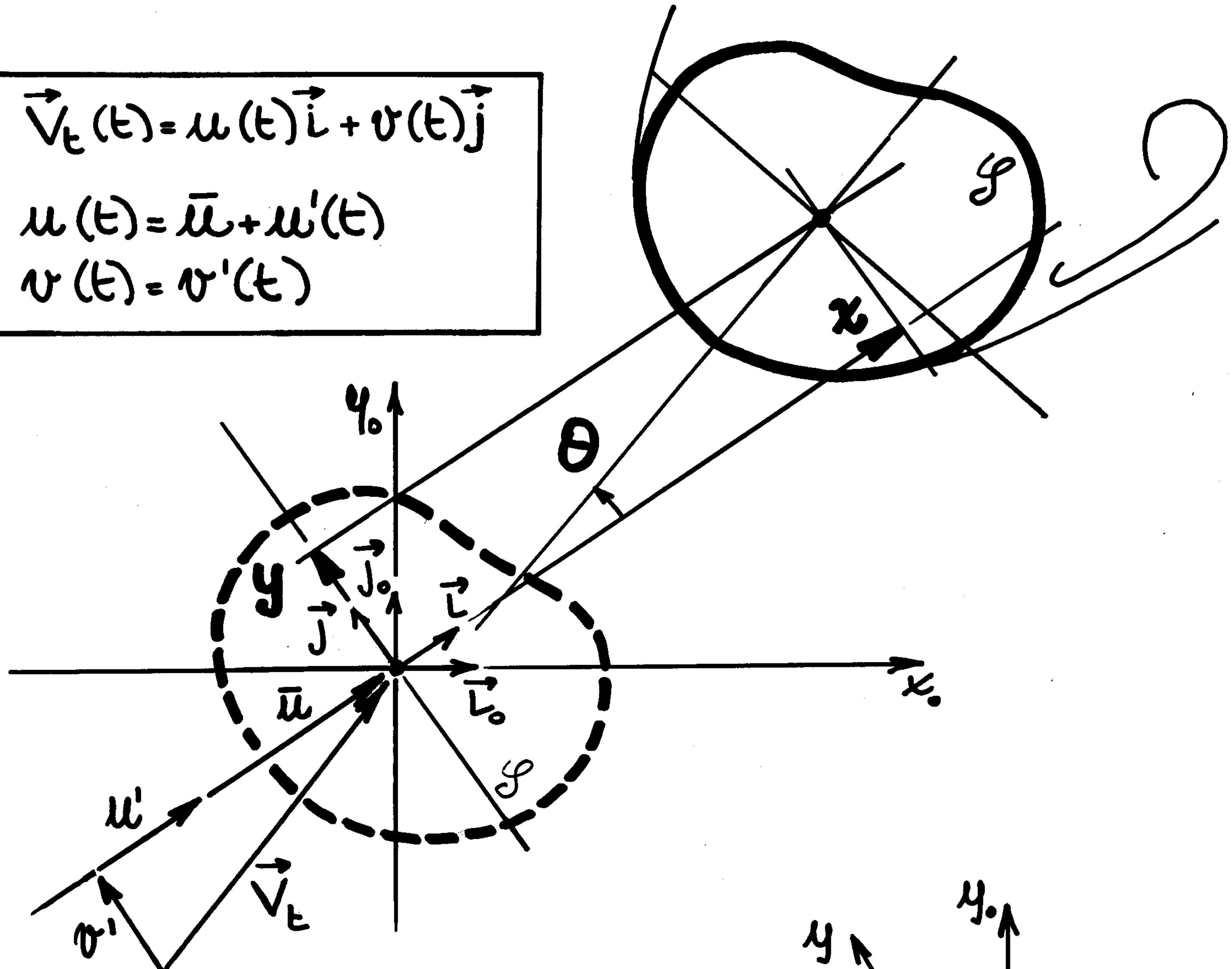
3. MOVING CYLINDER - TURBULENT FLOW

$$\vec{V}_t(t) = u(t)\vec{i} + v(t)\vec{j}$$

$$u(t) = \bar{u} + u'(t)$$

$$v(t) = v'(t)$$

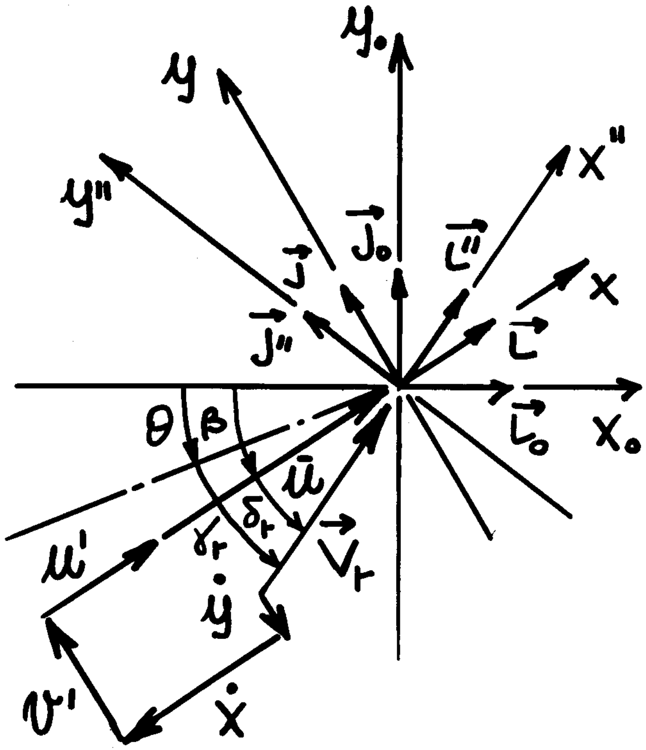


FIXED CYLINDER - RELATIVE FLOW

$$\vec{V}_r(t) = u_r(t)\vec{i} + v_r(t)\vec{j}$$

$$u_r(t) = \bar{u} + u'(t) - \dot{x}(t)$$

$$v_r(t) = v'(t) - \dot{y}(t)$$



• G. Solari: Gust-excited vibrations, in "Wind excited vibrations of structures, H. Sockel Ed., Springer, 1994.

Fixed cylinder · turbulent flow ⇒

$$\mathbb{F}(t) = \bar{\mathbb{F}} + \mathbb{F}'(t) + \mathbb{R}^{\circ}_{t}(t)$$
$$\mathbb{F}'(t) = \mathbb{F}'_{u}(t) + \mathbb{F}'_{v}(t) + \mathbb{F}'_{w}(t)$$
$\mathbb{R}^{\circ}_{t}(t) \sim \mathbb{O}$ for small turbulence

Moving cylinder · turbulent flow ⇒
Fixed cylinder · relative flow ⇒

$$\mathbb{F}(t) = \bar{\mathbb{F}} + \mathbb{F}'(t) + \mathbb{F}_{a}(t) + \mathbb{R}^{\circ}(t)$$
$$\mathbb{F}'(t) = \mathbb{F}'_{u}(t) + \mathbb{F}'_{v}(t) + \mathbb{F}'_{w}(t)$$
$\mathbb{R}^{\circ}(t) \sim \mathbb{O}$ for small turbulence and small motions
$\mathbb{F}_{a}(t) = -\mathbb{C}^{\circ}\dot{\mathbb{q}}(t) - \mathbb{K}^{\circ}\mathbb{q}(t)$ = linear aeroelastic forces

$$\mathbb{q}(t) = \begin{Bmatrix} x(t) \\ y(t) \\ \theta(t) \end{Bmatrix} ; \quad \dot{\mathbb{q}}(t) = \begin{Bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \end{Bmatrix} ; \quad \mathbb{R}^{\circ}(t) = \begin{Bmatrix} R^{\circ}_{x}(t) \\ R^{\circ}_{y}(t) \\ R^{\circ}_{\theta}(t) \end{Bmatrix}$$

$$\mathbb{C}^{\circ} = \frac{1}{2}\rho\bar{u}b \begin{bmatrix} 2c_d & (c'_d - c_\ell) & -R_o(c'_d - c_\ell) \\ 2c_\ell & (c_d + c'_\ell) & -R_o(c_d + c'_\ell) \\ 2bc_m & bc'_m & -bR_o c'_m \end{bmatrix}$$

$$\mathbb{K}^{\circ} = \frac{1}{2}\rho\bar{u}^2 b \begin{bmatrix} \emptyset & \emptyset & c'_d \\ \emptyset & \emptyset & c'_\ell \\ \emptyset & \emptyset & bc'_m \end{bmatrix}$$

R_o = Characteristic radius ~ 0 for compact sections
(Blevins & Iwan 1974, Nakamura & Mizota 1975)

$$
\begin{aligned}
R^0_x(t) = {} & \tfrac{1}{2}\rho b\{c_d[(u'-\dot{x})^2+(v'-\dot{y}+R\dot{\theta})^2]+(c'_d-c_l)[(v'-\dot{y}+R\dot{\theta})\cdot \\
& \cdot(u'-\dot{x})-2\theta u'U+2\theta\dot{x}U+\frac{2(v'-\dot{y}+R\dot{\theta})^3}{3(U+u'-\dot{x})}-\theta(u'-\dot{x})^2+ \\
& -\theta(v'-\dot{y}+R\dot{\theta})^2]-c_l[2\theta u'U-2\theta\dot{x}U+\theta(u'-\dot{x})^2+\theta(v'-\dot{y}+R\dot{\theta})^2]+ \\
& +(c''_d-c_d-2c'_l)[(v'-\dot{y}+R\dot{\theta})^2/2+\theta^2U^2/2-\theta U(v'-\dot{y}+R\dot{\theta})+\theta^2Uu'+ \\
& -\theta^2U\dot{x}-\theta(v'-\dot{y}+R\dot{\theta})(u'-\dot{x})]-(c_d+c'_l)[\theta(v'-\dot{y}+R\dot{\theta})U-\theta^2U^2+ \\
& +\theta(v'-\dot{y}+R\dot{\theta})(u'-\dot{x})-2\theta^2Uu'+2\theta^2U\dot{x}]-c_d[\theta^2U^2/2+\theta^2Uu'+ \\
& -\theta^2U\dot{x}]+(c'''_d-3c'_d-3c''_l+c_l)[\frac{(v'-\dot{y}+R\dot{\theta})^3}{6(U+u'-\dot{x})}-\theta^3U^2/6+ \\
& -\theta(v'-\dot{y}+R\dot{\theta})^2/2+U\theta^2(v'-\dot{y}+R\dot{\theta})/2]+(-2c'_d-c''_l+c_l)\cdot \\
& \cdot[(v'-\dot{y}+R\dot{\theta})^2\theta/2+\theta^3U^2/2-\theta^2U(v'-\dot{y}+R\dot{\theta})]+(-c'_d+c_l)\cdot \\
& \cdot[\theta^2U(v'-\dot{y}+R\dot{\theta})/2-\theta^3U^2/2]+c_l\theta^3U^2/6\}+\cdots
\end{aligned}
$$

$$
\begin{aligned}
R^0_y(t) = {} & \tfrac{1}{2}\rho b\{c_l[(u'-\dot{x})^2+(v'-\dot{y}+R\dot{\theta})^2]+(c_d+c'_l)[(v'-\dot{y}+R\dot{\theta})\cdot \\
& \cdot(u'-\dot{x})-2\theta u'U+2\theta\dot{x}U+\frac{2(v'-\dot{y}+R\dot{\theta})^3}{3(U+u'-\dot{x})}-\theta(u'-\dot{x})^2+ \\
& -\theta(v'-\dot{y}+R\dot{\theta})^2]+c_d[2\theta u'U-2\theta\dot{x}U+\theta(u'-\dot{x})^2+\theta(v'-\dot{y}+R\dot{\theta})^2]+ \\
& +(c''_l-c_l+2c'_d)[(v'-\dot{y}+R\dot{\theta})^2/2+\theta^2U^2/2-\theta U(v'-\dot{y}+R\dot{\theta})+\theta^2Uu'+ \\
& -\theta^2U\dot{x}-\theta(v'-\dot{y}+R\dot{\theta})(u'-\dot{x})]+(c'_d-c_l)[\theta(v'-\dot{y}+R\dot{\theta})U-\theta^2U^2+ \\
& +\theta(v'-\dot{y}+R\dot{\theta})(u'-\dot{x})-2\theta^2Uu'+2\theta^2U\dot{x}]-c_l[\theta^2U^2/2+\theta^2Uu'+ \\
& -\theta^2U\dot{x}]+(c'''_l-3c'_l+3c''_d-c_d)[\frac{(v'-\dot{y}+R\dot{\theta})^3}{6(U+u'-\dot{x})}-\theta^3U^2/6+ \\
& -\theta(v'-\dot{y}+R\dot{\theta})^2/2+U\theta^2(v'-\dot{y}+R\dot{\theta})/2]+(-2c'_l+c''_d-c_d)\cdot \\
& \cdot[(v'-\dot{y}+R\dot{\theta})^2\theta/2+\theta^3U^2/2-\theta^2U(v'-\dot{y}+R\dot{\theta})]-(c'_l+c_d)\cdot \\
& \cdot[\theta^2U(v'-\dot{y}+R\dot{\theta})/2-\theta^3U^2/2]-c_d\theta^3U^2/6\}+\cdots
\end{aligned}
$$

$$
\begin{aligned}
R^0_\theta(t) = {} & \tfrac{1}{2}\rho b^2\{c_m[(u'-\dot{x})^2+(v'-\dot{y}+R\dot{\theta})^2]+c'_m[(v'-\dot{y}+R\dot{\theta})\cdot \\
& \cdot(u'-\dot{x})-2\theta u'U+2\theta\dot{x}U+\frac{2(v'-\dot{y}+R\dot{\theta})^3}{3(U+u'-\dot{x})}-\theta(u'-\dot{x})^2+ \\
& -\theta(v'-\dot{y}+R\dot{\theta})^2]+c''_m[(v'-\dot{y}+R\dot{\theta})^2/2+\theta^2U^2/2+ \\
& -\theta U(v'-\dot{y}+R\dot{\theta})+\theta^2Uu'-\theta^2U\dot{x}-\theta(v'-\dot{y}+R\dot{\theta})(u'-\dot{x})]+ \\
& +c'''_m[\frac{(v'-\dot{y}+R\dot{\theta})^3}{6(U+u'-\dot{x})}-\theta^3U^2/6-\theta(v'-\dot{y}+R\dot{\theta})^2/2+U\theta^2(v'-\dot{y}+R\dot{\theta})/2]\}+\cdots
\end{aligned}
$$

4. EQUATIONS OF MOTION

Hp: linear system (3DOFs) with viscous damping

$$\boxed{\mathbb{M}\ddot{\mathbb{q}}(t)+\mathbb{C}\dot{\mathbb{q}}(t)+\mathbb{K}\mathbb{q}(t)=\mathbb{F}(t)}$$

$\mathbb{q}, \dot{\mathbb{q}}, \ddot{\mathbb{q}}$ = displacement, velocity, acceleration vectors
$\mathbb{M}, \mathbb{C}, \mathbb{K}$ = mass, damping, stiffness matrices

$$\mathbb{F}(t)=\bar{\mathbb{F}}+\mathbb{F}'(t)+\mathbb{F}_a(t)+\mathbb{R}^\circ(t)$$
$$\mathbb{F}'(t)=\mathbb{F}'_u(t)+\mathbb{F}'_v(t)+\mathbb{F}'_w(t)$$
$$\mathbb{F}_a(t)=-\mathbb{C}^\circ\dot{\mathbb{q}}(t)-\mathbb{K}^\circ\mathbb{q}(t)$$

Non-linear equations of motion

$$\boxed{\mathbb{M}\ddot{\mathbb{q}}(t)+(\mathbb{C}+\mathbb{C}^\circ)\dot{\mathbb{q}}(t)+(\mathbb{K}+\mathbb{K}^\circ)\mathbb{q}(t)=\bar{\mathbb{F}}+\mathbb{F}'(t)+\mathbb{R}^\circ(t)}$$

$\mathbb{C}^*=\mathbb{C}+\mathbb{C}^\circ$; $\quad\mathbb{C}^\circ$ = aerodynamic damping matrix
$\mathbb{K}^*=\mathbb{K}+\mathbb{K}^\circ$; $\quad\mathbb{K}^\circ$ = aerodynamic stiffness matrix

Linearized equations of motion

$\mathbb{R}^\circ(t)\sim\mathbb{O}$ for small turbulence and small motions $\Rightarrow$

$$\boxed{\mathbb{M}\ddot{\mathbb{q}}(t)+(\mathbb{C}+\mathbb{C}^\circ)\dot{\mathbb{q}}(t)+(\mathbb{K}+\mathbb{K}^\circ)\mathbb{q}(t)=\bar{\mathbb{F}}+\mathbb{F}'(t)}$$

$\mathbb{C}^\circ, \mathbb{K}^\circ = f(\rho, \bar{u}, b, R_o;$ aerod. parameters$) \Rightarrow$
$\mathbb{C}^*, \mathbb{K}^* = f($struct. prop.; $\rho, \bar{u}, b, R_o;$ aerod. param.$) \Rightarrow$

in principle, $\mathbb{C}^*, \mathbb{K}^*$ are non-symmetric, non-positive definite matrices

Any shape

$$\mathbb{C}^\circ = \frac{1}{2}\varrho\bar{u}b\begin{bmatrix} 2c_d & (c'_d - c_e) & -R_o(c'_d - c_e) \\ 2c_e & (c_d + c'_e) & -R_o(c_d + c'_e) \\ 2bc_m & bc'_m & -bR_o c'_m \end{bmatrix}$$

$$\mathbb{K}^\circ = \frac{1}{2}\varrho\bar{u}^2 b\begin{bmatrix} 0 & 0 & c'_d \\ 0 & 0 & c'_e \\ 0 & 0 & bc'_m \end{bmatrix}$$

X symmetry axis $\Rightarrow$ $c'_d = c_e = c_m = 0$

$$\mathbb{C}^\circ = \frac{1}{2}\varrho\bar{u}b\begin{bmatrix} 2c_d & 0 & 0 \\ 0 & c_d + c'_e & -R_o(c_d + c'_e) \\ 0 & bc'_m & -bR_o c'_m \end{bmatrix}$$

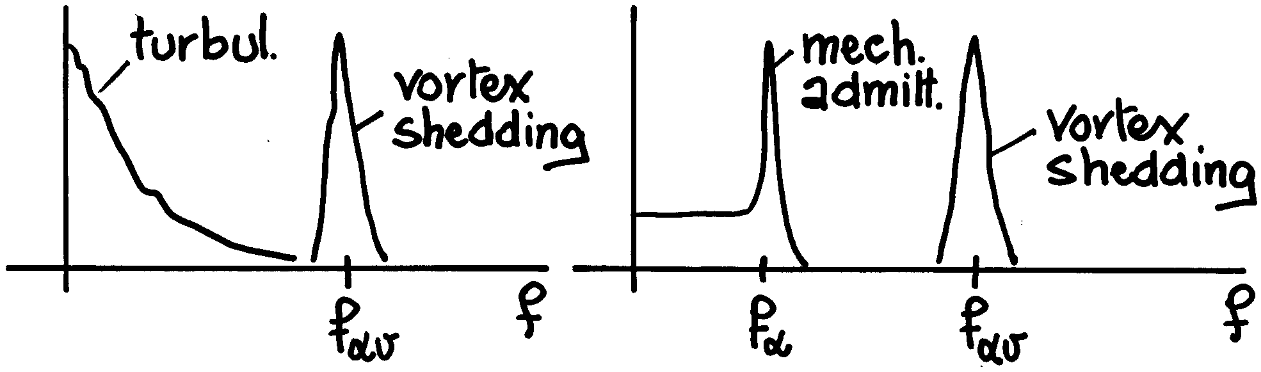
$$\mathbb{K}^\circ = \frac{1}{2}\varrho\bar{u}^2 b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c'_e \\ 0 & 0 & bc'_m \end{bmatrix}$$

Polar symmetry $\Rightarrow$ $c'_d = c_e = c_m = 0$ & $c'_e = c'_m = 0$

$$\mathbb{C}^\circ = \frac{1}{2}\varrho\bar{u}b\begin{bmatrix} 2c_d & 0 & 0 \\ 0 & c_d & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (symmetric, non-negative definite matrix)

$$\mathbb{K}^\circ = \frac{1}{2}\varrho\bar{u}^2 b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ (null matrix)

Experience teaches that this formulation provides a reliable model of the physical phenomenon, provided that structural oscillations be confined to the range of low frequencies f or, more precisely, to the range of reduced frequencies $fb/\bar{u} \ll S$



<u>harmonic content of motion within the range of low reduced frequencies</u>

$$\mathbb{M}\ddot{\mathbb{q}}(t) + (\mathbb{C}+\mathbb{C}^\circ)\dot{\mathbb{q}}(t) + (\mathbb{K}+\mathbb{K}^\circ)\mathbb{q}(t) = \bar{\mathbb{F}} + \mathbb{F}'(t)$$

<u>harmonic content of motion outside the range of low reduced frequencies</u>

$$\mathbb{M}\ddot{\mathbb{q}}(t) + [\mathbb{C}+\tilde{\mathbb{C}}^\circ(f)]\dot{\mathbb{q}}(t) + [\mathbb{K}+\tilde{\mathbb{K}}^\circ(f)]\mathbb{q}(t) = \bar{\mathbb{F}} + \mathbb{F}'(t)$$

$$\lim_{f\to 0} \tilde{\mathbb{C}}^\circ(f) = \mathbb{C}^\circ \quad ; \quad \lim_{f\to 0} \tilde{\mathbb{K}}^\circ(f) = \mathbb{K}^\circ$$

$\mathbb{C}^\circ, \mathbb{K}^\circ$ = aerodynamic damping and stiffness matrices
$\tilde{\mathbb{C}}^\circ(f), \tilde{\mathbb{K}}^\circ(f)$ = aerodynamic derivative matrices

Lagrangian space equations

$$\boxed{\mathbb{M}\ddot{\mathbb{q}}(t)+(\mathbb{C}+\bar{\mathbb{C}}^{\circ})\dot{\mathbb{q}}(t)+(\mathbb{K}+\bar{\mathbb{K}}^{\circ})\mathbb{q}(t)=\bar{\mathbb{F}}+\mathbb{F}'(t)=\mathbb{F}(t)}$$

system of $n=3$ linear differential equations of the second order

<u>State space equations</u>

$$\begin{cases}\dot{\mathbb{q}}(t)-\dot{\mathbb{q}}(t)=\mathbb{O}\\ \ddot{\mathbb{q}}(t)+\mathbb{M}^{-1}(\mathbb{C}+\bar{\mathbb{C}}^{\circ})\dot{\mathbb{q}}(t)+\mathbb{M}^{-1}(\mathbb{K}+\bar{\mathbb{K}}^{\circ})\mathbb{q}(t)=\mathbb{M}^{-1}\mathbb{F}(t)\end{cases}$$

$$\begin{Bmatrix}\dot{\mathbb{q}}(t)\\ \ddot{\mathbb{q}}(t)\end{Bmatrix}=\begin{bmatrix}\mathbb{O} & \mathbb{I}\\ -\mathbb{M}^{-1}(\mathbb{K}+\bar{\mathbb{K}}^{\circ}) & -\mathbb{M}^{-1}(\mathbb{C}+\bar{\mathbb{C}}^{\circ})\end{bmatrix}\begin{Bmatrix}\mathbb{q}(t)\\ \dot{\mathbb{q}}(t)\end{Bmatrix}+\begin{Bmatrix}\mathbb{O}\\ \mathbb{M}^{-1}\mathbb{F}(t)\end{Bmatrix}$$

$$\boxed{\dot{\mathbb{z}}(t)=\mathbb{G}\,\mathbb{z}(t)+\mathbb{p}(t)}$$

system of $2n=6$ linear differential equations of the first order

$$\mathbb{z}(t)=\begin{Bmatrix}\mathbb{q}(t)\\ \dot{\mathbb{q}}(t)\end{Bmatrix}=\text{state vector}$$

$$\mathbb{G}=\begin{bmatrix}\mathbb{O} & \mathbb{I}\\ -\mathbb{M}^{-1}(\mathbb{K}+\bar{\mathbb{K}}^{\circ}) & -\mathbb{M}^{-1}(\mathbb{C}+\bar{\mathbb{C}}^{\circ})\end{bmatrix}=\text{dynamic matrix}$$

$$\mathbb{p}(t)=\begin{Bmatrix}\mathbb{O}\\ \mathbb{M}^{-1}\mathbb{F}(t)\end{Bmatrix}=\text{force vector in the state space}$$

- State variables equation

$$\dot{\mathbb{Z}}(t) = \mathbb{G}\mathbb{Z}(t) + \mathbb{P}(t) \qquad (1)$$

- Homogeneous equation $\cdot\ \mathbb{P}(t) = \mathbb{O} \Rightarrow$

$$\dot{\mathbb{Z}}(t) = \mathbb{G}\mathbb{Z}(t) \qquad (2)$$

$\mathbb{Z}(t) = \mathbb{U}e^{\lambda t}$ particular integral of Eq. 2 provided that

$$\lambda \mathbb{U} e^{\lambda t} = \mathbb{G}\mathbb{U}e^{\lambda t} \Rightarrow (\mathbb{G} - \lambda \mathbb{I})\mathbb{U} = \mathbb{O} \qquad (3)$$

trivial solution $\mathbb{U} = \mathbb{O}$

non-trivial solution provided that $\mathrm{Det}(\mathbb{G} - \lambda\mathbb{I}) = \emptyset \qquad (4)$

Eq. 4 leads to a set of $2n$ roots or eigenvalues $\lambda_1, .. \lambda_{2n}$ to which $2n$ eigenvectors $\mathbb{U}_1, .. \mathbb{U}_{2n}$ correspond ($n = 3$). If there are real eigenvalues, also the corresponding eigenvectors are real; if there are complex eigenvalues, they occur in conjugate pairs and the corresponding eigenvectors have the same property.

The general integral of Eq. 2 is provided by the linear combination of its $2n$ particular integrals, that is:

$$\boxed{\mathbb{Z}(t) = \sum_{1}^{2n}{}_K A_K \mathbb{U}_K e^{\lambda_K t}}$$

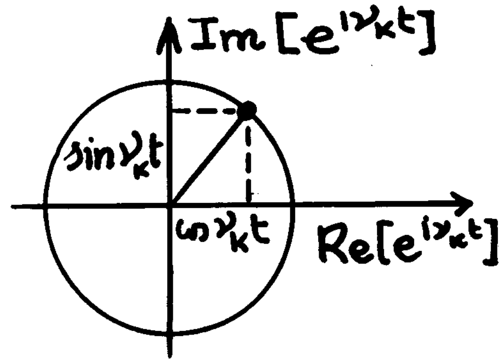
$A_1, .. A_{2n}$ are constant depending on the initial conditions.

$$\boxed{\mathbf{z}(t)=\sum_{1}^{2n}{}_K A_K \mathbf{u}_K e^{\lambda_K t}} \qquad \boxed{\lambda_K=\mu_K+i\nu_K} \Rightarrow$$

$$\boxed{\mathbf{z}(t)=\sum_{1}^{2n}{}_K A_K \mathbf{u}_K e^{\mu_K t} e^{i\nu_K t}}$$

The eigenvalues λ_K are called the POLES of the system and define its stability.

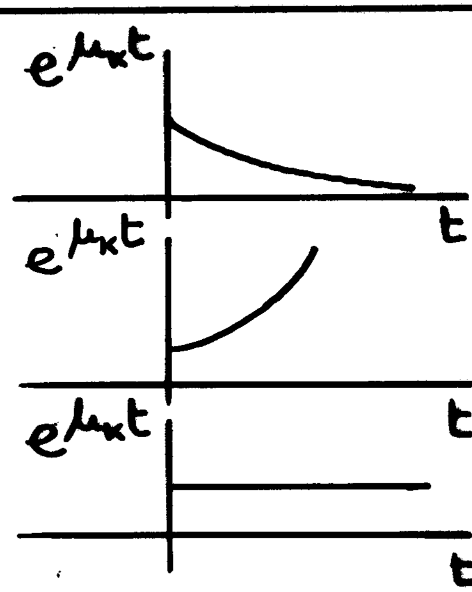
$e^{i\nu_K t}=\cos\nu_K t+i\sin\nu_K t$ corresponds to an oscillatory motion with circular frequency ν_K. For $\nu_K=0$ (static case) the motion is non-oscillatory



$\mu_K<0 \Rightarrow$ a damping effect

$\mu_K>0 \Rightarrow$ an amplification

$\mu_K=0 \Rightarrow$ a neutral condition



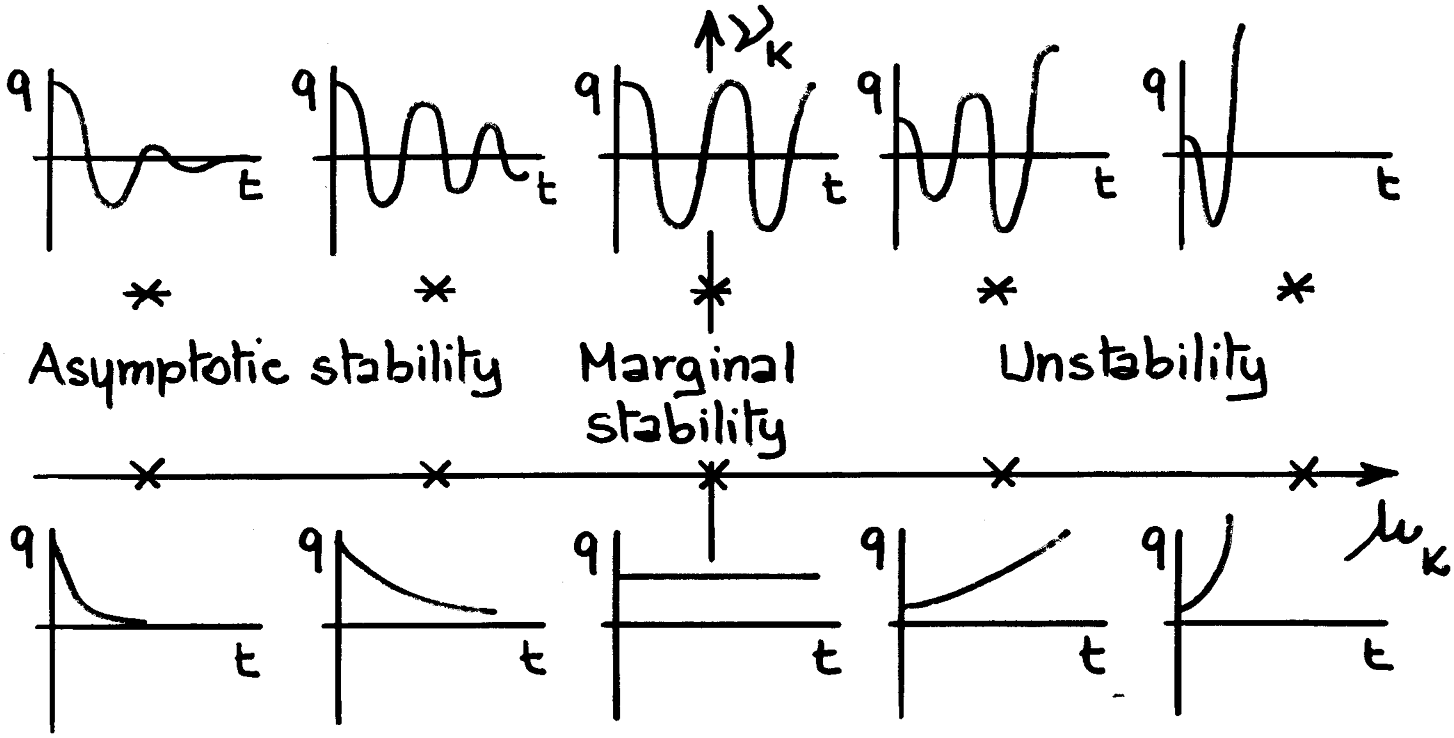
The system results:

- ASYMPTOTICALLY STABLE if all $\mu_K<0$
- UNSTABLE if at least one $\mu_K>0$
- MARGINALLY STABLE if at least one $\mu_K=0$ and no $\mu_K>0$

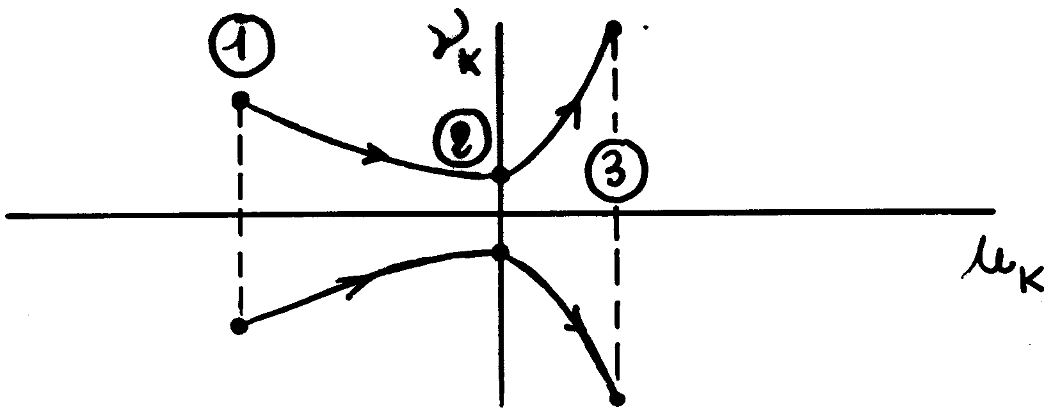
Since μ_K depends on $\bar{u}$, the stability depends on $\bar{u}$.

$$\boxed{z(t)=\sum_{K=1}^{2m} A_K \Phi_K e^{\mu_K t} e^{i\nu_K t}} \qquad \boxed{\lambda_K=\mu_K+i\nu_K}$$

ARGAND-GAUSS PLANE OF THE POLES



The evolution of λ_K in the plane μ_K, ν_K as a function of $\bar{u}$ defines the evolution of the system stability on varying the mean wind velocity



1. $\bar{u}=\bar{u}_1 \Rightarrow$ asymptotic stability
2. $\bar{u}=\bar{u}_2 \Rightarrow$ marginal stability = bifurcation point
3. $\bar{u}=\bar{u}_3 \Rightarrow$ unstability

State-space equation of motion

$$\dot{\mathbf{z}}(t) = \mathbf{G}\mathbf{z}(t)$$

$$\mathbf{z}(t) = \mathbf{d}e^{\lambda t}$$

$$\mathbf{d}\lambda e^{\lambda t} = \mathbf{G}\mathbf{d}e^{\lambda t}$$

$$(\mathbf{G} - \lambda\mathbf{I})\mathbf{d} = \mathbf{0}$$

Eigenvalues

$$\lambda_k = \mu_k + i\nu_k = -\xi_k\omega_k + i\omega_k \quad (k = 1, 2, ...2n;\ n = 3)$$

$$\mu_k = \mathrm{Re}(\lambda_k) = -\xi_k\omega_k \Rightarrow \xi_k = -\frac{\mu_k}{\omega_k} = -\frac{\mu_k}{\nu_k}$$

$$\nu_k = \mathrm{Im}(\lambda_k) = \omega_k$$

General integral of motion

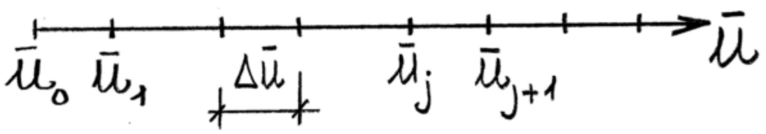
$$\mathbf{z}(t) = \sum_{1}^{2n} A_k\mathbf{d}_k e^{\lambda_k t} = \sum_{1}^{2n} A_k\mathbf{d}_k e^{-\xi_k\omega_k t}\mathbf{e}^{i\omega_k t}$$

The dynamic system results:

- ASYMPTOTICALLY STABLE if all $\xi_k > 0$
- UNSTABLE if at least one $\xi_k < 0$
- MARGINALLY STABLE if at least one $\xi_k = 0$ and no $\xi_k < 0$

Solution of the problem

$$\mathbf{G} = \mathbf{G}(\bar{u}) \Rightarrow \lambda_k = \lambda_k(\bar{u}) = -\xi_k(\bar{u})\omega_k(\bar{u}) + i\omega_k(\bar{u}) \quad (k = 1, 2, ...2n)$$



$$j = 1 \Rightarrow \bar{u} = \bar{u}_j \Rightarrow \mathbf{G} = \mathbf{G}(\bar{u}_j) \Rightarrow \lambda_k = \lambda_k(\bar{u}_j) \Rightarrow \xi_k = \xi_k(\bar{u}_j), \omega_k = \omega_k(\bar{u}_j)$$

$\exists\xi_k(\bar{u}) < 0$? If YES, $\bar{u}_c \in (\bar{u}_{j-1}, \bar{u}_j)$

If NO, $j = j + 1$

Decoupled aeroelastic equations of motion

$$m\ddot{x}(t)+\left[c+\rho\bar{u}c_D A\right]\dot{x}(t)+kx(t)=f_x(t)$$

$$m\ddot{y}(t)+\left[c+\frac{1}{2}\rho\bar{u}b\left(c_d+c_l'\right)\right]\dot{y}(t)+ky(t)=f_y(t)$$

$$I\ddot{\theta}(t)+\left[c-\frac{1}{2}\rho\bar{u}b^2R_0c_m'\right]\dot{\theta}(t)+\left[k+\frac{1}{2}\rho\bar{u}^2b^2c_m'\right]\theta(t)=m_\theta(t)$$

1 - D.O.F. aeroelastic equation of motion

$$m\ddot{q}(t)+\left[c+c^0\right]\dot{q}(t)+\left[k+k^0\right]q(t)=f(t)$$

3 - D.O.F. coupled aeroelastic equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\left[\mathbf{C}+\mathbf{C}^0\right]\dot{\mathbf{q}}(t)+\left[\mathbf{K}+\mathbf{K}^0\right]\mathbf{q}(t)=\mathbf{f}(t)$$

3 - D.O.F. coupled aeroelastic equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\left[\mathbf{C}+\mathbf{C}^0\right]\dot{\mathbf{q}}(t)+\left[\mathbf{K}+\mathbf{K}^0\right]\mathbf{q}(t)=\mathbf{f}(t)$$

3 - D.O.F. coupled dynamic equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t)$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} c_{xx} & c_{xy} & c_{x\theta} \\ c_{yx} & c_{yy} & c_{y\theta} \\ c_{\theta x} & c_{\theta y} & c_{\theta\theta} \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}+\begin{bmatrix} k_{xx} & k_{xy} & k_{x\theta} \\ k_{yx} & k_{yy} & k_{y\theta} \\ k_{\theta x} & k_{\theta y} & k_{\theta\theta} \end{bmatrix}\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}=\begin{Bmatrix} f_x \\ f_y \\ m_\theta \end{Bmatrix}$$

3 - D.O.F. uncoupled dynamic equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\mathbf{C}\dot{\mathbf{q}}(t)+\mathbf{K}\mathbf{q}(t)=\mathbf{f}(t)$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{Bmatrix}+\begin{bmatrix} c_{xx} & 0 & 0 \\ 0 & c_{yy} & 0 \\ 0 & 0 & c_{\theta\theta} \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}+\begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{\theta\theta} \end{bmatrix}\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}=\begin{Bmatrix} f_x \\ f_y \\ m_\theta \end{Bmatrix}$$

3 - D.O.F. mechanically coupled aeroelastic equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\left[\mathbf{C}+\mathbf{C}^{0}\right]\dot{\mathbf{q}}(t)+\left[\mathbf{K}+\mathbf{K}^{0}\right]\mathbf{q}(t)=\mathbf{f}(t)$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{Bmatrix}+$$

$$+\left(\begin{bmatrix} c_{xx} & c_{xy} & c_{x\theta} \\ c_{yx} & c_{yy} & c_{y\theta} \\ c_{\theta x} & c_{\theta y} & c_{\theta\theta} \end{bmatrix}+\frac{1}{2}\rho\bar{u}b\begin{bmatrix} 2c_d & (c_d'-c_l) & -R_0(c_d'-c_l) \\ 2c_l & (c_d+c_l') & -R_0(c_d+c_l') \\ 2bc_m & bc_m' & -R_0bc_m' \end{bmatrix}\right)\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}+$$

$$+\left(\begin{bmatrix} k_{xx} & k_{xy} & k_{x\theta} \\ k_{yx} & k_{yy} & k_{y\theta} \\ k_{\theta x} & k_{\theta y} & k_{\theta\theta} \end{bmatrix}+\frac{1}{2}\rho\bar{u}^2b\begin{bmatrix} 0 & 0 & c_d' \\ 0 & 0 & c_l' \\ 0 & 0 & bc_m' \end{bmatrix}\right)\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}=\begin{Bmatrix} f_x \\ f_y \\ m_\theta \end{Bmatrix}$$

3 - D.O.F. mechanically uncoupled aeroelastic equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\left[\mathbf{C}+\mathbf{C}^{0}\right]\dot{\mathbf{q}}(t)+\left[\mathbf{K}+\mathbf{K}^{0}\right]\mathbf{q}(t)=\mathbf{f}(t)$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{Bmatrix}+$$

$$+\left(\begin{bmatrix} c_{xx} & 0 & 0 \\ 0 & c_{yy} & 0 \\ 0 & 0 & c_{\theta\theta} \end{bmatrix}+\frac{1}{2}\rho\bar{u}b\begin{bmatrix} 2c_d & (c_d'-c_l) & -R_0(c_d'-c_l) \\ 2c_l & (c_d+c_l') & -R_0(c_d+c_l') \\ 2bc_m & bc_m' & -R_0bc_m' \end{bmatrix}\right)\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}+$$

$$+\left(\begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{\theta\theta} \end{bmatrix}+\frac{1}{2}\rho\bar{u}^2b\begin{bmatrix} 0 & 0 & c_d' \\ 0 & 0 & c_l' \\ 0 & 0 & bc_m' \end{bmatrix}\right)\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}=\begin{Bmatrix} f_x \\ f_y \\ m_\theta \end{Bmatrix}$$

3 - D.O.F. fully uncoupled aeroelastic equations of motion

$$\mathbf{M}\ddot{\mathbf{q}}(t)+\left[\mathbf{C}+\mathbf{C}^{0}\right]\dot{\mathbf{q}}(t)+\left[\mathbf{K}+\mathbf{K}^{0}\right]\mathbf{q}(t)=\mathbf{f}(t)$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{Bmatrix}+$$

$$+\left(\begin{bmatrix} c_{xx} & 0 & 0 \\ 0 & c_{yy} & 0 \\ 0 & 0 & c_{\theta\theta} \end{bmatrix}+\frac{1}{2}\rho\bar{u}b\begin{bmatrix} 2c_d & 0 & 0 \\ 0 & (c_d+c_l') & 0 \\ 0 & 0 & -R_0bc_m' \end{bmatrix}\right)\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{Bmatrix}+$$

$$+\left(\begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{\theta\theta} \end{bmatrix}+\frac{1}{2}\rho\bar{u}^2b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & bc_m' \end{bmatrix}\right)\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}=\begin{Bmatrix} f_x \\ f_y \\ m_\theta \end{Bmatrix}$$

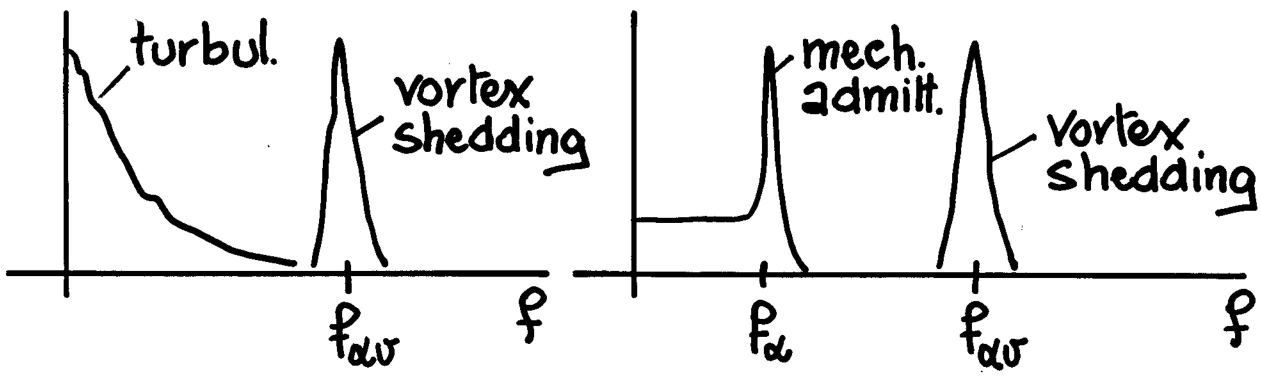
Decoupled aeroelastic equations of motion

$$m\ddot{x}(t)+\left[c+\rho\bar{u}c_DA\right]\dot{x}(t)+kx(t)=f_x(t)$$

$$m\ddot{y}(t)+\left[c-\frac{1}{2}\rho\bar{u}b(c_d+c_l')\right]\dot{y}(t)+ky(t)=f_y(t)$$

$$I\ddot{\theta}(t)+\left[c-\frac{1}{2}\rho\bar{u}b^2R_0c_m'\right]\dot{\theta}(t)+\left[k+\frac{1}{2}\rho\bar{u}^2b^2c_m'\right]\theta(t)=m_\theta(t)$$

Experience teaches that this formulation provides a reliable model of the physical phenomenon, provided that structural oscillations be confined to the range of low frequencies f or, more precisely, to the range of reduced frequencies $fb/\bar{u} \ll S$



harmonic content of motion within the range of low reduced frequencies

$$\mathbb{M}\ddot{\mathbf{q}}(t)+(\mathbb{C}+\mathbb{C}^{\circ})\dot{\mathbf{q}}(t)+(\mathbb{K}+\mathbb{K}^{\circ})\mathbf{q}(t)=\bar{\mathbb{F}}+\mathbb{F}'(t)$$

harmonic content of motion outside the range of low reduced frequencies

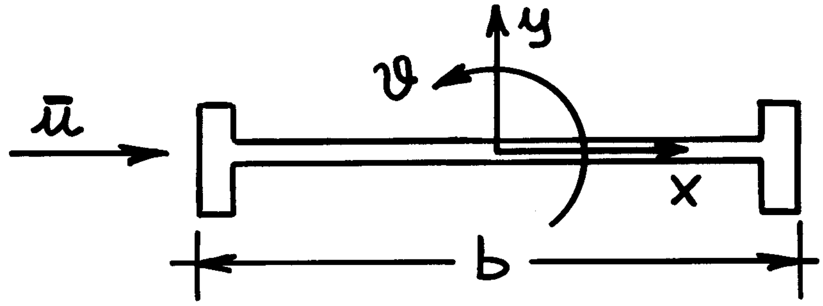
$$\mathbb{M}\ddot{\mathbf{q}}(t)+[\mathbb{C}+\tilde{\mathbb{C}}^{\circ}(f)]\dot{\mathbf{q}}(t)+[\mathbb{K}+\tilde{\mathbb{K}}^{\circ}(f)]\mathbf{q}(t)=\bar{\mathbb{F}}+\mathbb{F}'(t)$$

$$\lim_{f\to 0}\tilde{\mathbb{C}}^{\circ}(f)=\mathbb{C}^{\circ}\ ; \ \lim_{f\to 0}\tilde{\mathbb{K}}^{\circ}(f)=\mathbb{K}^{\circ}$$

$\mathbb{C}^{\circ}, \mathbb{K}^{\circ}$ = aerodynamic damping and stiffness matrices

$\tilde{\mathbb{C}}^{\circ}(f), \tilde{\mathbb{K}}^{\circ}(f)$ = aerodynamic derivative matrices

BRIDGE AERODYNAMICS



$$\tilde{\mathbb{C}}^{o}(f) = -\frac{1}{2}\varrho\bar{u}b\begin{bmatrix} KP_1^*(K) & KP_5^*(K) & bKP_2^*(K) \\ KH_5^*(K) & KH_1^*(K) & bKH_2^*(K) \\ bKA_5^*(K) & bKA_1^*(K) & b^2KA_2^*(K) \end{bmatrix}$$

$$\tilde{\mathbb{K}}^{o}(f) = -\frac{1}{2}\varrho\bar{u}^2b\begin{bmatrix} K^2P_4^*(K)/b & K^2P_6^*(K)/b & K^2P_3^*(K) \\ K^2H_6^*(K)/b & K^2H_4^*(K)/b & K^2H_3^*(K) \\ K^2A_6^*(K) & K^2A_4^*(K) & K^2A_3^*(K)b \end{bmatrix}$$

A_i^*, H_i^*, P_i^* $(i=1,2,..6)$ = aerodynamic derivatives (or flutterderivatives)

$K = 2\pi f b/\bar{u}$ = reduced frequency

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