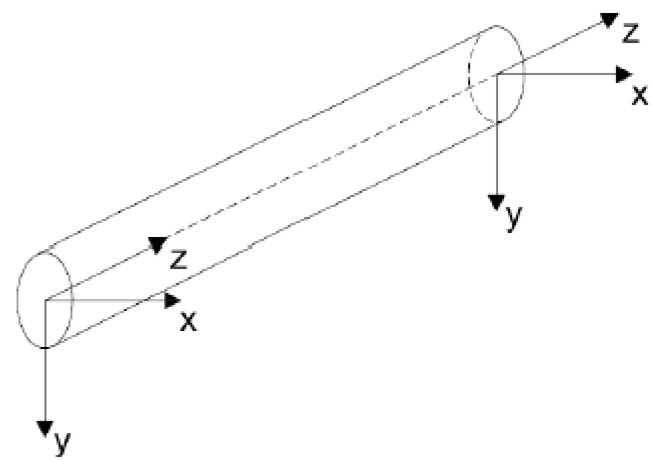
VIGAS RECTAS CONTINUAS, ECUACIONES ANGULARES, ECUACIÓN DE TRES MOMENTOS Y APLICACIONES

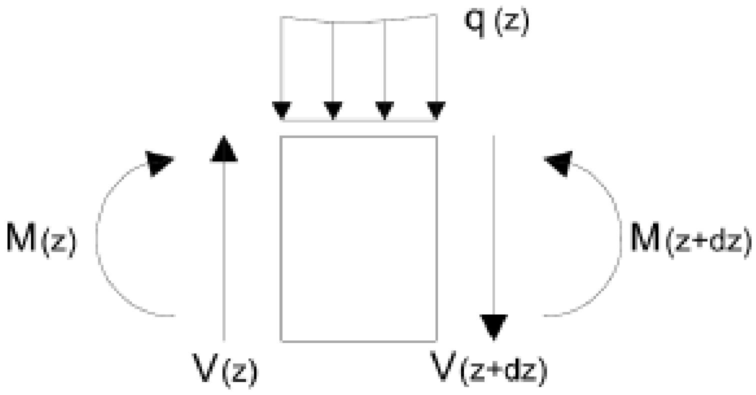
Convenciones e Hipótesis

Estructuras compuestas por **vigas rectas**, para simplificar la representación gráfica que tienen el eje de la viga horizontal

a) Vigas rectas
b) Materiales elásticos y lineales
c) Secciones simétricas respecto del eje y.
d) Cargas distribuidas q o concentradas P según el eje y.



Equilibrio de un tramo



Suma de Fuerzas verticales =0

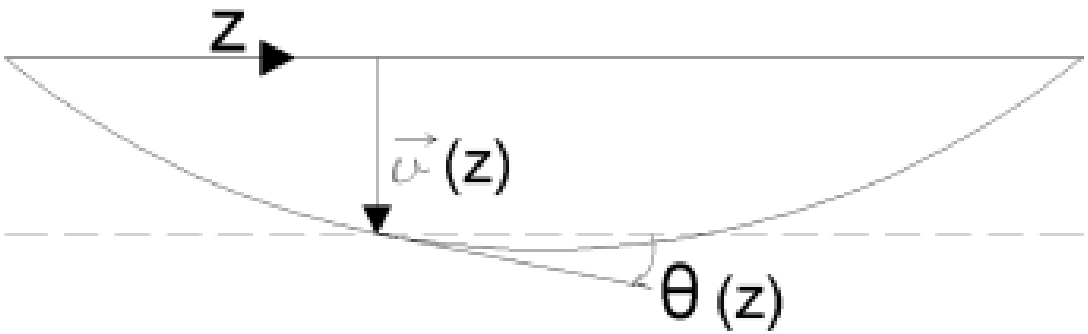
$$\frac{dV}{dz} + q(z) = 0$$

Suma de Momentos en z+dz =0

$$\frac{dM}{dz} = V(z)$$

$$\frac{d^2M}{dz^2} = \frac{dV}{dz} = -q$$

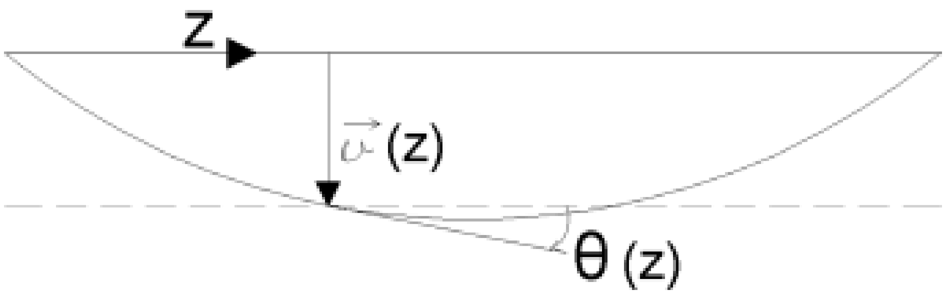
Relación Momento-Curvatura



$$\frac{d^2v}{dz^2} = \frac{d\theta}{dz} = -\frac{M}{EI_x}$$

Analogía de Mohr

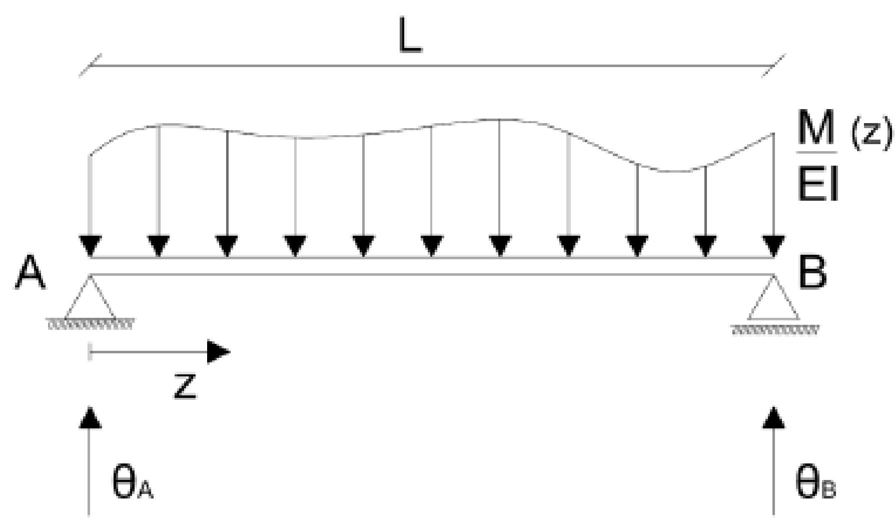
$$\nu \leftrightarrow M$$
$$\theta \leftrightarrow V$$
$$\frac{M}{EI_x} \leftrightarrow q$$



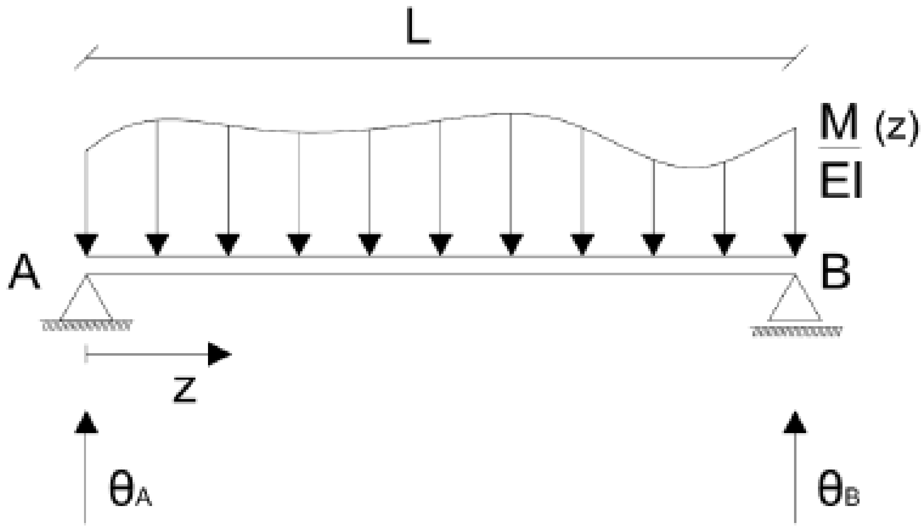
$$\nu(0) = \nu(L) = 0$$
$$M(0) = M(L) = 0$$

Angulos de giro en los apoyos

$$\theta_A = \theta(0)$$
$$\theta_B = -\theta(L)$$
$$v_A = v(0)$$
$$v_B = v(L)$$



Reacción en los apoyos

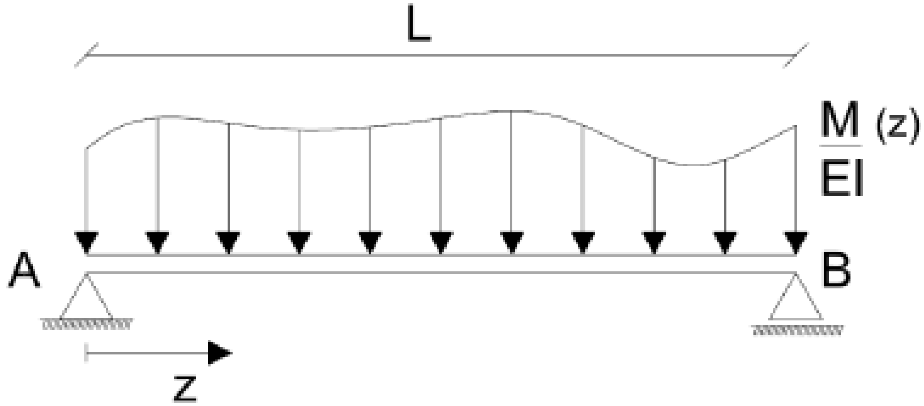


$$\theta_A = \frac{1}{L}\int_0^L \frac{M}{EI_x}(L-z)dz$$

$$\theta_B = \frac{1}{L}\int_0^L \frac{M}{EI_x} z dz$$

Considerando una viga S.A.

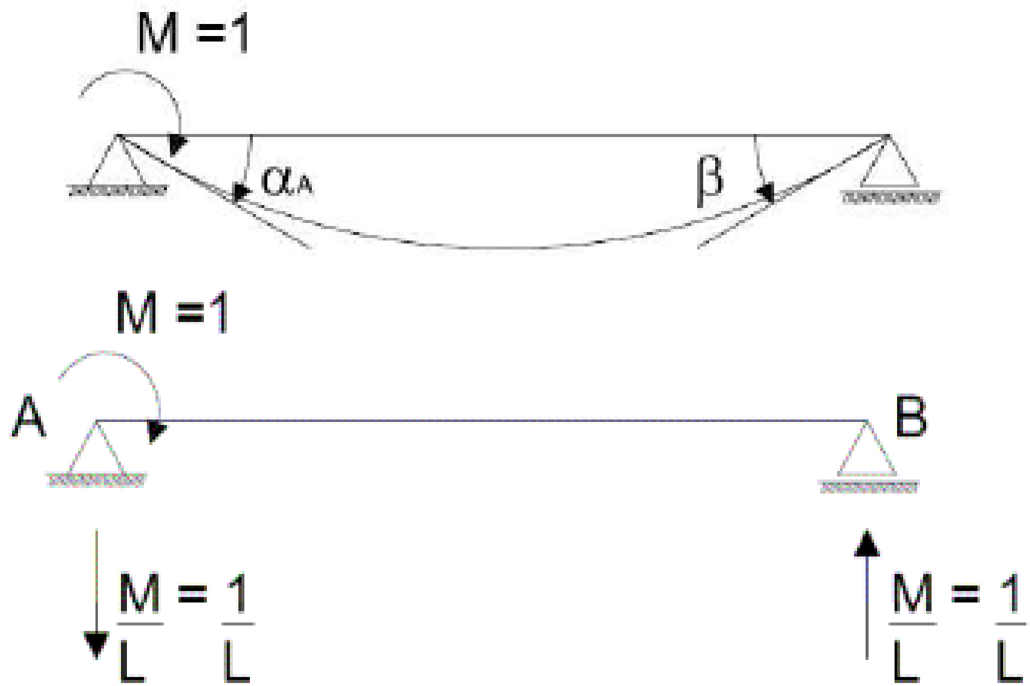
Para las cargas aplicadas en el vano



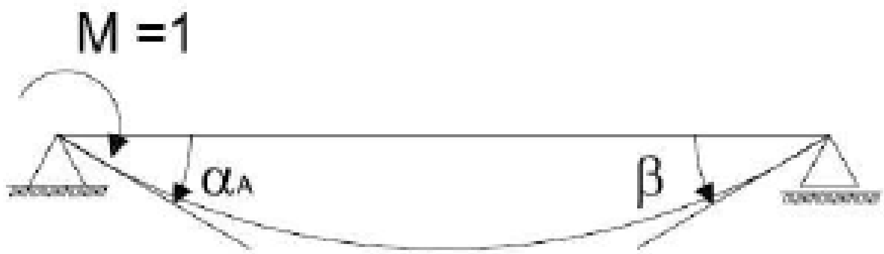
$$\alpha_{0A} = \frac{1}{L}\int_0^L \frac{M}{EI_x}(L-z)dz \qquad \alpha_{0B} = \frac{1}{L}\int_0^L \frac{M}{EI_x} z dz$$

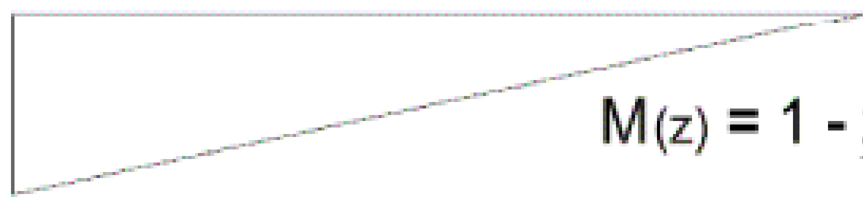
Giros

Considerando un momento unitario aplicado en A:



Giros (cont.)



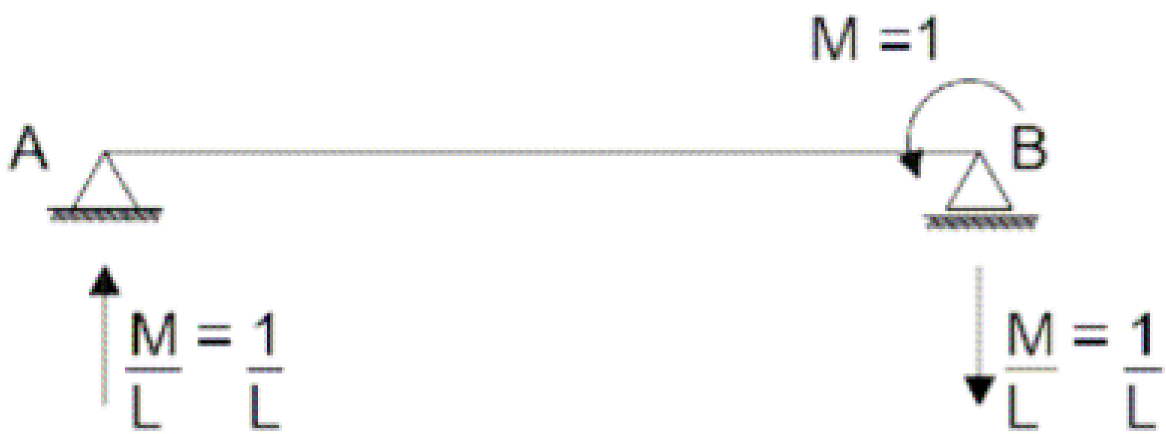


$$M_{(z)} = 1 - \frac{z}{L} = \frac{L - z}{L}$$

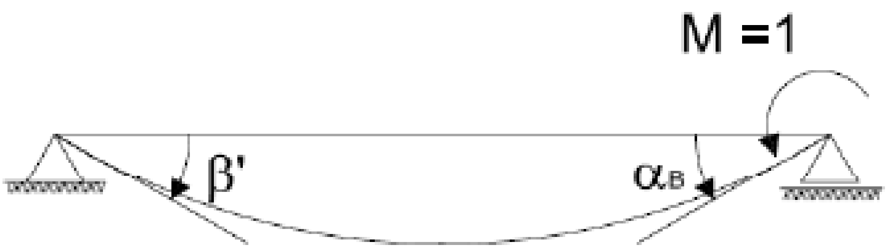
$$\alpha_A = \frac{1}{L}\int_0^L \frac{\left(1 - \frac{z}{L}\right)(L - z)}{EI_x} dz$$

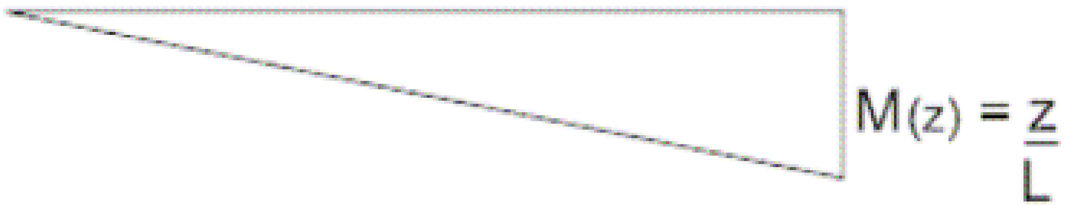
$$\beta = \frac{1}{L}\int_0^L \frac{\left(1 - \frac{z}{L}\right)}{EI_x} z dz$$

Giros (cont.)



Giros (cont.)

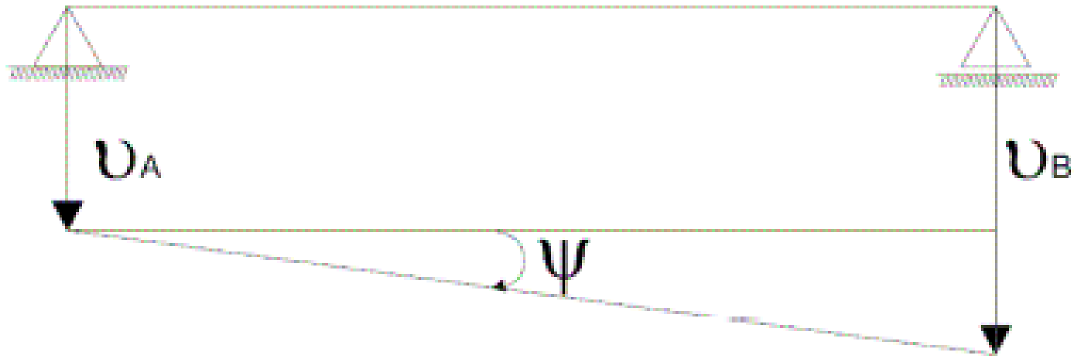




$$\alpha_B = \frac{1}{L}\int_0^L \frac{z}{L} \times \frac{z}{EI_x} dz$$

$$\beta' = \frac{1}{L}\int_0^L \frac{\frac{z}{L}(L-z)}{EI_x} dz = \beta$$

Descenso de apoyos

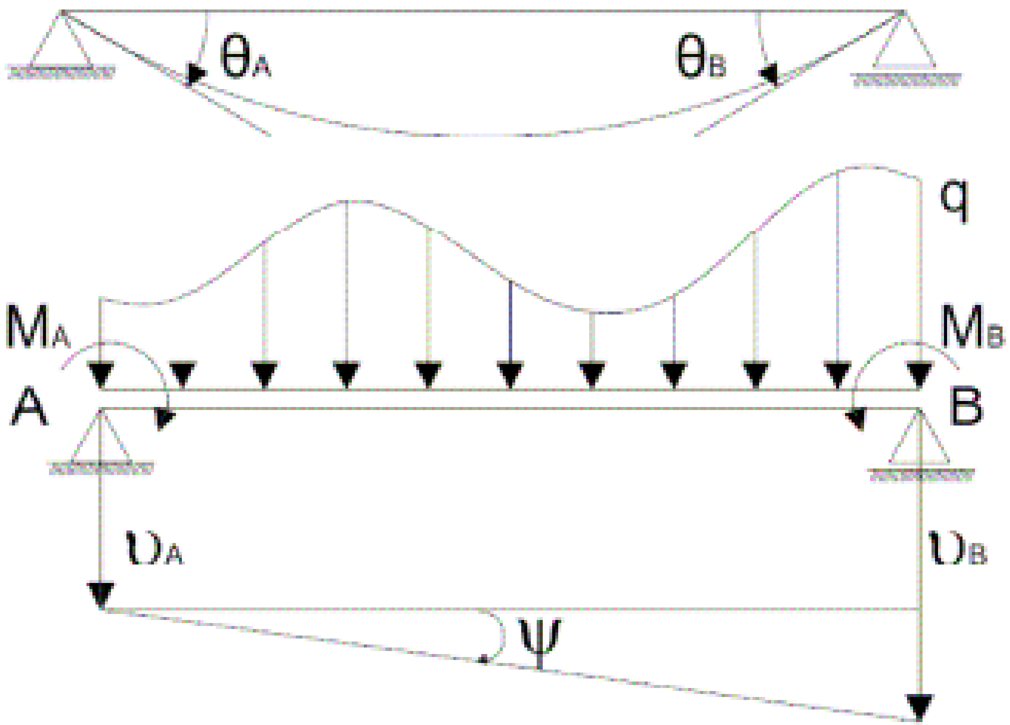


$$v_A = v(0)$$
$$v_B = v(L)$$

$$\psi \cong tg\,\psi = \frac{v_B - v_A}{L}$$

$$\theta_A = \psi$$
$$\theta_B = -\psi$$

Caso General (EI cualquiera)



$$\theta_A = \alpha_{0A} + M_A \cdot \alpha_A + M_B \cdot \beta + \psi$$

$$\theta_B = \alpha_{0B} + M_A \cdot \beta + M_B \cdot \alpha_B - \psi$$

Caso EI constante

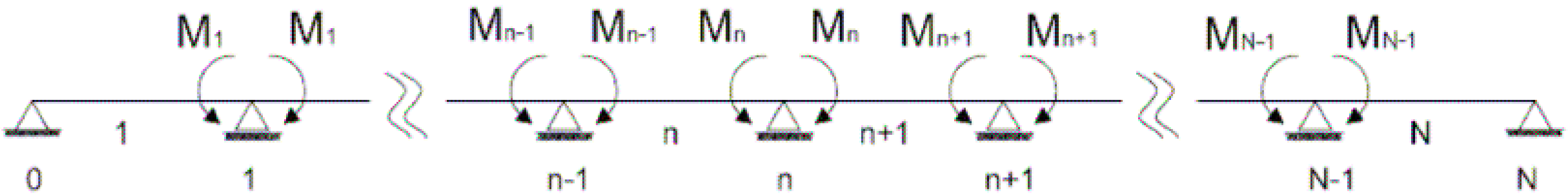
$$\alpha_A = \frac{L}{EI_x}\int_0^1 \left(1 - 2\cdot u + u^2\right)du = \frac{L}{EI_x}\left(u - u^2 + \frac{u^3}{3}\right)\Bigg|_0^1 \Rightarrow \alpha_A = \frac{L}{3EI_x}$$

$$\alpha_B = \frac{L}{EI_x}\int_0^1 u^2 du = \frac{L}{EI_x}\left(\frac{u^3}{3}\right)\Bigg|_0^1 \Rightarrow \alpha_B = \frac{L}{3EI_x}$$

$$\beta = \frac{L}{EI_x}\int_0^1 (1-u)u\,du = \frac{L}{EI_x}\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\Bigg|_0^1 = \frac{L}{EI_x}\left(\frac{1}{2} - \frac{1}{3}\right) \Rightarrow \beta = \frac{L}{6EI_x}$$

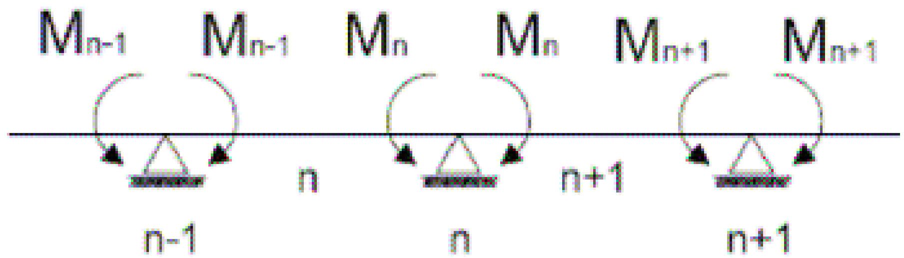
$$\alpha = \alpha_A = \alpha_B = \frac{L}{3EI_x} \qquad \beta = \frac{L}{6EI_x} = \frac{1}{2}\alpha$$

Viga continua de N tramos



En un apoyo genérico n

- confluyen los tramos n y n+1
- hay trasmisión de momento,
- no hay trasmisión de cortante (ya que cada tramo descarga de forma independiente en el apoyo).



θ_{n-1}^{n} Barra n, Nudo n-1. En el caso anterior era θ_A

θ_{n}^{n} Barra n, Nudo n. En el caso anterior era θ_B

M_n Depende solo del nudo n

ψ^{n} Depende solo de la barra

Ecuaciones angulares θ_{n-1}^{n} Barra n Nudo n-1

Análoga situación se presenta para el coeficiente β donde para el tramo n es β^n. Los términos α_{0A} y α_{0B} de la barra n se denominan $\alpha_{0_{n-1}}^{n}$ y $\alpha_{0_n}^{n}$. Los coeficientes α_A y α_B de la barra n se llaman α_{n-1}^{n} y α_n^n. En el caso que la sección de la viga sea constante estos coeficientes dependen solo del tramo y se denomina a ambos α^n.

$$\theta_{n-1}^{n} = \alpha_{0_{n-1}}^{n} + M_{n-1}\alpha_{n-1}^{n} + M_n\beta^n + \psi^n$$

$$\theta_{n}^{n} = \alpha_{0_{n}}^{n} + M_{n-1}\beta^{n} + M_n\alpha_n^n - \psi^n$$

$$\theta_{n}^{n+1} = \alpha_{0_{n}}^{n+1} + M_{n}\alpha_{n}^{n+1} + M_{n+1}\beta^{n+1} + \psi^{n+1}$$

$$\theta_{n+1}^{n+1} = \alpha_{0_{n+1}}^{n+1} + M_{n}\beta^{n+1} + M_{n+1}\alpha_{n+1}^{n+1} - \psi^{n+1}$$

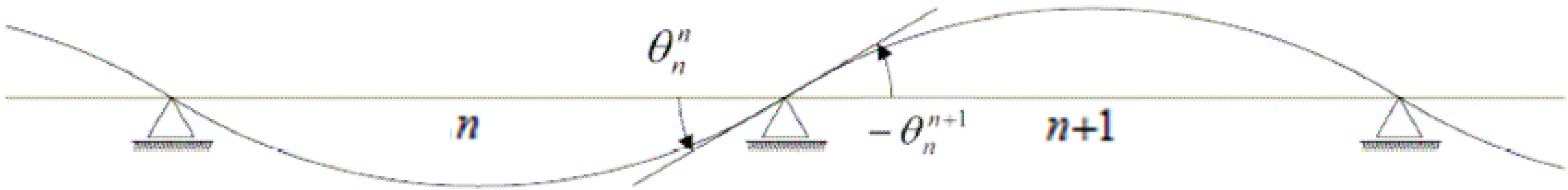
Donde sabemos que:

$$\psi^n = \frac{\upsilon_n - \upsilon_{n-1}}{L^n}$$

Ecuación de tres momentos

θ_{n-1}^{n} Barra n
Nudo n-1

$$\theta_n^n = -\theta_n^{n+1}$$



$$\theta_n^n + \theta_n^{n+1} = 0$$

$$\theta_n^n = \alpha_{0_n}^{\ n} + M_{n-1}\beta^n + M_n\alpha_n^n - \psi^n$$

$$\theta_n^{n+1} = \alpha_{0_n}^{\ n+1} + M_n\alpha_n^{n+1} + M_{n+1}\beta^{n+1} + \psi^{n+1}$$

$$M_{n-1}\beta^n + M_n\left(\alpha_n^n + \alpha_n^{n+1}\right) + M_{n+1}\beta^{n+1} + \alpha_{0_n}^{\ n} + \alpha_{0_n}^{\ n+1} - \psi^n + \psi^{n+1} = 0$$

Ecuación de tres momentos

θ_{n-1}^{n} Barra n
Nudo n-1

$$M_{n-1}\beta^{n} + M_{n}\left(\alpha_{n}^{n} + \alpha_{n}^{n+1}\right) + M_{n+1}\beta^{n+1} + \alpha_{0_n}^{\;n} + \alpha_{0_n}^{\;n+1} - \psi^{n} + \psi^{n+1} = 0$$

$\alpha_{0_n}^{\;n}$ y $\alpha_{0_n}^{\;n+1}$ Dependen de la geometría de la estructura y de las cargas aplicadas

ψ^{n} y ψ^{n+1} Dependen de los descensos de los apoyos y de la geometría de la estructura

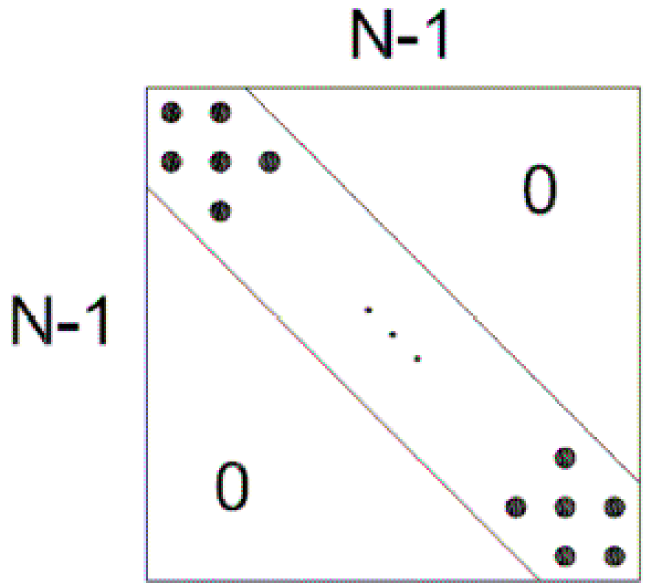
α_{n}^{n}, α_{n}^{n+1}, β^{n} y β^{n+1} Dependen de la geometría y de los Momentos:

$$M_{n-1},\ M_{n}\ y\ M_{n+1}$$

N+1 apoyos, N tramos

$$M_{n-1}\beta^{n} + M_{n}\left(\alpha_{n}^{n} + \alpha_{n}^{n+1}\right) + M_{n+1}\beta^{n+1} + \alpha_{0_n}^{\ n} + \alpha_{0_n}^{\ n+1} - \psi^{n} + \psi^{n+1} = 0$$

Tenemos n-1 momentos M como incognita.



Viga continua con empotramientos

θ_0^1 Barra 1

Nudo 0

Empotramiento en el primer apoyo:

$$\theta_0^1 = 0 = \alpha_{0_0}^{\ 1} + M_0\alpha_0^1 + M_1\beta^1 + \psi^1$$

Tenemos una incognita más que sería M_0 y n ecuaciones

$$M_{n-1}\beta^n + M_n\left(\alpha_n^n + \alpha_n^{n+1}\right) + M_{n+1}\beta^{n+1} + \alpha_{0_n}^{\ n} + \alpha_{0_n}^{\ n+1} - \psi^n + \psi^{n+1} = 0$$

Si n=0, entonces no tienen sentido los subindices (apoyos) negativos, ni los tramos igual a cero (superindices).

Viga continua con empotramientos

θ_{n-1}^{n} Barra n

Nudo n-1

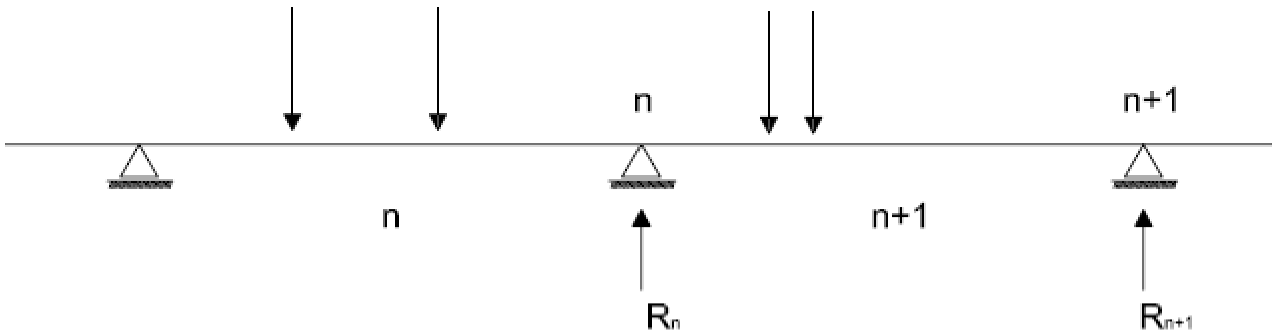
Empotramiento en el apoyo n:

$$\theta_N^N = 0 = \alpha_{0_N}^{\;N} + M_{N-1}\beta^N + M_N \alpha_N^N - \psi^N$$

Tenemos una incognita más que sería M_n y n ecuaciones

$$M_{n-1}\beta^n + M_n\left(\alpha_n^n + \alpha_n^{n+1}\right) + M_{n+1}\beta^{n+1} + \alpha_{0_n}^{\;n} + \alpha_{0_n}^{\;n+1} - \psi^n + \psi^{n+1} = 0$$

Reacciones



$$R_n = R_n^n + R_n^{n+1}$$

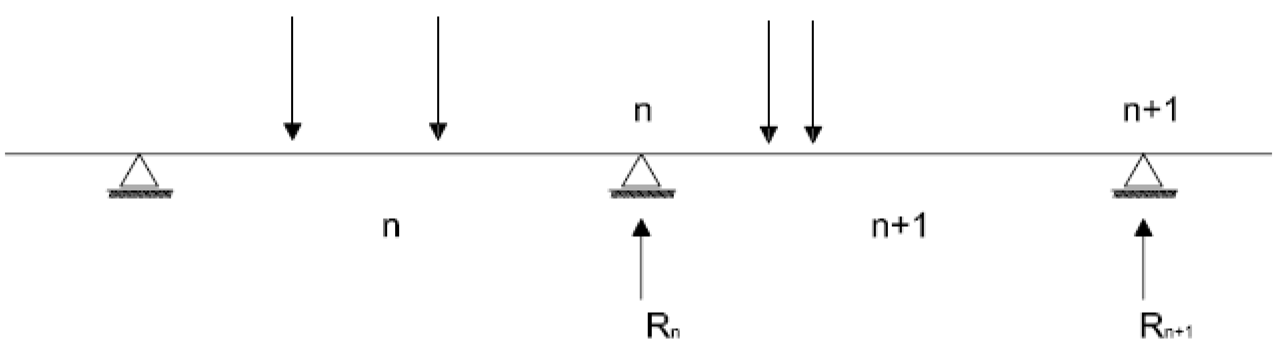
$$R_n^n = R_{0_n}^{\ n} + \frac{M_{n-1}}{L^n} - \frac{M_n}{L^n}$$

$$R_n^{n+1} = R_{0_n}^{\ n+1} - \frac{M_n}{L^{n+1}} + \frac{M_{n+1}}{L^{n+1}}$$

Reacciones

$$R_n = R_{0\,n}^{n} + R_{0n}^{n+1} + \frac{M_{n-1}}{L^n} - \frac{M_n}{L^n} - \frac{M_n}{L^{n+1}} + \frac{M_{n+1}}{L^{n+1}}$$

$$R_{0n} + \frac{1}{L^n}\left(M_{n-1} - M_n\right) + \frac{1}{L^{n+1}}\left(M_{n+1} - M_n\right)$$



Vigas con EI

$$\beta^n = \frac{L^n}{6EI_x}$$

$$\alpha^n = \alpha_n^n = \alpha_{n-1}^n = \frac{L^n}{3EI_x}$$

Sustituyendo en la Ec. de 3 Momentos y multiplicando por 6 EI:

$$M_{n-1}L^n + 2M_n\left(L^n + L^{n+1}\right) + M_{n+1}L^{n+1} + 6EI_x\left(\alpha_{0_n}^{\;n} + \alpha_{0_n}^{\;n+1}\right) + 6EI_x\left(\psi^{n+1} - \psi^n\right) = 0$$

$$\alpha_{0_n}^{\;n} = \frac{1}{L^n}\int_0^{L^n}\frac{M}{EI_x}z\,dz \qquad 6EI_x\alpha_{0_n}^{\;n} = \frac{6}{L^n}\int_0^{L^n}Mz\,dz$$

$$\alpha_{0_n}^{\;n+1} = \frac{1}{L^{n+1}}\int_0^{L^{n+1}}\frac{M}{EI_x}\left(L^{n+1} - z\right)dz \qquad 6EI_x\alpha_{0_n}^{\;n+1} = \frac{6}{L^{n+1}}\int_0^{L^{n+1}}M\left(L^{n+1} - z\right)dz$$

Términos de Carga

$$M_{n-1}L^n + 2M_n\left(L^n + L^{n+1}\right) + M_{n+1}L^{n+1} + 6EI_x\left(\alpha_{0_n}^{\;n} + \alpha_{0_n}^{\;n+1}\right) + 6EI_x\left(\psi^{n+1} - \psi^n\right) = 0$$

$$6EI_x\alpha_{0_n}^{\;n} = \frac{6}{L^n}\int_0^{L^n} Mzdz \qquad \mathscr{R}^{\mathrm{n}} = \frac{6}{\left(L^n\right)^2}\int_0^{L^n} Mzdz \qquad \mathscr{L}^{\mathrm{n}} = \frac{6}{\left(L^n\right)^2}\int_0^{L^n} M\left(L^n - z\right)dz$$

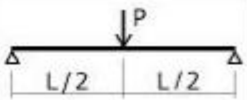
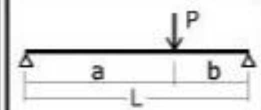
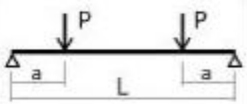
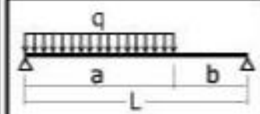
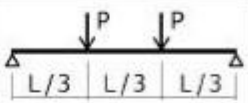
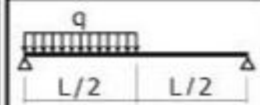
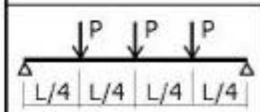
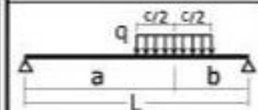
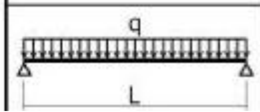
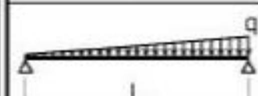
$$6EI_x\alpha_{0_n}^{\;n+1} = \frac{6}{L^{n+1}}\int_0^{L^{n+1}} M\left(L^{n+1} - z\right)dz \qquad \mathscr{L}^{\mathrm{n+1}} = \frac{6}{\left(L^{n+1}\right)^2}\int_0^{L^{n+1}} M\left(L^{n+1} - z\right)dz$$

$$M_{n-1}L^n + 2M_n\left(L^n + L^{n+1}\right) + M_{n+1}L^{n+1} + \mathscr{R}^{\mathrm{n}}L^n + \mathscr{L}^{\mathrm{n+1}}L^{n+1} + 6EI_x\left(\psi^{n+1} - \psi^n\right) = 0$$

Términos de Carga

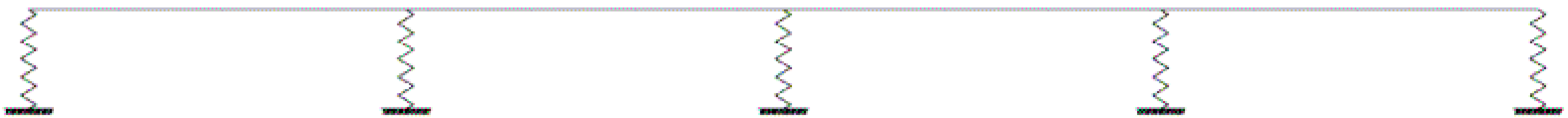
Instituto de Estructuras y Transporte
Prof. Julio Ricaldoni

TÉRMINOS DE CARGA

CARGAS SIMÉTRICAS		CARGAS NO SIMÉTRICAS		
ESTADO DE CARGA	$\mathcal{L} = \mathcal{R}$	ESTADO DE CARGA	$\mathcal{L}$	$\mathcal{R}$
	$\frac{3}{8}PL$		$\frac{Qab}{L^2}(L+b)$	$\frac{Qab}{L^2}(L+a)$
	$3Pa\left(1-\frac{a}{L}\right)$		$\frac{qa^2}{4}\left(2-\frac{a}{L}\right)^2$	$\frac{qa^2}{4}\left(2-\frac{a^2}{L^2}\right)$
	$\frac{2}{3}PL$		$\frac{9}{64}qL^2$	$\frac{7}{64}qL^2$
	$\frac{15}{16}PL$		$\frac{qabc}{L^2}\left(L+b-\frac{c^2}{4a}\right)$	$\frac{qabc}{L^2}\left(L+a-\frac{c^2}{4b}\right)$
	$\frac{1}{4}qL^2$		$\frac{7}{60}qL^2$	$\frac{2}{15}qL^2$

Viga sobre apoyos elásticos

$$R_n = k_n \cdot \upsilon_n$$



$$\psi^n = \frac{\upsilon_n - \upsilon_{n-1}}{L^n} \qquad \psi^{n+1} = \frac{\upsilon_{n+1} - \upsilon_n}{L^{n+1}}$$

$$\psi^n = \frac{\dfrac{R_n}{k_n} - \dfrac{R_{n-1}}{k_{n-1}}}{L^n} =$$

Viga sobre apoyos elásticos

$$\psi^{n} = f_1\left(M_{n-2}, M_{n-1}, M_{n}, M_{n+1}, R_{0n}, R_{0n-1}\right)$$
$$\psi^{n+1} = f_2\left(M_{n-1}, M_{n}, M_{n+1}, M_{n+2}, R_{0n}, R_{0n+1}\right)$$

siendo ambas ecuaciones lineales.
Sustituyendo en la ecuación de tres momentos,

$$M_{n-1}\beta^{n} + M_{n}\left(\alpha_{n}^{n} + \alpha_{n}^{n+1}\right) + M_{n+1}\beta^{n+1} + \alpha_{0_n}^{\ n} + \alpha_{0_n}^{\ n+1} - \psi^{n} + \psi^{n+1} = 0$$

Ecuación de 5 Momentos

Sustituyendo en la Ec. de 3 M

$$A \cdot M_{n-2} + B \cdot M_{n-1} + C \cdot M_{n} + D \cdot M_{n+1} + E \cdot M_{n+2} + F \cdot R_{0,n-1} + G \cdot R_{0,n} + H \cdot R_{0,n+1} + \alpha_{0\,n}^{\;n} + \alpha_{0\,n}^{\;n+1} = 0$$

$$A = \frac{1}{k_{n-1} L^{n-1} L^{n}}$$

$$B = \beta^{n} - \frac{1}{k_{n}\left(L^{n}\right)^{2}} - \frac{1}{k_{n-1} L^{n-1} L^{n}} - \frac{1}{k_{n-1}\left(L^{n}\right)^{2}} - \frac{1}{k_{n} L^{n} L^{n+1}}$$

$$C = \alpha_{n}^{n} + \alpha_{n}^{n+1} + \frac{1}{k_{n}\left(L^{n}\right)^{2}} + \frac{2}{k_{n} L^{n} L^{n+1}} + \frac{1}{k_{n-1}\left(L^{n}\right)^{2}} + \frac{1}{k_{n+1}\left(L^{n+1}\right)^{2}} + \frac{1}{k_{n}\left(L^{n+1}\right)^{2}}$$

$$D = \beta^{n+1} - \frac{1}{k_{n} L^{n} L^{n+1}} - \frac{1}{k_{n+1}\left(L^{n+1}\right)^{2}} - \frac{1}{k_{n+1} L^{n+1} L^{n+2}} - \frac{1}{k_{n}\left(L^{n+1}\right)^{2}}$$

$$E = \frac{1}{k_{n+1} L^{n+1} L^{n+2}}$$

$$F = \frac{1}{k_{n-1} L^{n}}$$

$$G = -\frac{1}{k_{n} L^{n}} - \frac{1}{k_{n} L^{n+1}}$$

$$H = \frac{1}{k_{n+1} L^{n+1}}$$

- Notas realizadas en base a los apuntes de RII (Autores: A. Morquio y L. Delacoste)

Apoyo

Si un apoyo es fijo (no es elástico) $k_n = \infty$

$$R_{n-1} = R_0 = R_{00} + \frac{M_1}{L^1}$$

n=1

$$R_n = R_1 = R_{0,1} - \frac{M_1}{L^1} - \frac{M_1}{L^2} + \frac{M_2}{L^2}$$

no existe el tramo 0, ni el momento M_0, ni tampoco el apoyo –1