Sistemas Lineales 2

1 Solución Examen Julio 2010

a

$$v_0 = \frac{3}{2} L \dot{i}_{L_2},$$

$$v_{L_1} = -R\, i_{L_2}, \qquad v_{L_1} = \frac{2}{3} L \dot{i}_{L_1},$$

$$(v_s + R\, i_{L_2})\frac{1}{R} + (v_0 + R\, i_{L_2})\frac{1}{R} + i_{L_2} = i_{L_1}$$

$$\Longrightarrow \quad v_0 = R\, i_{L_1} - 3R\, i_{L_2} - v_s$$

$$\Longrightarrow \quad \dot{i}_{L_1} = -\frac{3}{2}\frac{1}{\tau} i_{L_2}$$

$$\dot{i}_{L_2} = \frac{2}{3}\frac{1}{\tau} i_{L_1} - 2\frac{1}{\tau} i_{L_2} - \frac{2}{3}\frac{1}{L} v_s$$

$$\Longrightarrow \quad \frac{d}{dt}\begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{2}\frac{1}{\tau} \\ \frac{2}{3}\frac{1}{\tau} & -2\frac{1}{\tau} \end{pmatrix} \begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} + \begin{pmatrix} 0 \\ -\frac{2}{3}\frac{1}{L} \end{pmatrix} v_s ,$$

$$v_0 = \begin{pmatrix} R & -3R \end{pmatrix} \begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} + (-1)\, v_s .$$

τR

(b)

$$(sI-A) = \begin{pmatrix} s & \frac{3}{2}\frac{1}{\tau} \\ -\frac{2}{3}\frac{1}{\tau} & \left(s+\frac{2}{\tau}\right) \end{pmatrix}$$

$$\det\{(sI-A)\} = s\left(s+\frac{2}{\tau}\right)+\frac{1}{\tau^2} = \left(s+\frac{1}{\tau}\right)^2$$

$$(sI-A)^T = \begin{pmatrix} s & -\frac{2}{3}\frac{1}{\tau} \\ \frac{3}{2}\frac{1}{\tau} & \left(s+\frac{2}{\tau}\right) \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{\left(s+\frac{1}{\tau}\right)^2} \begin{pmatrix} \left(s+\frac{2}{\tau}\right) & -\frac{3}{2}\frac{1}{\tau} \\ \frac{2}{3}\frac{1}{\tau} & s \end{pmatrix}$$

$$H(s) = E\,(sI-A)^{-1}B + D =$$

$$= \frac{1}{\left(s+\frac{1}{\tau}\right)^2} \begin{pmatrix} 1 & -3 \end{pmatrix} \begin{pmatrix} \frac{1}{\tau^2} \\ -\frac{2}{3}\frac{1}{\tau}s \end{pmatrix} - 1 =$$

$$= \frac{1}{\left(s+\frac{1}{\tau}\right)^2} \left(\frac{2}{\tau}s+\frac{1}{\tau^2}\right) - 1 = -\frac{s^2}{\left(s+\frac{1}{\tau}\right)^2}$$

$$\frac{\left(\frac{2}{\tau}s+\frac{1}{\tau^2}\right)}{\left(s+\frac{1}{\tau}\right)^2}=\frac{C_1}{\left(s+\frac{1}{\tau}\right)}+\frac{C_2}{\left(s+\frac{1}{\tau}\right)^2}$$

$$C_1 = \frac{2}{\tau} \qquad C_2 = -\frac{1}{\tau^2}$$

$$\boxed{\Longrightarrow H(s)=\frac{-s^2}{\left(s+\frac{1}{\tau}\right)^2}=-1+\frac{2/\tau}{\left(s+\frac{1}{\tau}\right)}-\frac{1/\tau^2}{\left(s+\frac{1}{\tau}\right)^2}}$$

$$\boxed{\Longrightarrow h(t)=-\delta(t)+\mu(t)\left(\frac{2}{\tau}e^{-\frac{t}{\tau}}-\frac{t}{\tau^2}e^{-\frac{t}{\tau}}\right)}$$

$$\left(\mu(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}\right)$$

Sigue inmediatamente de la definición que

$h \in \mathbb{A} \subset \mathbb{A}_e$.

Notemos sin embargo que $h \notin \mathcal{L}_{1,e}$. Por tanto, dado que $\mathcal{L}_1 \subset \mathcal{L}_{1,e}$, $h \notin \mathcal{L}_1$.

(c) (i) Dado que los valores propios de

$$A = \begin{pmatrix} 0 & -\frac{3}{2}\frac{1}{\tau} \\ \frac{2}{3}\frac{1}{\tau} & -2\frac{1}{\tau} \end{pmatrix}$$

son $\lambda_1 = \lambda_2 = -\frac{1}{\tau} < 0$,

concluimos que el sistema bajo consideracion <u>es internamente estable</u>. (<u>Teorema 4</u>)

(ii) Dado que $h \in \mathbb{A}$, sigue entonces (<u>Teorema 3</u>) que el sistema <u>es BIBO estable</u>.

Alternativamente, podemos arribar a la misma conclusion usando (i) y <u>Corolario 5</u>.

d

$$Y(s) = \frac{1}{\left(s+\frac{1}{\tau}\right)^2} \begin{pmatrix} 1 & -3 \end{pmatrix} \begin{pmatrix} V_1\left(s+\frac{2}{\tau}\right) - \frac{3}{2}\frac{1}{\tau}V_2 \\ \frac{2}{3}\frac{1}{\tau}V_1 + sV_2 \end{pmatrix}$$

$$- \frac{Vs}{\left(s+\frac{1}{\tau}\right)^2} =$$

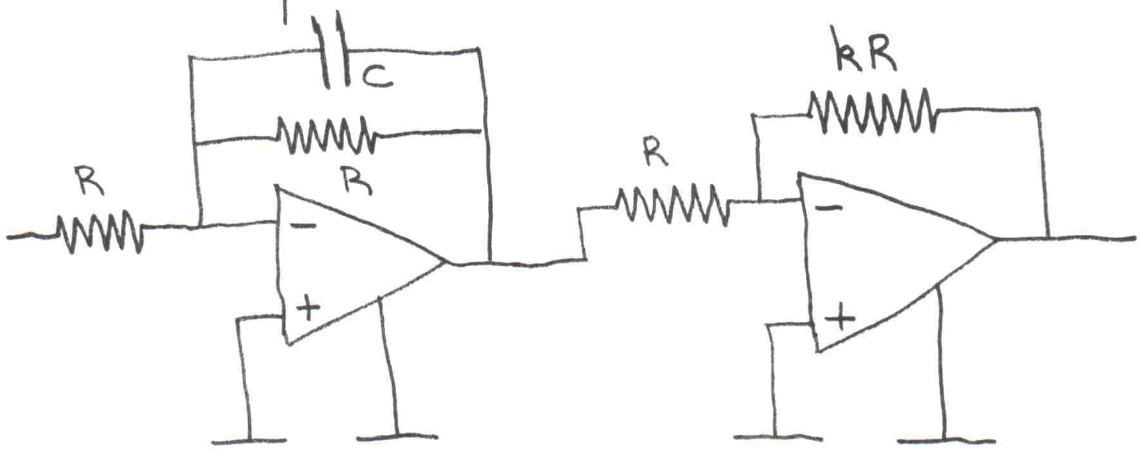
$$= \frac{1}{\left(s+\frac{1}{\tau}\right)^2}\left[\left(V_1 - 3V_2 - V\right)s - \frac{3}{2}\frac{1}{\tau}V_2\right] =$$

$$= \frac{\overset{(V_1-3V_2-V)}{\tilde{C}_1}}{\left(s+\frac{1}{\tau}\right)} + \frac{\overset{\frac{1}{\tau}\left(V+\frac{3}{2}V_2-V_1\right)}{\tilde{C}_2}}{\left(s+\frac{1}{\tau}\right)^2}$$

$$\Longrightarrow \boxed{y(t) = \mu(t)\, e^{-\frac{t}{\tau}}\left[\left(V_1 - 3V_2 - V\right) + \frac{t}{\tau}\left(V + \frac{3}{2}V_2 - V_1\right)\right]}$$

(e)

(i) Notemos que la función de transferencia correspondiente a



es
$$\frac{k}{(1+\tau s)} = \frac{1}{\left(s+\frac{1}{\tau}\right)}\frac{k}{\tau}.$$

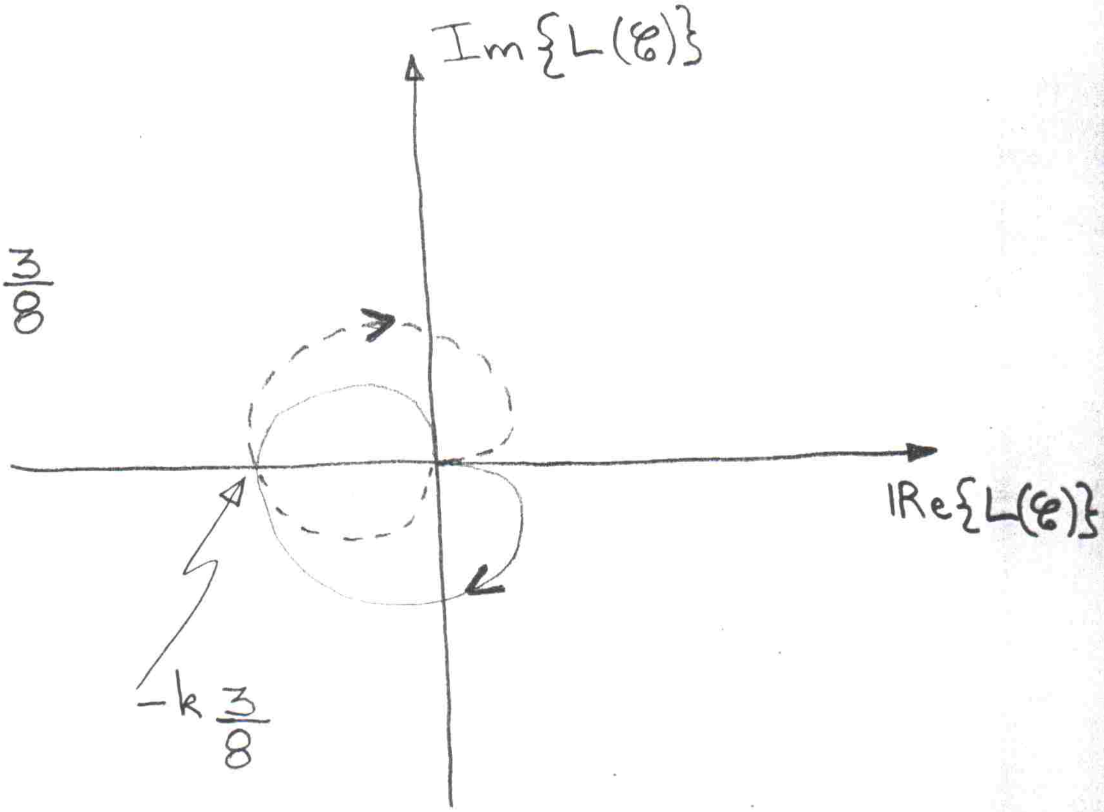
Sigue entonces que

$$\boxed{L(s) = -\frac{k}{\tau}\,\frac{s^2}{\left(s+\frac{1}{\tau}\right)^3}}$$

(ii)

$$\omega_{cr.} = \frac{\sqrt{3}}{\tau}$$

$$L(j\omega_{cr.}) = -k\,\frac{3}{8}$$



(Notemos que L es real-racional y estrictamente propia, lo cual implica que la interconexion esta <u>bien definida</u>.) En este caso tenemos que ($k>0$, y) $P=0$. Así, invocando el Criterio de Estabilidad de Nyquist y usando el grafico de Nyquist sigue que <u>la interconexion es BIBO estable si y solo si</u>

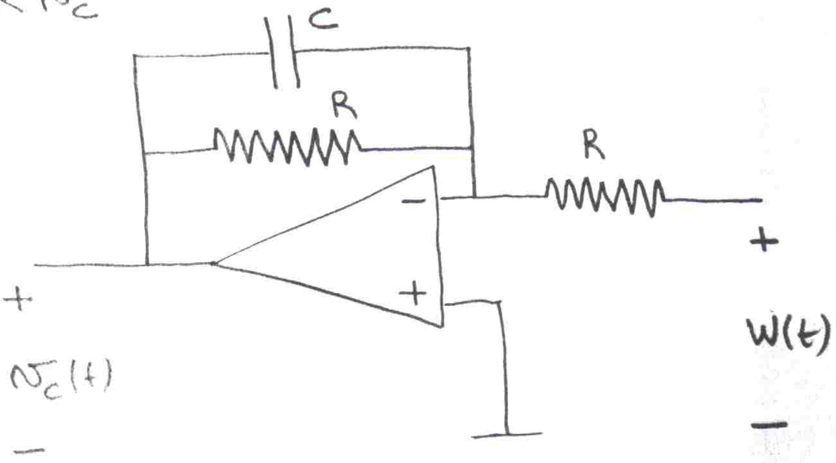
$$\boxed{0 < k < \frac{8}{3}}\ .$$

(f)

(i) $$\dot{\begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix}} = A \begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} + B v$$

$$w = E \begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} + D v$$

$$v = -u + k v_c$$



$$\frac{w}{R} + \frac{v_c}{R} + C \dot{v}_c = 0$$

$$\Longrightarrow \quad \dot{v}_c = -\frac{1}{\tau} v_c - \frac{1}{\tau} w$$

$$\Longrightarrow \quad \dot{v}_c = -\frac{1}{\tau} v_c - \frac{1}{\tau} E \begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} - \frac{1}{\tau} D k v_c + \frac{1}{\tau} D u$$

$$\Longrightarrow \dot{v}_c = -\frac{1}{\tau} \begin{pmatrix} E & (1+Dk) \end{pmatrix} \begin{pmatrix} i_{L_1} \\ i_{L_2} \\ v_c \end{pmatrix} + \frac{1}{\tau} D u$$

$$\begin{pmatrix} \dot{i}_{L_1} \\ \dot{i}_{L_2} \end{pmatrix} = A \begin{pmatrix} i_{L_1} \\ i_{L_2} \end{pmatrix} + k B v_c - B u .$$

Así,

$$\boxed{\begin{aligned} \begin{pmatrix} \dot{i}_{L_1} \\ \dot{i}_{L_2} \\ \dot{v}_c \end{pmatrix} &= \overbrace{\left(\begin{array}{c|c} A & kB \\ \hline -\frac{1}{\tau} E & -\frac{1}{\tau}(1+Dk) \end{array}\right)}^{A_{cl}} \begin{pmatrix} i_{L_1} \\ i_{L_2} \\ v_c \end{pmatrix} + \overbrace{\left(\begin{array}{c} -B \\ \hline \frac{1}{\tau} D \end{array}\right)}^{B_{cl}} u \\ w &= \underbrace{\left(\begin{array}{c|c} E & kD \end{array}\right)}_{E_{cl}} \begin{pmatrix} i_{L_1} \\ i_{L_2} \\ v_c \end{pmatrix} + \underbrace{(-D)}_{D_{cl}} u \end{aligned}}$$

$$\Longrightarrow \boxed{A_{cl} = \begin{pmatrix} 0 & -\frac{3}{2}\frac{1}{\tau} & 0 \\ \frac{2}{3}\frac{1}{\tau} & -\frac{2}{\tau} & -\frac{2}{3} k \frac{1}{\tau}\frac{1}{R} \\ -\frac{R}{\tau} & \frac{3R}{\tau} & \frac{(k-1)}{\tau} \end{pmatrix}}$$

(ii)

$$\boxed{T_{cl}(s) = \frac{-H(s)}{1+L(s)} = \frac{s^2\left(s+\frac{1}{\tau}\right)}{\left[\left(s+\frac{1}{\tau}\right)^3 - \frac{k}{\tau}s^2\right]}}$$

(iii) Dado que $A_{cl} \in \mathbb{R}^{3\times 3}$ y $T_{cl}(s) = \frac{P_{Num,3}(s)}{P_{Den,3}(s)}$,

donde los polinomios (de grado 3) $P_{Num,3}(s)$ y $P_{Den,3}(s)$ son coprimos, se verifica entonces en virtud de Proposición 6 que el sistema de Figura 2 es internamente estable si y solo si es BIBO estable.

(Recordemos que confinamos aquí nuestro análisis solo para $k>0$.)

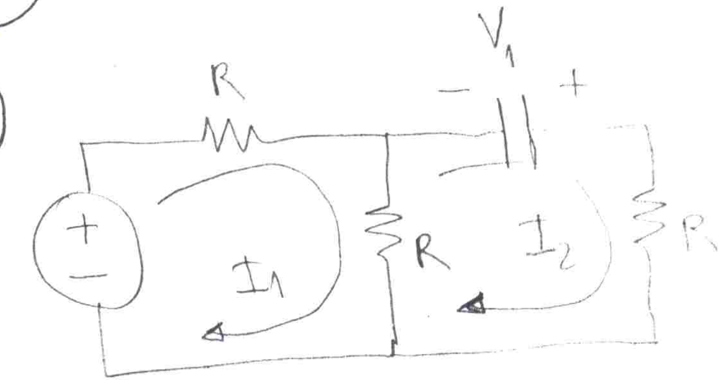
Sabemos de nuestro análisis de la parte (e) que el sistema es BIBO estable si y solo si $0<k<\frac{8}{3}$.

Así, invocando el Teorema 4, sigue que (cuando confinamos nuestro análisis para $k>0$) todos los valores propios de A_{cl} tienen parte real negativa si y solo si

$$0<k<\frac{8}{3}.$$

2

a



$$V_s = R I_1 + R(I_1 - I_2)$$

$$V_1 = R I_2 + R(I_2 - I_1)$$

$$i_c = C \dot{v}_c \Longrightarrow I_c = C s V_c - C v_c(0^-)$$

$$\Longrightarrow V_c = \frac{1}{Cs} I_c + \frac{1}{s} v_{01}$$

$$\Longrightarrow V_1 = -\frac{1}{Cs} I_2 + \frac{1}{s} v_{01}$$

$$\Longrightarrow \begin{pmatrix} 2R & -R \\ -R & 2R + \frac{1}{Cs} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} V_s \\ \frac{1}{s} v_{01} \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{1}{\left[2R\left(2R+\frac{1}{cs}\right)-R^2\right]} \begin{pmatrix} \left(2R+\frac{1}{cs}\right) & R \\ R & 2R \end{pmatrix} \begin{pmatrix} V_s \\ \frac{1}{s} v_{01} \end{pmatrix}$$

$$E_{oc}(s) = R I_2 = \frac{1}{\left(3+\frac{2}{\tau s}\right)} \left(V_s + \frac{2}{s} v_{01}\right)$$

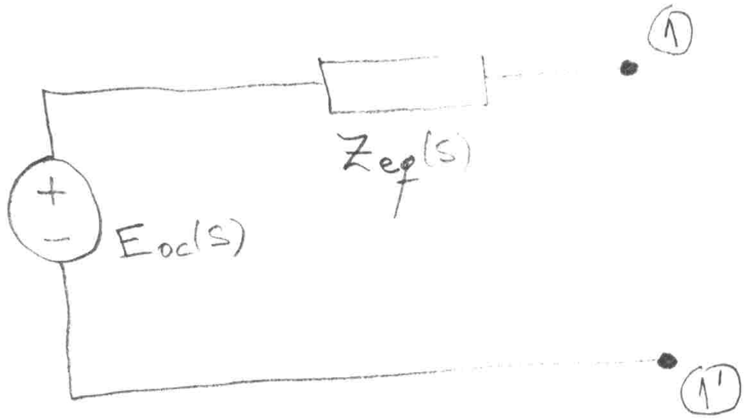
$$\Longrightarrow \boxed{E_{oc}(s) = \frac{\tau}{(2+3\tau s)} \left(s V_s(s) + 2 v_{01}\right)} \quad , \quad \tau = RC.$$

$$Z_{eq}(s) = \frac{R\left(\frac{R}{2}+\frac{1}{cs}\right)}{\frac{3R}{2}+\frac{1}{cs}} + R = \frac{R\,(\tau s+2)}{(3\tau s+2)} + R =$$

$$= 4R \, \frac{(\tau s+1)}{(3\tau s+2)}$$

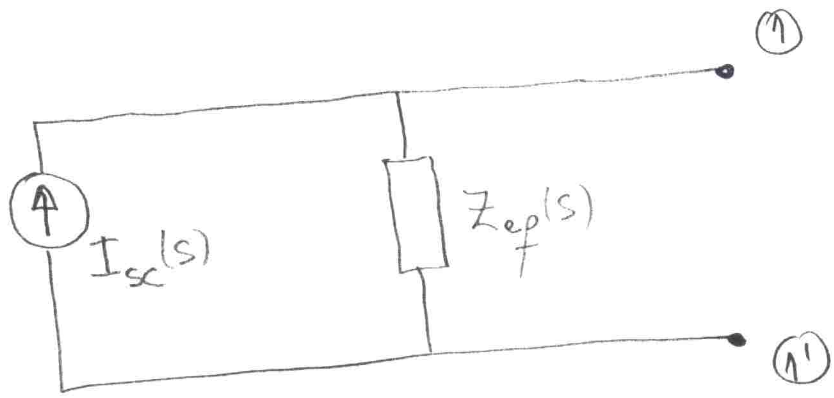
$$\Longrightarrow \boxed{Z_{eq}(s) = 4R \, \frac{(\tau s+1)}{(3\tau s+2)}}$$

Circuito Equivalente de Thévenin



(b)

Circuito Equivalente de Norton



$$\boxed{I_{sc}(s) = \frac{E_{oc}(s)}{Z_{eq}(s)} = \frac{\tau}{4R}\,\frac{1}{(\tau s+1)}\left(s V_s(s) + 2 v_{01}\right)}$$

(c) Άρα

$$E_{oc}(s) = \frac{1}{(2+3\tau s)} \tau(V_0 + 2v_{01}).$$

(i) $$V(s) = E_{oc}(s) \frac{\frac{4}{cs}}{\left(Z_{ef}(s) + \frac{4}{cs}\right)} =$$

$$= 4E_{oc}(s) \frac{1}{(4 + cs\, Z_{ef}(s))} =$$

$$= \frac{\tau(V_0 + 2v_{01})}{\cancel{(2+3\tau s)}} \frac{1}{\frac{(2+3\tau s) + \tau s(\tau s + 1)}{\cancel{(2+3\tau s)}}} =$$

$$= \frac{\tau(V_0 + 2v_{01})}{(\tau^2 s^2 + 4\tau s + 2)}$$

$$\Rightarrow \quad V(s) = \frac{(V_0 + 2v_{01})}{\tau} \frac{1}{\left(s^2 + \frac{4}{\tau}s + \frac{2}{\tau^2}\right)}$$

$$\lambda_{1,2} = \frac{-\frac{4}{\tau} \pm \sqrt{\frac{8}{\tau^2}}}{2} = -\frac{2}{\tau} \pm \frac{\sqrt{2}}{\tau}$$

$$\lambda_1 = -\frac{1}{\tau}(2+\sqrt{2}) \quad , \quad \lambda_2 = -\frac{1}{\tau}(2-\sqrt{2})$$

$$V(s) = \frac{(V_0 + 2v_{01})}{\tau} \frac{1}{(s-\lambda_1)(s-\lambda_2)} =$$

$$= \frac{(V_0 + 2v_{01})}{\tau} \left[\frac{1}{(\lambda_1 - \lambda_2)} \frac{1}{(s-\lambda_1)} + \frac{1}{(\lambda_2 - \lambda_1)} \frac{1}{(s-\lambda_2)} \right]$$

$$(\lambda_1 - \lambda_2) = -\frac{1}{\tau} 2\sqrt{2}$$

$$\Rightarrow V(s) = \frac{(V_0 + 2v_{01})}{2\sqrt{2}} \left[\frac{1}{(s-\lambda_2)} - \frac{1}{(s-\lambda_1)} \right]$$

$$\Rightarrow \boxed{v(t) = \mu(t) \frac{(V_0 + 2v_{01})}{2\sqrt{2}} \left[e^{-(2-\sqrt{2})\frac{t}{\tau}} - e^{-(2+\sqrt{2})\frac{t}{\tau}} \right]}$$

(ii)

$$V(s) = E_{oc}(s) \frac{R}{(Z_{ep}(s) + R)} =$$

$$= \frac{\tau(V_0 + 2v_{01})}{\cancel{(2+3\tau s)}} \frac{1}{\frac{(4\tau s + 4 + 3\tau s + 2)}{\cancel{(2+3\tau s)}}} =$$

$$= \frac{\tau(V_0 + 2v_{01})}{(7\tau s + 6)} = \frac{(V_0 + 2v_{01})}{7} \frac{1}{\left(s + \frac{6}{7\tau}\right)}$$

$$\boxed{\Longrightarrow v(t) = \mu(t) \frac{(V_0 + 2v_{01})}{7} e^{-\frac{6}{7}\frac{t}{\tau}}}$$

(d) Sigue de la parte (c) (i) que

$$v_{ZSR}(t) = \frac{V_0}{2\sqrt{2}}\left(e^{-(2-\sqrt{2})\frac{t}{\tau}} - e^{-(2+\sqrt{2})\frac{t}{\tau}}\right), \quad t \geq 0 .$$

Así, invocando el Teorema de Reciprocidad obtenemos que

$$\boxed{i_R(t) = \frac{I_0}{2\sqrt{2}}\left(e^{-(2-\sqrt{2})\frac{t}{\tau}} - e^{-(2+\sqrt{2})\frac{t}{\tau}}\right), \quad t \geq 0 .}$$

(e)

(i)

$$\tilde{H}(s) = (1+\Gamma_L(s))\, e^{-Ts} \frac{1}{(1-\Gamma_g(s)\Gamma_L(s)e^{-2Ts})} \frac{Z_0}{(Z_0+Z_g(s))}$$

$$\Gamma_L(s) = \frac{1}{2} \quad , \quad Z_g(s) = Z_{eq}(s) = 4R \, \frac{(\tau s+1)}{(3\tau s+2)}$$

$$\frac{Z_0}{(Z_0+Z_g(s))} = \frac{1}{(7\tau s+6)}(3\tau s+2)$$

$$\Gamma_g(s) = \frac{(Z_g(s)-Z_0)}{(Z_g(s)+Z_0)} = \frac{(\tau s+2)}{(7\tau s+6)}$$

$$V(l,s) = \tilde{H}(s)\, E_{oc}(s) = \tilde{H}(s) \frac{\tau s}{(3\tau s+2)} V_s(s)$$

$$\Longrightarrow \quad H(s) = \tilde{H}(s) \frac{\tau s}{(3\tau s+2)}$$

$$\Longrightarrow H(s) = \frac{3}{2} e^{-Ts} \frac{1}{\left(1 - \frac{1}{2}\frac{(\tau s+2)}{(7\tau s+6)} e^{-2Ts}\right)} \frac{\tau s}{(7\tau s+6)}$$

(ii) Dado que se verifican las hipótesis del Corolario 2 (de "Notas Complementarias Sobre Líneas de Transmisión")

(Note que $|L(j\omega)| = |\Gamma_L(j\omega)||\Gamma_g(j\omega)| \leq \frac{1}{2} \quad \forall \omega \in \mathbb{R}^+$.)

tenemos entonces que

$$\frac{1}{(1+L)} \in \hat{\mathcal{A}} ,$$

donde $L(s) = -\Gamma_L(s)\Gamma_g(s) e^{-2Ts} = -\frac{1}{2}\frac{(\tau s+2)}{(7\tau s+6)} e^{-2Ts}$.

Así, $H \in \hat{\mathcal{A}}$. Es decir $h \in \mathcal{A}$, lo cual implica que <u>el sistema bajo consideración es BIBO estable</u>.

(iii) Sigue también del Corolario 2 (usado anteriormente) que

$$\frac{1}{\left(1-\Gamma_0(s)\,\Gamma_\ell(s)\,e^{-2Ts}\right)} = \sum_{k=0}^{+\infty} \left(\Gamma_0(s)\,\Gamma_\ell(s)\,e^{-2Ts}\right)^k, \quad s\in\mathbb{C}^+$$

Así,

$$V(\ell,s) = \frac{3}{2} V_0 \frac{\tau}{(7\tau s+6)} \sum_{k=0}^{+\infty} \left(\frac{1}{2}\right)^k \left(\frac{(\tau s+2)}{(7\tau s+6)}\right)^k e^{-T(2k+1)s},$$

$$s\in\mathbb{C}^+ .$$

Tenemos entonces que

$$\boxed{\begin{aligned} v(\ell,t) &= \frac{3}{2}\,\frac{1}{7} V_0\, e^{-\frac{6}{7}\frac{(t-T)}{\tau}} \mu(t-T) \\ &+ \frac{3}{4}\,\frac{1}{7^2} V_0\, e^{-\frac{6}{7}\frac{(t-3T)}{\tau}} \left(1+\frac{8}{7}\frac{(t-3T)}{\tau}\right) \mu(t-3T), \\ & t\in[0,5T) . \end{aligned}}$$