5. *a)* Consideremos dos series de potencias $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$, $\sum_{n=0}^{+\infty} b_n(z-z_0)^n$ con radios de convergencia $R_a, R_b > 0$. Probar que si existe un abierto U entorno de z_0, con $U \subset B(z_0, R_a) \cap B(z_0, R_b)$ tal que

$$\sum_{n=0}^{+\infty} a_n(z-z_0)^n = \sum_{n=0}^{+\infty} b_n(z-z_0)^n \quad \forall z \in U$$

entonces $a_n = b_n$ $\forall n$.

Datos: Las series de potencias son holomorfas

Si f es una s.d.p. con radio de convergencia R, f' también.

Si $f = \sum_{n=0}^{\infty} a_n (z-z_0)^n \Rightarrow f' = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$

$= \sum_{n=0}^{\infty} (n+1) a_{n+1} (z-z_0)^n$

$f(z_0) = a_0, \quad f'(z_0) = a_1, \quad f''(z_0) = 2a_2, \ldots, f^{(n)}(z_0) = n! a_n$

$a_0 + a_1 z + a_2 z^2 + a_3 z^3$

$a_1 + 2a_2 z + 3a_3 z^2$

$2a_2 + 3! a_3 z$

$3! a_3$

$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n \Rightarrow g^{(n)}(z_0) = n! b_n = n! a_n$

$\Rightarrow a_n = b_n$

3. Consideremos la serie de potencia de la forma

$$\frac{1}{2}z + z^2 + \frac{1}{8}z^3 + \frac{1}{4}z^4 + \frac{1}{32}z^5 + \frac{1}{16}z^6 + \frac{1}{128}z^7 + \frac{1}{64}z^8 + \ldots$$

$\frac{1}{2^1}, \frac{1}{2^0}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2^5}, \frac{1}{2^4}$

$a_n = \begin{cases} \frac{1}{2^{n+1}}, & n \text{ par} \\ \frac{1}{2^{n-1}}, & n \text{ impar} \end{cases}$

$L = \lim_n \frac{|a_{n+1}|}{|a_n|} = \lim \frac{1/2^n}{1/2^{n+1}}$ (n par)

$= \lim 2 = 2$

n impar

$\lim \frac{1/2^{n+2}}{1/2^{n-1}} = \lim \frac{1}{8}$

$$\Rightarrow \limsup \frac{|a_{n+1}|}{|a_n|} = 2 \Rightarrow R = \frac{1}{2}$$

4. Sea $T:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ una transformación de Mobius tal que

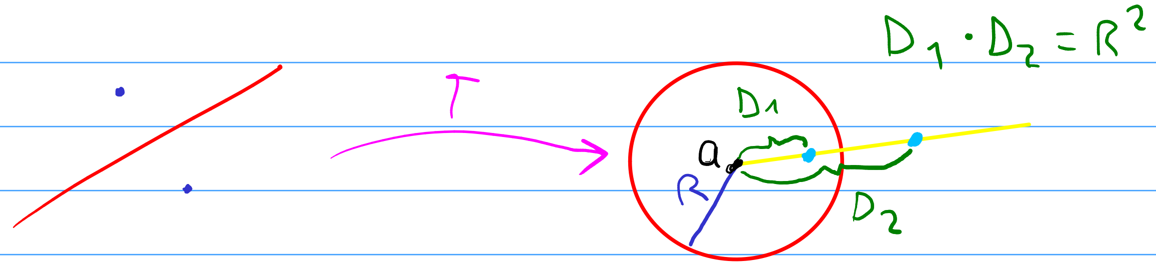
$$T(z)=\frac{2z-i}{-iz-2}$$

a) Sea $D=\{z\in\mathbb{C}:\ |z|<1\}$. Probar que $T(D)=D$.

$$|T(e^{i\theta})| = \left|\frac{2e^{i\theta}-i}{-ie^{i\theta}-2}\right| = \left|\frac{2\cos\theta + i(2\sin\theta-1)}{\sin\theta-2+i(-\cos\theta)}\right|$$

$$= \frac{4\cos^2\theta+4\sin^2\theta-4\sin\theta+1}{\sin^2\theta-4\sin\theta+4+\cos^2\theta} = \frac{5-4\sin\theta}{5-4\sin\theta} = 1$$

$T(0)=i/2\in D$



z_1 y z_2 son inversos resp. de $\partial B(a,R) \Leftrightarrow$

$$(z_1-a)\overline{(z_2-a)} = R^2$$

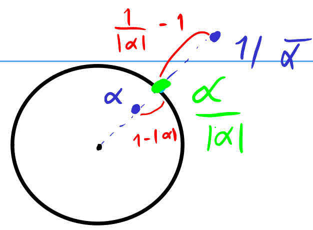
8) *b*) Probar que las tranformaciones de Möbius que llevan el disco unidad en el disco unidad son de la forma $T(z)=e^{i\beta}\dfrac{z-\alpha}{\bar{\alpha}z-1}$ con $\beta\in\mathbb{R}$ y $|\alpha|<1$.

Sea $\alpha\in D\ /\ T(\alpha)=0 \Rightarrow T(z)=\dfrac{a(z-\alpha)}{cz+d}$

Centro: $0+i\cdot 0=0 \Rightarrow (\alpha-0)\overline{(\beta-0)}=1^2 \Rightarrow \alpha\bar{\beta}=1 \Rightarrow \beta=\dfrac{1}{\bar{\alpha}}$

$$\Rightarrow T(1/\bar{\alpha})=\infty \Rightarrow T(z)=\frac{a(z-\alpha)}{c(z-1/\bar{\alpha})} = \frac{a}{c/\bar{\alpha}}\cdot\frac{z-\alpha}{\bar{\alpha}z-1}$$

$w:=\dfrac{\alpha}{|\alpha|} \Rightarrow |w|=1$



$$T(w) = \frac{a}{c} \cdot \frac{w-\alpha}{w - 1/\bar{\alpha}} \Rightarrow |T(w)| = \left|\frac{a}{c}\right| \frac{|w-\alpha|}{|w-1/\bar{\alpha}|}$$

$$= \left|\frac{a}{c}\right| \frac{1-|\alpha|}{\frac{1}{|\alpha|}-1} = \left|\frac{a}{c}\right| |\alpha| \frac{\cancelto{1}{1-|\alpha|}}{\cancel{1-|\alpha|}} = \left|\frac{a}{c}\right| |\alpha| = 1$$

$$\Rightarrow 1 = \left|\frac{a\alpha}{c}\right| = \left|\frac{a\bar{\alpha}}{c}\right| = \left|\frac{a}{c/\bar{\alpha}}\right|$$

$$\Rightarrow \exists \beta \in \mathbb{R},\ \frac{a}{c/\bar{\alpha}} = e^{i\beta} \Rightarrow T(z) = e^{i\beta} \frac{z-\alpha}{\bar{\alpha}z - 1}$$