Sistemas Lineales 2

Solucion Examen Febrero 2010

Problema 1.—

a

$$\frac{v_s - v_{c2}}{R} + C\,\dot{v}_{c1} = \frac{v_{c2}}{R} + C\,\dot{v}_{c2}$$

$$\frac{v_{c2} + v_{c1}}{R} = -C\,\dot{v}_{c2}$$

$$\Longrightarrow \quad \dot{v}_{c2} = -\frac{1}{\tau} v_{c1} - \frac{1}{\tau} v_{c2}$$

$$C\,\dot{v}_{c1} = -\frac{v_s}{R} + \frac{\cancel{2}}{R} v_{c2} - \cancel{\frac{v_{c2}}{R}} - \frac{v_{c1}}{R}$$

$$\Longrightarrow \quad \dot{v}_{c1} = -\frac{1}{\tau} v_{c1} + \frac{1}{\tau} v_{c2} - \frac{1}{\tau} v_s$$

$$\boxed{\Longrightarrow \quad A = \begin{pmatrix} -\frac{1}{\tau} & \frac{1}{\tau} \\ -\frac{1}{\tau} & -\frac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{\tau} \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad D = 0}$$

(b) $H(s) = E(sI-A)^{-1}B$

$$(sI-A) = \begin{pmatrix} \left(s+\frac{1}{\tau}\right) & -\frac{1}{\tau} \\ \frac{1}{\tau} & \left(s+\frac{1}{\tau}\right) \end{pmatrix}$$

$$\det\{(sI-A)\} = \left(s+\frac{1}{\tau}\right)^2 + \frac{1}{\tau^2}$$

$$(sI-A)^T = \begin{pmatrix} \left(s+\frac{1}{\tau}\right) & \frac{1}{\tau} \\ -\frac{1}{\tau} & \left(s+\frac{1}{\tau}\right) \end{pmatrix}$$

$$(sI-A)^{-1} = \frac{1}{\left[\left(s+\frac{1}{\tau}\right)^2 + \frac{1}{\tau^2}\right]} \begin{pmatrix} \left(s+\frac{1}{\tau}\right) & \frac{1}{\tau} \\ -\frac{1}{\tau} & \left(s+\frac{1}{\tau}\right) \end{pmatrix}$$

$$\Longrightarrow \boxed{H(s) = -\frac{1}{\tau} \frac{s}{\left[\left(s+\frac{1}{\tau}\right)^2 + \frac{1}{\tau^2}\right]}}$$

$$H(s) = -\frac{1}{\tau}\left\{ \frac{\left(s+\frac{1}{\tau}\right)}{\left[\left(s+\frac{1}{\tau}\right)^2 + \frac{1}{\tau^2}\right]} - \frac{\frac{1}{\tau}}{\left[\left(s+\frac{1}{\tau}\right)^2 + \frac{1}{\tau^2}\right]} \right\}$$

$$\Longrightarrow \boxed{h(t) = -\frac{1}{\tau} e^{-\frac{t}{\tau}} \left\{ \cos \frac{1}{\tau} t - \sin \frac{1}{\tau} t \right\}, \quad t \geq 0}$$

(i) y (ii) $\boxed{h \in \mathcal{L}_1 \subset \mathcal{L}_{1,e}\,.}$

(c) El polinomio caract. de la matriz A es

$$P_A(s) = \det\{(sI-A)\} = s^2 + \frac{2}{\tau}s + \frac{2}{\tau^2}$$

cuyas raíces tienen partes reales negativas.

(i) El sistema es entonces <u>internamente estable</u>.

(ii) Dado que el sistema es internamente estable, entonces, dicho sistema (con $x_0 = 0$) es <u>BIBO estable</u>.

(d) $$Y(s) = E\,(sI-A)^{-1}x_0 + H(s)\,V_s(s)$$

$$\Longrightarrow Y(s) = \frac{1}{\left[\left(s+\frac{1}{\tau}\right)^2 + \frac{1}{\tau^2}\right]}\left[\left(s+\frac{1}{\tau}\right)(V_1+V_2) + \left(\frac{1}{\tau}\right)(V_2-V_1-V)\right]$$

$$\boxed{\Longrightarrow y(t) = e^{-\frac{t}{\tau}}\left[(V_1+V_2)\cos\frac{1}{\tau}t + (V_2-V_1-V)\sin\frac{1}{\tau}t\right],\quad t \geq 0.}$$

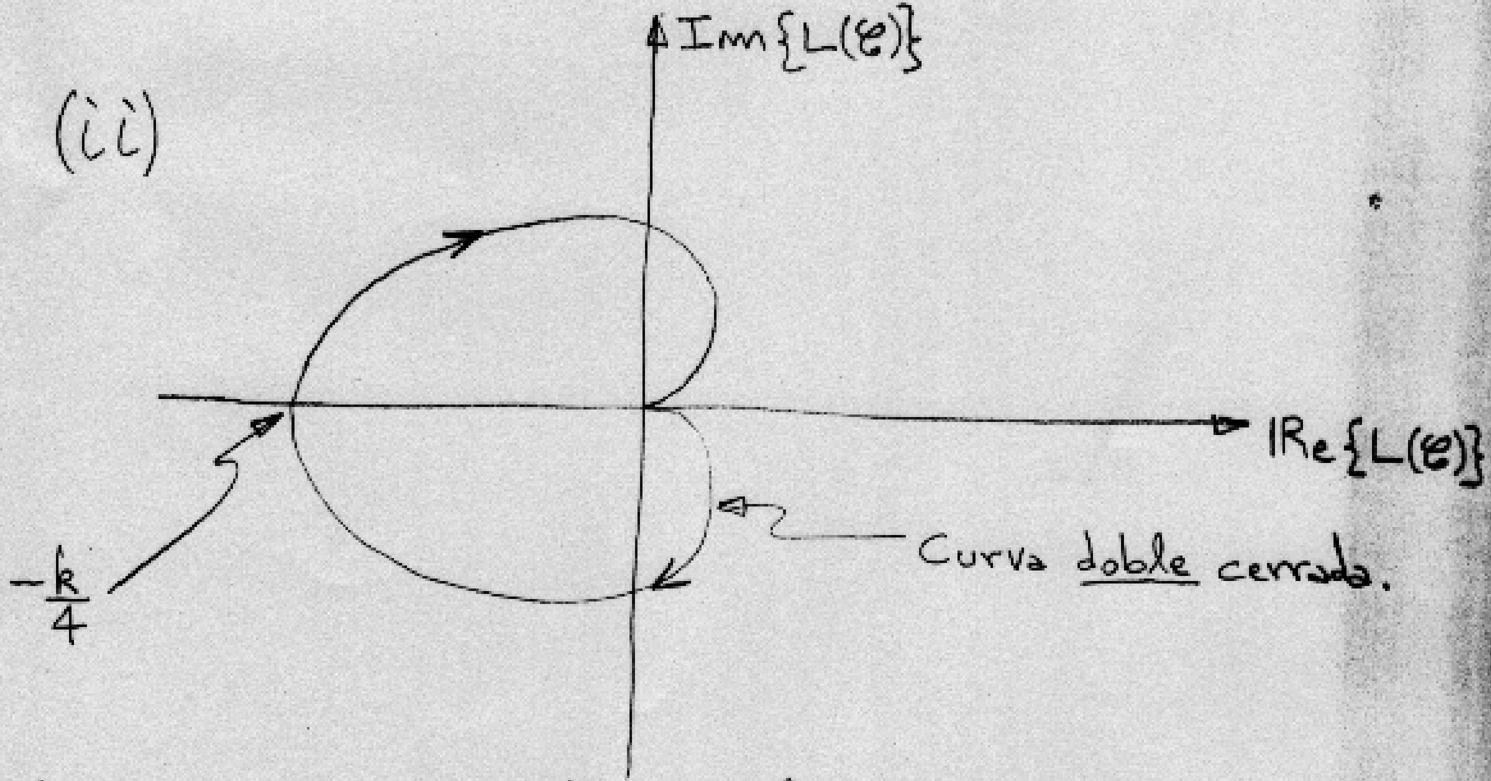
(e)

(i) $L(s) = -k\, H^2(s) = -k\left[\dfrac{\frac{1}{\tau}s}{\left[\left(s+\frac{1}{\tau}\right)^2 + \frac{1}{\tau^2}\right]}\right]^2 =$

$$= -k\left[\frac{\zeta\omega_o s}{\left(s^2 + 2\zeta\omega_o s + \omega_o^2\right)}\right]^2 ,$$

$$\omega_o = \frac{\sqrt{2}}{\tau} , \quad \zeta = \frac{1}{\sqrt{2}} .$$

(ii)



(Notemos que L es real-racional y estrictamente propia, lo cual implica que la interconexión está bien definida.) En este caso tenemos que $(k>0, y)$ $P=0$. Así, invocando el Criterio de Estabilidad de Nyquist y usando el gráfico de Nyquist sigue que la interconexion es BIBO estable si y solo si

$$\boxed{0 < k < 4}.$$

(f)

$$\dot{x}_1(t) = A x_1(t) + B w(t),$$

$$v(t) = -u(t) + k E x_1(t),$$

$$\dot{x}_2(t) = A x_2(t) + B v(t),$$

$$w(t) = E x_2(t)$$

$$x_1(t) = \begin{pmatrix} v_{c_1}(t) \\ v_{c_2}(t) \end{pmatrix}, \quad x_2(t) = \begin{pmatrix} v_{c_3}(t) \\ v_{c_4}(t) \end{pmatrix}, \quad z(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

$$\boxed{\begin{aligned} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} &= \begin{pmatrix} A & BE \\ kBE & A \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -B \end{pmatrix} u(t), \\ w(t) &= \begin{pmatrix} 0 & E \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \end{aligned}}$$

(i)

$$\boxed{A_{cl} = \begin{pmatrix} A & BE \\ kBE & A \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau} & \frac{1}{\tau} & -\frac{1}{\tau} & -\frac{1}{\tau} \\ -\frac{1}{\tau} & -\frac{1}{\tau} & 0 & 0 \\ -\frac{k}{\tau} & -\frac{k}{\tau} & -\frac{1}{\tau} & \frac{1}{\tau} \\ 0 & 0 & -\frac{1}{\tau} & -\frac{1}{\tau} \end{pmatrix}}$$

(ii)

$$T_{cl}(s) = \frac{-H(s)}{1+L(s)} = \frac{\frac{1}{\tau}s\left[\left(s+\frac{1}{\tau}\right)^2+\frac{1}{\tau^2}\right]}{\left[\left(s+\frac{1}{\tau}\right)^2+\frac{1}{\tau^2}\right]^2 - k\frac{1}{\tau^2}s^2}$$

(iii) Notemos que $A_{cl} \in \mathbb{R}^{4\times 4}$ y que

$T_{cl}(s) = \frac{P_{Num,3}(s)}{P_{Denom,4}(s)}$ donde los polinomios $P_{Num,3}$ y $P_{Denom,4}$ son <u>coprimos</u>. Sigue entonces invocando un resultado del teorico (Proposicion 6 en Notas Sobre Estabilidad BIBO y Estabilidad Interna...), y usando nuestro analisis de estabilidad de la parte e (confinando nuestro estudio para $k>0$) que la matriz A_{cl} tendrá todos sus valores propios con parte real negativa si y solo si

$$\boxed{0<k<4}.$$

(iv)

$$T_{cl}\left(j\frac{\sqrt{2}}{\tau}\right) = \frac{\frac{1}{2}}{1-\frac{k}{4}} = \frac{2}{(4-k)}$$

$$\Longrightarrow \boxed{w_{ssr}(t) = \frac{2U_0}{(4-k)} \cos\frac{\sqrt{2}}{\tau}t \quad , \quad t \geq 0}\ .$$

Problema 2.

(a)

$$V_1 = V_+ + V_-$$

$$I_1 = \frac{1}{Z_0}\left(V_+ - V_-\right)$$

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{Z_0} & -\frac{1}{Z_0} \end{pmatrix} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$$\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \begin{pmatrix} e^{-T_d s} & e^{T_d s} \\ \frac{1}{Z_0} e^{-T_d s} & -\frac{1}{Z_0} e^{T_d s} \end{pmatrix} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$$\boxed{\Longrightarrow \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh T_d s & Z_0 \sinh T_d s \\ \frac{1}{Z_0} \sinh T_d s & \cosh T_d s \end{pmatrix}}_{T(s)} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix}}$$

(i) $A(s) = D(s) = \cosh T_d s$, $B(s) = Z_0 \sinh T_d s$,

$C(s) = \frac{1}{Z_0} \sinh T_d s$.

(ii) $\det\{T(s)\} = 1 \Longrightarrow$ <u>$\mathcal{N}$ es Reciproco</u>.

(b)

(i) $Z_g = Z_L = Z_0 \Longrightarrow \Gamma_g = \Gamma_L = 0.$

$$\Longrightarrow V_0(s) = V(\ell, s) = V_+(s)\, e^{-T_d s} =$$

$$= \frac{1}{2} V_s(s)\, e^{-T_d s}$$

$$\Longrightarrow \boxed{H(s) = \frac{1}{2} e^{-T_d s}}.$$

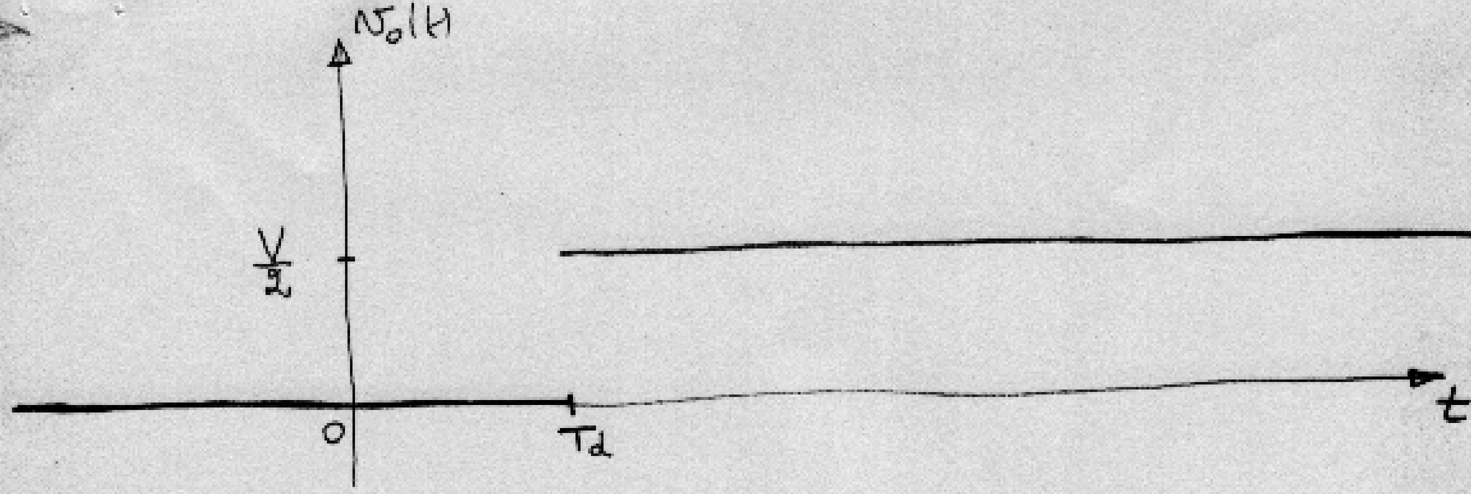
$$\Longrightarrow \boxed{h(t) = \frac{1}{2} \delta(t - T_d)}.$$

(ii) $h \in \mathbb{A} \subset \mathbb{A}_e.$

$h \notin \mathcal{L}_1 \quad , \quad h \notin \mathcal{L}_{1,e}.$

(iii) Dado que $h \in \mathbb{A}$, el sistema es entonces <u>BIBO estable</u>.

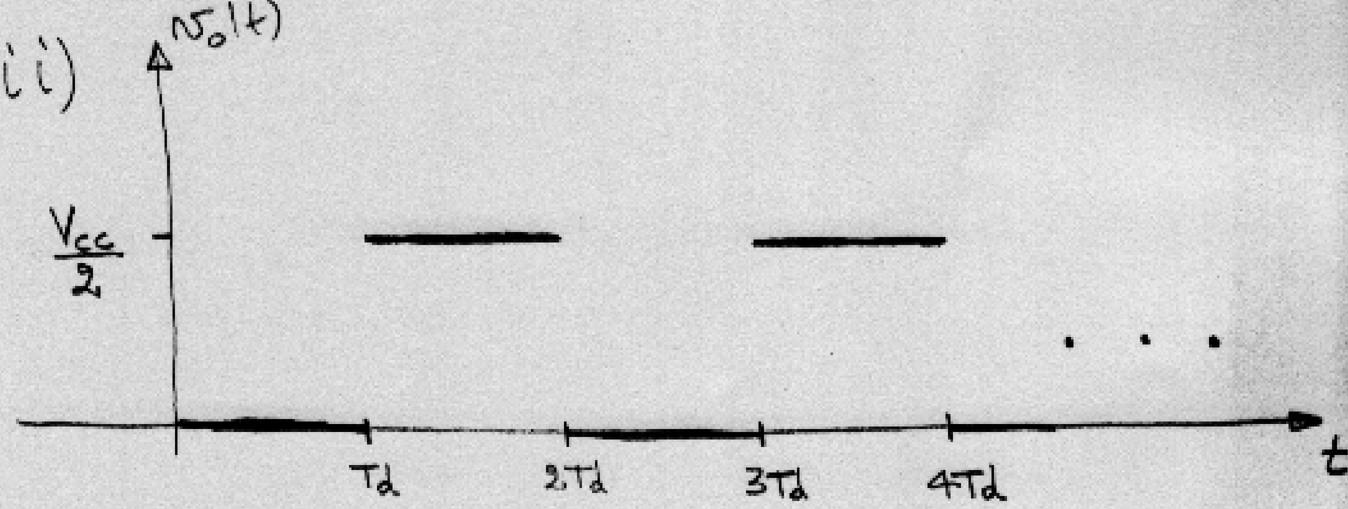
(iv) $$\boxed{v_0(t) = \frac{V}{2} \mu(t - T_d) \quad , \quad t \in \mathbb{R}.}$$



$v_o \notin \mathcal{L}_1$, $v_o \in \mathcal{L}_{1,e}$.

(c) (i) El circuito de Figura 5 opera como un Multivibrador Astable.

(iii)



(ii)

$$v_o(t) = \begin{cases} 0 , & t \in [0, T_d) \\ \frac{V_{cc}}{2} , & t \in [T_d, 2T_d) \end{cases} ,$$

$$v_o(t + 2T_d) = v_o(t) , \quad \forall t \geq 0 .$$

(d)

(i) $$\boxed{i_2(t) = \frac{V_{cc}}{4R} \quad , \quad t \geqslant 0}$$

$$v_1(t) = L\frac{di_1(t)}{dt} + \frac{1}{2}L\frac{di_2(t)}{dt} = L\frac{di_1(t)}{dt}$$

$$v_o(t) = R\,i_1(t) + v_1(t)$$

$$\Longrightarrow \quad \frac{di_1(t)}{dt} = -\frac{1}{T_d}\,i_1(t) + \frac{1}{T_d}\,\frac{v_o(t)}{R}$$

$$\boxed{\begin{gathered} i_1(t) = \begin{cases} \dfrac{V_{cc}}{2R}\,\dfrac{(1-e^{-1})}{(1-e^{-2})}\,e^{-\frac{t}{T_d}} & , \; t \in [0, T_d) \\[2ex] -\dfrac{V_{cc}}{2R}\,\dfrac{(1-e^{-1})}{(1-e^{-2})}\,e^{-\frac{(t-T_d)}{T_d}} + \dfrac{V_{cc}}{2R} & , \; t \in [T_d, 2T_d) \end{cases} \\ i_1(t+2T_d) = i_1(t) \quad , \quad t \geqslant 0 . \end{gathered}}$$

$$v_1(t) = \frac{V_{cc}}{2}\,\frac{(1-e^{-1})}{(1-e^{-2})}\begin{cases} -e^{-\frac{t}{T_d}} & , \; t\in[0,T_d) \\ e^{-\frac{(t-T_d)}{T_d}} & , \; t\in[T_d,2T_d) \end{cases}$$

$$v_1(t+2T_d) = v_1(t) \quad , \quad t \geq 0 .$$

$$v_2(t) = \frac{1}{2}\, v_1(t) \quad , \quad t \geq 0 .$$

(ii)