] = 0 \Rightarrow$$

Two real equations of the 4th degree in the real unknown X as a function of K

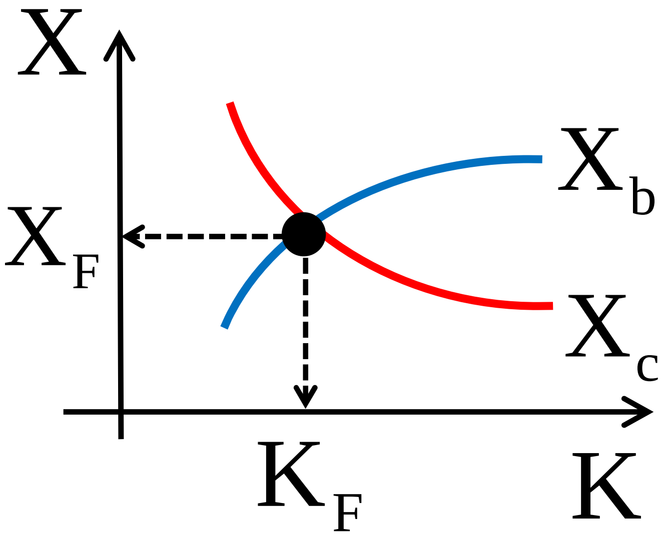
$$\begin{cases} b_4(K)X^4 + b_3(K)X^3 + b_2(K)X^2 + b_1(K)X + b_0(K) = 0 \\ c_4(K)X^4 + c_3(K)X^3 + c_2(K)X^2 + c_1(K)X + c_0(K) = 0 \end{cases}$$

Two real equations of the 4th degree in the real unknown X as a function of K

$$\begin{cases} b_4(K)X^4 + b_3(K)X^3 + b_2(K)X^2 + b_1(K)X + b_0(K) = 0 \\ c_4(K)X^4 + c_3(K)X^3 + c_2(K)X^2 + c_1(K)X + c_0(K) = 0 \end{cases}$$

Complex and negative solutions are disregarded $\Rightarrow$

$$X_b = X_b(K);\ X_c = X_c(K)$$



Flutter circular frequency

$$\omega_F = \omega_y X_F$$

Flutter critical velocity

$$\bar{u}_F = \frac{\omega_F b}{K_F}$$

CLASSICAL FLUTTER

2 D.O.F. coupled vertical and torsional flutter

$$\begin{cases} m\left[\ddot{y}+2\xi_y\omega_y\dot{y}+\omega_y^2 y\right]=\dfrac{1}{2}\rho\bar{u}^2 b\left[KH_1^*\dfrac{\dot{y}}{\bar{u}}+KH_2^*\dfrac{b\dot{\theta}}{\bar{u}}+K^2H_3^*\theta+K^2H_4^*\dfrac{y}{b}\right] \\ I\left[\ddot{\theta}+2\xi_\theta\omega_\theta\dot{\theta}+\omega_\theta^2\theta\right]=\dfrac{1}{2}\rho\bar{u}^2 b^2\left[KA_1^*\dfrac{\dot{y}}{\bar{u}}+KA_2^*\dfrac{b\dot{\theta}}{\bar{u}}+K^2A_3^*\theta+K^2A_4^*\dfrac{y}{b}\right] \end{cases}$$

STALL FLUTTER

1 D.O.F. torsional flutter

$$I\left[\ddot{\theta}+2\xi_\theta\omega_\theta\dot{\theta}+\omega_\theta^2\theta\right]=\frac{1}{2}\rho\bar{u}^2 b^2 KA_2^*\frac{b\dot{\theta}}{\bar{u}}\Rightarrow$$

$$\ddot{\theta}+2\left[\xi_\theta-\frac{\rho b^4}{4I}\frac{\omega}{\omega_\theta}A_2^*\right]\omega_\theta\dot{\theta}+\omega_\theta^2\theta=0$$

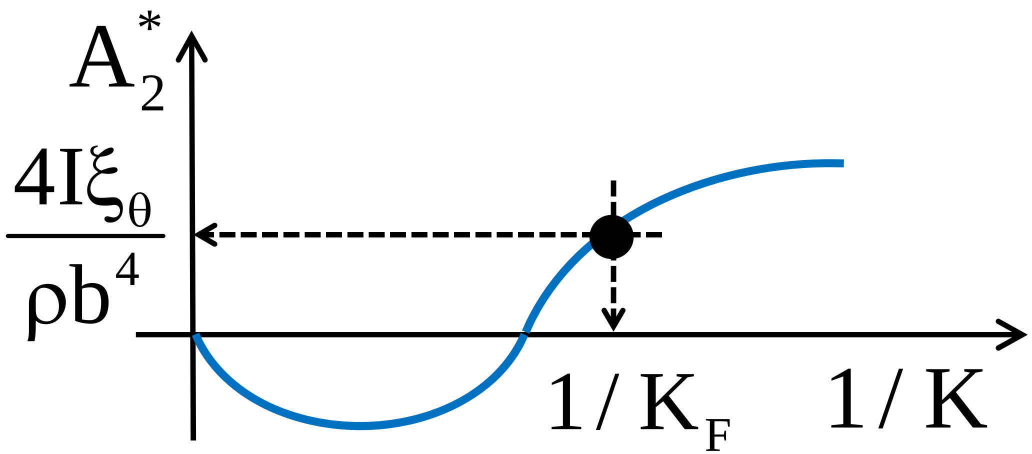
STALL FLUTTER – 1 D.O.F. torsional flutter

$$\ddot{\theta} + 2\left[\xi_\theta - \frac{\rho b^4}{4I}\frac{\omega}{\omega_\theta}A_2^*\right]\omega_\theta\dot{\theta} + \omega_\theta^2\theta = 0$$

Flutter critical condition

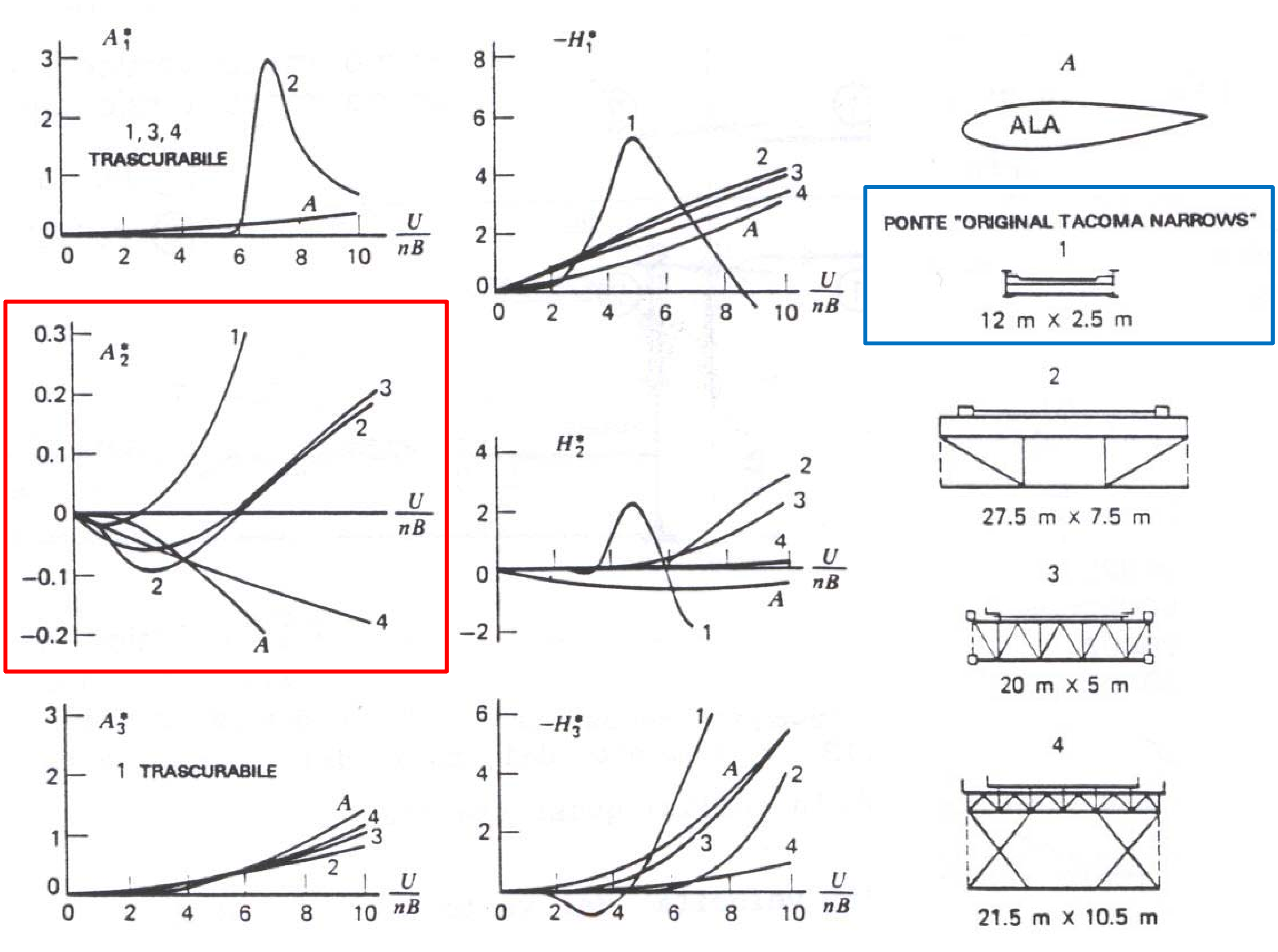
$$\omega = \omega_\theta$$

$$A_2^* = \frac{4I\xi_\theta}{\rho b^4}$$



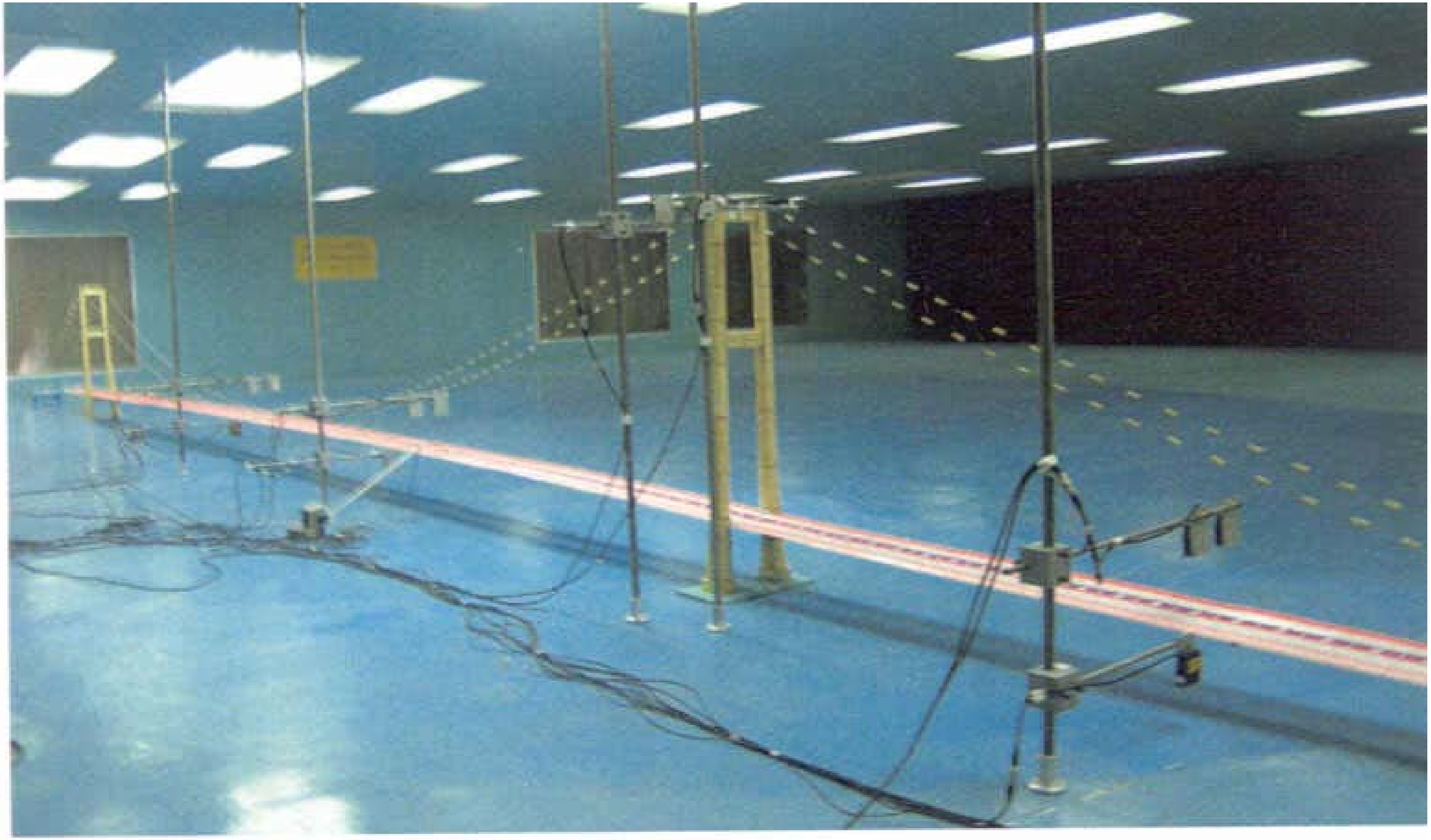
Flutter critical velocity

$$\bar{u}_F = \frac{b\omega_\theta}{K_F}$$



Flutter derivatives

Great Belt East Bridge, 1997



Xihoumen Bridge, China, 2008

Akashi-Kaykio Bridge, 1998

Akashi-Kaykio Bridge, 1998

Akashi-Kaykio Bridge, 1998