CLASE 3

10/08/25

NÚMEROS COMPLEJOS.

$$z = a + ib \xrightarrow{f} f(z) = e^{z}$$

$$e^{z} = e^{a}(\cos b + i \operatorname{sen} b).$$

$$\mathbb{C} \xrightarrow{f} \mathbb{C}.$$

Obs f coincide con la función exponencial real si tomamos

$$f\big|_{\mathbb{R}}$$

$$z = a + ib \quad (b = 0) \qquad \overset{b}{\overset{\shortparallel}{}}$$

$$z = a \in \mathbb{R} \quad (z = a + 0i)$$

$$f(z) = e^{z} = e^{a}(\cos 0 + i \operatorname{sen} 0)$$

$$= e^{a}$$

Obs $z = a + ib \quad (a = 0)$

↓

Imaginario Puro

$$z = ib$$

$$f(z) = e^{z} = e^{ib} = \cos b + i \operatorname{sen} b.$$

Obs: Las propiedades aritméticas de la exponencial real e^t, $t \in \mathbb{R}$ se cumple para la exponencial compleja

$$e^{z+w} = e^z \cdot e^w \qquad z, w \in \mathbb{C}.$$

$z = a + ib$ $\qquad$ $z + w = a + c + i(b+d)$

$w = c + id$

$$e^{z+w} = e^{a+c}\left(\cos(b+d) + i\,\text{sen}(b+d)\right).$$

$$= e^a \cdot e^c$$

$$e^z \cdot e^w = e^a(\cos b + i\,\text{sen}\, b) \cdot e^c(\cos d + i\,\text{sen}\, d)$$

$$= e^a e^c(\cos b + i\,\text{sen}\, b)(\cos d + i\,\text{sen}\, d)$$

$$= e^a \cdot e^c\left(\cos b \cos d - \text{sen}\, b\, \text{sen}\, d + i(\text{sen}\, b \cos d + \text{sen}\, d \cos b)\right)$$

$$\boxed{e^{z+w} = e^z \cdot e^w}$$

Obs: (Ejemplo)

Si $z = a + ib$ con $a = 0$ $b = \pi$

$z = ib = i\pi$

$$e^z = e^a(\cos b + i\sin b) = e^0(\cos\pi + i\sin\pi)$$

$$e^{i\pi} = e^0(\cos\pi + i\sin\pi)$$

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

<u>Obs 5:</u> Las propiedades de la exponencial real relativos a desigualdades, crecimiento, etc <u>no</u> son trasladables a la exponencial compleja. (No tienen sentido).

<u>Obs 6:</u> Módulo y el Argumento de e^z.

$z = a + ib$

- Módulo de e^z

$$|e^z| = |e^a(\cos b + i\sin b)|$$
$$= |e^a|\,|\cos b + i\sin b|$$
$$= e^a\,\underbrace{|\cos b + i\sin b|}_{1}$$
$$= e^a$$

El módulo de e^z depende solamente de $Re(z)$.

- El Argumento de e^z depende solamente de la parte imaginaria

$$\begin{tikzcd} & & e^a \\ z = a + ib \arrow[r, "f"] & e^z \arrow[ur, "\text{Modulo}"] \arrow[dr, "\text{Argumento}"'] & \\ & & b \end{tikzcd}$$

<u>Observación:</u> Si z es un número complejo de modulo $r = |z|$ y Argumento θ tenemos la notación más compacta

$$\boxed{z = re^{i\theta}} \; //$$

<u>Obs:</u> (Propiedades)

Sean $z = re^{i\theta}$ $\quad w = \rho e^{i\varphi}$

•) $z = w \iff \begin{cases} r = \rho \\ \theta - \varphi = 2k\pi \end{cases} \quad k \in \mathbb{Z}$

•) $\overline{z} = \overline{re^{i\theta}} = re^{-i\theta} = \dfrac{r}{e^{i\theta}}$

•) $z \cdot w = re^{i\theta}\rho e^{i\varphi} = r\rho e^{i(\theta+\varphi)}$

$w \neq 0$) $\dfrac{z}{w} = \dfrac{re^{i\theta}}{\rho e^{i\varphi}} = \dfrac{r}{\rho} e^{i(\theta-\varphi)}$

Obs Potencias de números complejos

$$z = re^{i\theta} = |z|e^{i\theta} \qquad r = |z|$$

$$z^0 = 1$$

$$z^1 = z$$

$$z^2 = z \cdot z = re^{i\theta} \cdot re^{i\theta} = r^2 e^{i(2\theta)}$$

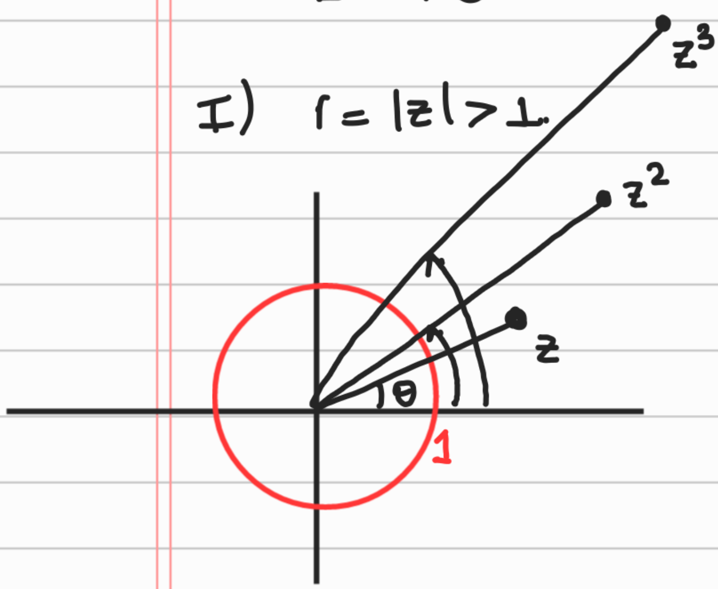
$\vdots$

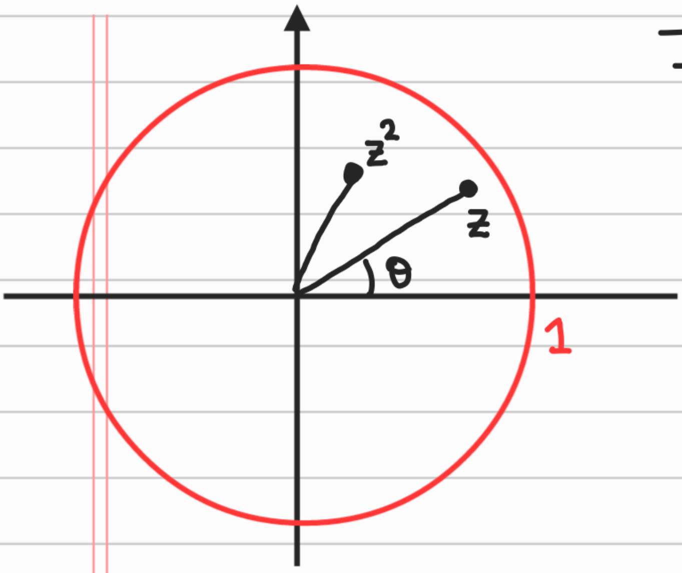
$$z^n = r^n e^{in\theta}$$

Consideremos las siguientes situaciones:

$$z = re^{i\theta} \qquad r = |z|$$

I) $r = |z| > 1$





II $\quad r=|z|<1$

$|z|>|z|^2$

<u>Ejemplo:</u> Calcular la parte imaginaria de

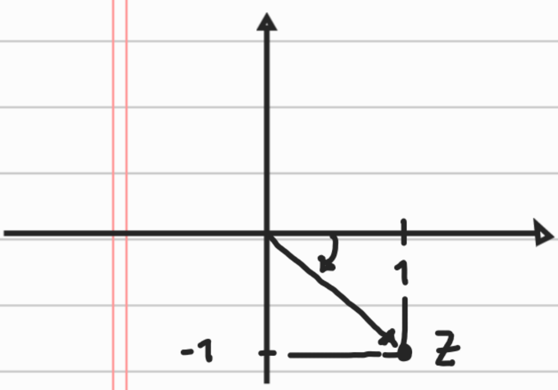
$(1-i)^5 = a+ib$

$z=1-i \longleftarrow$ Forma binómica

z^5

$$z=1-i \xrightarrow{\text{Forma exp}} z=|z|e^{i\theta}$$

$|z|=?\qquad \theta=?$



$|z|=\sqrt{2}\qquad \theta=-\pi/4$

$$z=\sqrt{2}\,e^{-i\pi/4}$$

$$z^5=(1-i)^5=\left(\sqrt{2}\,e^{-i\pi/4}\right)^5$$

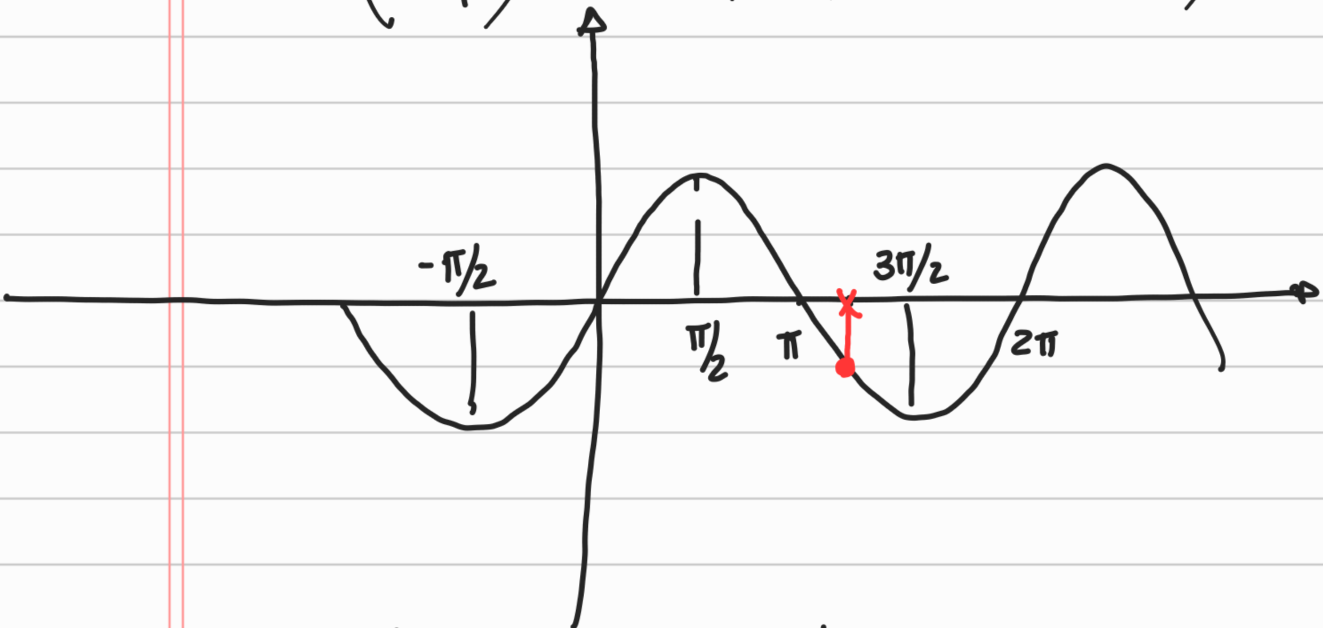
$$=\left(\sqrt{2}\right)^5 e^{-i5\pi/4}$$

¿Parte imaginaria de $z^5 = \sqrt{2}^5 e^{-i5\pi/4}$?

$$(\sqrt{2})^5 e^{-i5\pi/4} = (a + ib)(\sqrt{2})^5$$
$$= \underline{\underline{b(\sqrt{2})^5}}$$

$$e^{-i\frac{5\pi}{4}} = \cos\left(-\frac{5\pi}{4}\right) + i\,\text{Sen}\left(-\frac{5\pi}{4}\right)$$

$$\text{Sen}\left(-\frac{5\pi}{4}\right) = -\text{Sen}\left(\frac{5\pi}{4}\right) = -\text{Sen}\left(\pi + \frac{\pi}{4}\right)$$



$$\text{Sen}(-5\pi/4) = \frac{\sqrt{2}}{2}.$$

$\Rightarrow$ Parte imaginaria de $z^5 = \sqrt{2}^5 e^{-i5\pi/4}$

$$\frac{\sqrt{2}}{2}\cdot(\sqrt{2})^5 = \frac{\sqrt{2}^6}{2} = 4.$$

Raíces de un número complejo

Se trata de resolver la siguiente ecuación

$$w^n = z_0$$ donde n es un natural, $n \geq 2$ y $z_0 \neq 0$ un número complejo conocido.

Estrategia.

$$w = |w|(\cos\varphi + i\,\text{sen}\,\varphi) = |w|e^{i\varphi}.$$

$$w^n = |w|^n(\cos n\varphi + i\,\text{sen}\,n\varphi)$$

Si $z_0 = |z_0|(\cos\theta_0 + i\,\text{sen}\,\theta_0)$

La ecuación $w^n = z_0$

$$|w|^n(\cos n\varphi + i\,\text{sen}\,n\varphi) = |z_0|(\cos\theta_0 + i\,\text{sen}\,\theta_0)$$

De donde

1) $|w|^n = |z_0| \rightarrow$ $\boxed{|w| = \sqrt[n]{|z_0|}}$

2) $n\varphi_k - \theta_0 = 2k\pi$ $k \in \mathbb{Z}$ $\boxed{\varphi_k = \dfrac{\theta_0 + 2k\pi}{n} \quad k \in \mathbb{Z}}$

$$W^n = z_0$$

$$W_k = \sqrt[n]{|z_0|}\left(\cos\left(\frac{\theta_0+2k\pi}{n}\right) + i\,\text{Sen}\left(\frac{\theta_0+2k\pi}{n}\right)\right)$$

$k = 0, \ldots, n-1.$

__Ejemplo:__

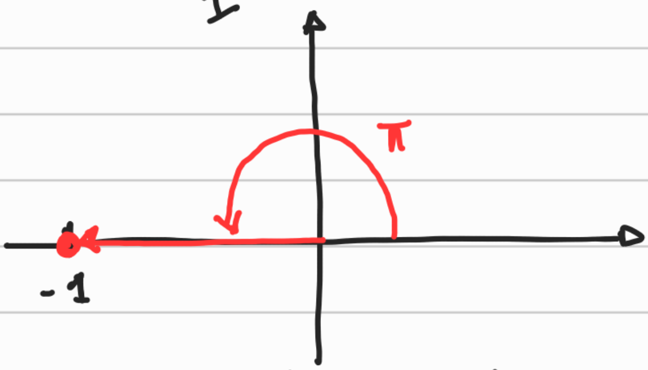
Calcular las raíces sextas de $\underline{z_0 = -1}$.

$W^6 = -1.$ $\qquad W = ?$ $\qquad |z_0| = 1$

$\theta_0 = \pi$

$z_0 = -1 = e^{i\pi}$ sus 6 raíces están dados por

$$W_k = \cancelto{1}{\sqrt[6]{|z_0|}}\left(\cos\left(\frac{\theta_0+2k\pi}{6}\right) + i\,\text{Sen}\left(\frac{\theta_0+2k\pi}{6}\right)\right)$$



$k=0$ $\qquad W_0 = \cos\left(\frac{\pi + 2 0 \pi}{6}\right) + i\,\text{Sen}\left(\frac{\pi + 2.0\pi}{6}\right)$

$$= \cos\left(\frac{\pi}{6}\right) + i\,\text{Sen}\left(\frac{\pi}{6}\right)$$

$k=1$ $\qquad W_1 = \cos\left(\frac{\pi + 2\pi}{6}\right) + i\,\text{Sen}\left(\frac{\pi + 2\pi}{6}\right)$

$$W_1 = \cos\left(\frac{3\pi}{6}\right) + i \operatorname{Sen}\left(\frac{3\pi}{6}\right)$$

$$= \cos\left(\frac{\pi}{2}\right) + i \operatorname{Sen}\left(\frac{\pi}{2}\right)$$

.

.

$K = 5$ $$W_5 = \cos\left(\frac{\pi + 10\pi}{6}\right) + i \operatorname{Sen}\left(\frac{\pi + 10\pi}{6}\right).$$

RESUMEN

Raices de un número complejo

Objetivo: Resolver la ecuación

$$z^n = z_0 \quad \text{donde } n \in \mathbb{N} \ (n \geq 2)$$

y z_0 es un complejo no nulo conocido.

$$z_0 = |z_0| (\cos \theta_0 + i \operatorname{sen} \theta_0).$$

Las n raíces n-ésimas de z_0 vienen dadas por la fórmula

$$z_k = |z_0|^{1/n} \left(\cos\left(\frac{\theta_0 + 2k\pi}{n}\right) + i \operatorname{Sen}\left(\frac{\theta_0 + 2k\pi}{n}\right)\right)$$

con $k = 0, 1, 2, \ldots, n-1$.

Retomamos el Ejemplo de la Clase.

Resolver $\quad z^6 = -1$.

Procedimiento:

1) Expresamos $z_0 = -1$ en notación polar

$$z_0 = e^{\pi i} = 1 (\cos \pi + i \operatorname{Sen} \pi)$$

$$|z_0| = 1 \ \text{ y } \ \theta_0 = \pi$$

2) Aplicamos la fórmula

$$z_k = \sqrt[6]{|z_0|}\left(\cos\left(\frac{\theta_0 + 2k\pi}{6}\right) + i\,\mathrm{Sen}\left(\frac{\theta_0 + 2k\pi}{6}\right)\right)$$

$k = 0, 1, \ldots 5$

$|z_0| = 1 \qquad \theta = \pi$

$$z_0 = \cos(\pi/6) + i\,\mathrm{Sen}(\pi/6)$$

$$z_1 = \cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$$

$$z_2 = \cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$$

$$z_3 = \cos\left(\frac{\pi}{6} + \pi\right) + i\,\mathrm{Sen}\left(\frac{\pi}{6} + \pi\right)$$

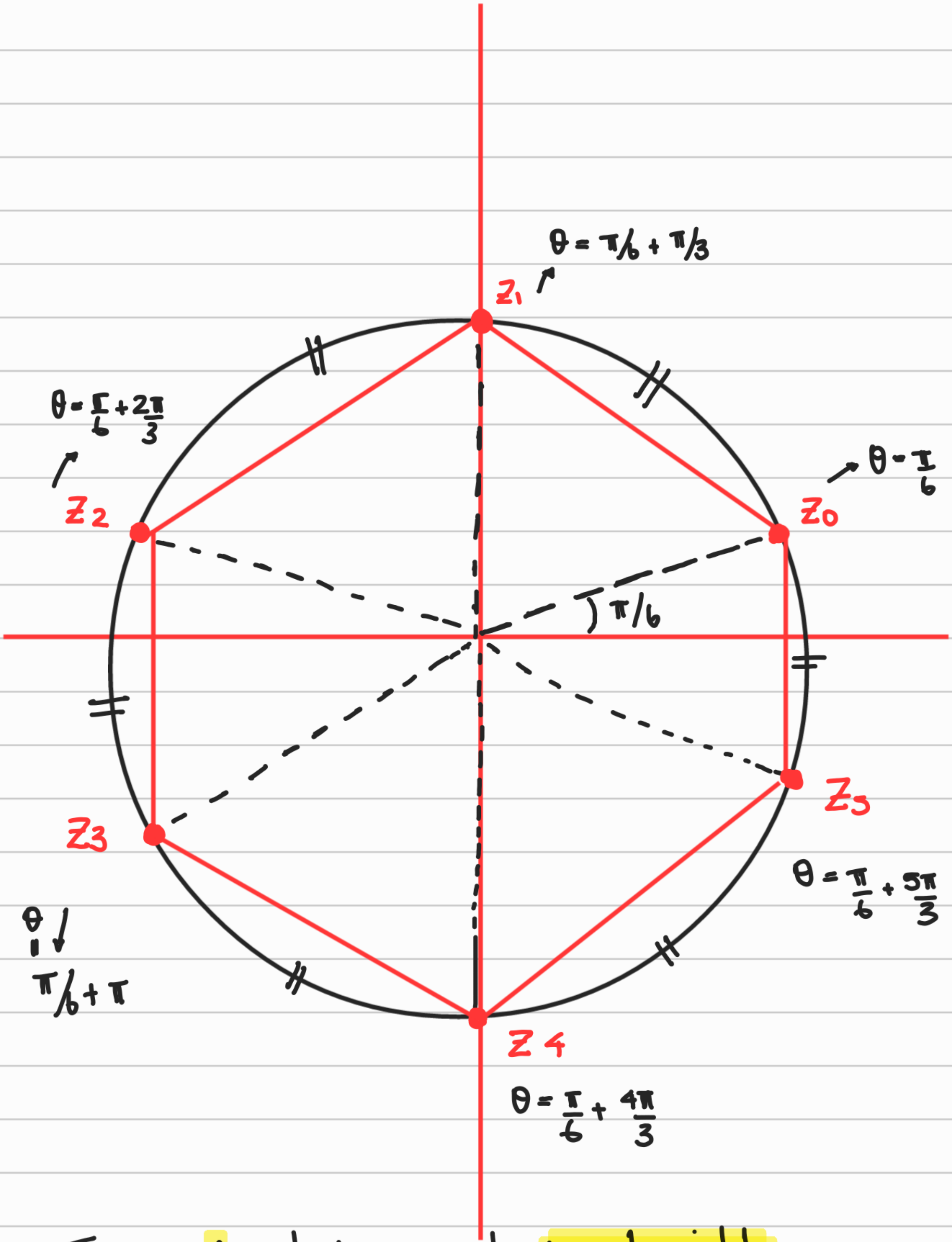
$$z_4 = \cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{\pi}{6} + \frac{4\pi}{3}\right)$$

$$z_5 = \cos\left(\frac{\pi}{6} + \frac{5\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{\pi}{6} + \frac{5\pi}{3}\right)$$

$$z_k = \pm i \,,\ \pm\left(\frac{\sqrt{3}}{2} \pm \frac{1}{2}i\right).$$

$$z_k = \sqrt[6]{|z_0|}\left(\cos\left(\frac{\theta_0}{6} + \frac{2k\pi}{6}\right) + i\,\mathrm{Sen}\left(\frac{\theta_0}{6} + \frac{2k\pi}{6}\right)\right)$$

$$= \sqrt[6]{|z_0|}\left(\cos\left(\frac{\theta_0}{6} + \frac{k\pi}{3}\right) + i\,\mathrm{Sen}\left(\frac{\theta_0}{6} + \frac{k\pi}{3}\right)\right)$$



Tenemos 6 soluciones, de igual módulo y formando un polígono regular.