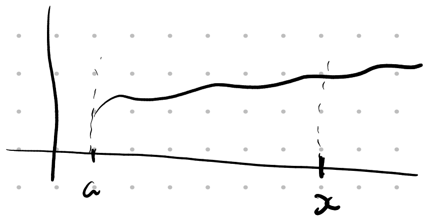
Integrales impropias de primera especie

$f: [a, +\infty) \to \mathbb{R}$

tal que f es integrable y acotada en todo intervalo de la forma $[a, x]$



$$\int_a^{+\infty} f(t)\,dt = \lim_{x \to +\infty} \int_a^x f(t)\,dt \quad \leftarrow \text{definición}$$

$\int_a^{+\infty} f(t)\,dt$ converge o no?

→ podemos calcular $\int_a^x f(t)\,dt$ y tomar el límite
Ejercicio 2 del práctico

→ criterios de convergencia

① Criterio de comparación

f y g tales que $0 \leq f(t) \leq g(t)$ para todo $t \geq a$

* si $\int_a^{+\infty} g(t)\,dt$ converge $\Rightarrow \int_a^{+\infty} f(t)\,dt$ converge
* si $\int_a^{+\infty} f(t)\,dt$ diverge $\Rightarrow \int_a^{+\infty} g(t)\,dt$ diverge

② Criterio de equivalentes

f y g tales que $\begin{cases} f(t) \geq 0 \text{ y } g(t) \geq 0 \text{ para todo } t \\ \lim\limits_{t \to +\infty} \dfrac{f(t)}{g(t)} = L > 0 \end{cases}$

entonces:

$\int_a^{+\infty} f(t)\,dt$ y $\int_a^{+\infty} g(t)\,dt$ son de la misma clase

③ Criterio de convergencia absoluta

si $\int_a^{+\infty} |f(t)|\,dt$ converge entonces $\int_a^{+\infty} f(t)\,dt$ converge

integral impropia que podemos usar para comparar:

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx \quad \begin{cases} \text{converge si } \alpha > 1 \\ \text{diverge si } \alpha \leq 1 \end{cases}$$

3. Sea $k > 0$. Hallar el valor de k para que la integral $\int_1^{+\infty} \left(\frac{x}{2x^2+2k} - \frac{k}{x+1} \right) dx$ sea convergente y calcularla.

$k > 0$

$$\int_1^{+\infty} \frac{x}{2x^2+2k} - \frac{k}{x+1}\,dx$$

* vamos a buscar k para que la integral sea convergente

$$\frac{x}{2x^2+2k} - \frac{k}{x+1} = \frac{x(x+1) - k(2x^2+2k)}{(2x^2+2k)(x+1)}$$

$$= \frac{x^2+x-2kx^2-2k^2}{2x^3+2x^2+2kx+2k}$$

$$= \frac{(1-2k)x^2+x-2k^2}{2x^3+2x^2+2kx+2k}$$

CASO 1: $1-2k=0 \iff k=\frac{1}{2}$

$$\frac{(1-2k)x^2+x-2k^2}{2x^3+2x^2+2kx+2k} \underset{k=\frac{1}{2}}{=} \frac{x-2\left(\frac{1}{2}\right)^2}{2x^3+2x^2+2\frac{1}{2}x+2\cdot\frac{1}{2}} \sim \frac{x}{2x^3} \sim \frac{1}{2x^2}$$

$$\int_1^{+\infty} \frac{1}{2x^2}\,dx = \frac{1}{2}\int_1^{+\infty} \frac{1}{x^2}\,dx \quad \text{converge}$$

entonces por el criterio de equivalente

$$\int_1^{+\infty} \frac{x}{2x^2+2k} - \frac{k}{x+1}\, dx \quad \text{converge si } k = \frac{1}{2}$$

CASO 2: $1-2k \neq 0 \iff k \neq \frac{1}{2}$

$$\frac{\overset{\neq 0}{(1-2k)}x^2 + x - 2k^2}{2x^3+2x^2+2kx+2k} \sim \frac{(1-2k)x^2}{2x^3} \sim \frac{1-2k}{2} \cdot \frac{1}{x}$$

como $\int_1^{+\infty} \frac{1}{x}\, dx$ diverge, por el criterio de equivalente

$$\int_1^{+\infty} \frac{x}{2x^2+2k} - \frac{k}{x+1}\, dx \quad \text{diverge si } k \neq \frac{1}{2}$$

6. Clasificar:

$a)\ \int_{-\infty}^{+\infty} e^{-x^2}\, dx \qquad b)\ \int_0^1 \frac{e^{-x}}{x}\, dx \qquad c)\ \int_0^1 \frac{\log(x)}{\sqrt{x}}\, dx \qquad d)\ \int_1^{+\infty} \frac{\sin(x)}{x^2}\, dx \qquad e)\ \int_0^{+\infty} \sin^2(t)\, dt$

$f)\ \int_0^{+\infty} \frac{x}{\sqrt{x^4+1}}\, dx \qquad g)\ \int_{-\infty}^{+\infty} \frac{x}{\cosh(x)}\, dx \qquad h)\ \int_{-1}^{1} \frac{1}{x^2-1}\, dx \qquad i)\ \int_0^{+\infty} \frac{\cos(x)}{\sqrt{x}}\, dx \qquad j)\ \int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$

$k)\ \int_0^{+\infty} e^{x^2-\frac{1}{x^2}}\, dx \qquad l)\ \int_{-1}^{1} \frac{e^{\frac{-1}{x-1}}}{(x-1)^2}\, dx \qquad m)\ \int_0^{\frac{\pi}{2}} \frac{dx}{\sin^{\alpha}(x)\cos^{\beta}(x)} \qquad (\alpha,\beta \in \mathbb{R})$

d) $\int_1^{+\infty} \frac{\sin(x)}{x^2}\, dx$

veamos si $\int_1^{+\infty} \left|\frac{\sin(x)}{x^2}\right| dx$ converge:

$$\left.\begin{array}{l} 0 \leq \left|\frac{\sin(x)}{x^2}\right| = \frac{|\sin(x)|}{x^2} \leq \frac{1}{x^2} \\ \int_1^{+\infty} \frac{1}{x^2}\, dx \text{ converg} \end{array}\right] \implies \text{por comparación } \int_1^{+\infty} \left|\frac{\sin(x)}{x^2}\right| dx \text{ converge}$$

entonces $\int_1^{+\infty} \frac{\sin(x)}{x^2}$ converge absolutamente d') $\int_1^{+\infty} \frac{\sin(x)}{x}\,dx$

veamos si $\int_1^{+\infty} \left|\frac{\sin(x)}{x}\right| dx$ converge

$$\left.\begin{array}{l} 0 \leq \left|\frac{\sin(x)}{x}\right| = \frac{|\sin(x)|}{x} \leq \frac{1}{x} \\ \int_1^{+\infty} \frac{1}{x}\,dx \text{ diverge} \end{array}\right\}$$ comparación no nos dice nada

$$\int_1^x \sin(t)\cdot\frac{1}{t}\,dt = -\frac{\cos(t)}{t}\Big|_1^x - \int_1^x \frac{\cos(t)}{t^2}\,dt$$

$f(t) = \frac{1}{t} \rightsquigarrow f'(t) = -\frac{1}{t^2}$

$g'(t) = \sin(t) \rightsquigarrow g(t) = -\cos(t)$

$$\int_1^x \sin(t)\cdot\frac{1}{t}\,dt = -\frac{\cos(x)}{x} + \cos(1) - \int_1^x \frac{\cos(t)}{t^2}\,dt$$

$$\Rightarrow \int_1^{+\infty} \frac{\sin(t)}{t}\,dt = \underbrace{\left(\lim_{x\to+\infty} -\frac{\cos(x)}{x}\right)}_{\searrow 0} + \cos(1) - \underbrace{\int_1^{+\infty} \frac{\cos(t)}{t^2}\,dt}_{\in \mathbb{R} \text{ porque la integral converge}}$$

$\rightarrow$ vamos a estudiar $\int_1^{+\infty} \frac{\cos(t)}{t^2}\,dt$

$$\left.\begin{array}{l} 0 \leq \left|\frac{\cos(t)}{t^2}\right| \leq \frac{1}{t^2} \\ \int_1^{+\infty} \frac{1}{t^2}\,dt \text{ converge} \end{array}\right\} \Rightarrow \text{por comparación } \int_1^{+\infty} \left|\frac{\cos(t)}{t^2}\right| dt \text{ converge}$$

$$\Rightarrow \int_1^{+\infty} \frac{\cos(t)}{t^2}\,dt \text{ converge}$$

$$\int_1^{+\infty} \frac{\sin(t)}{t}\,dt \text{ converge ?}$$

$$\int_1^{x} \frac{\sin(t)}{t}\,dt = -\frac{\cos(x)}{x} + \cos(1) - \int_1^{x} \frac{\cos(t)}{t^2}\,dt$$

$$\int_1^{+\infty} \frac{\sin(t)}{t}\,dt = \lim_{x\to+\infty} \int_1^{x} \frac{\sin(t)}{t}\,dt$$

$$= \lim_{x\to+\infty} \Big(\underbrace{-\frac{\cos(x)}{x}}_{\to 0} + \cos(1) - \int_1^{x} \frac{\cos(t)}{t^2}\,dt \Big)$$

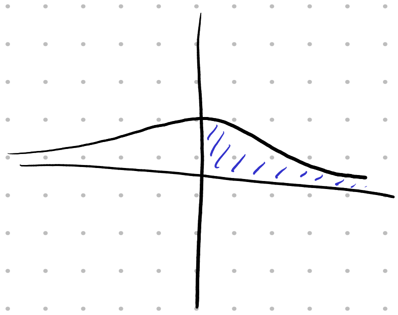
$$\underbrace{= \cos(1) - \underbrace{\int_1^{+\infty} \frac{\cos(t)}{t^2}\,dt}_{\text{converge}}}_{\text{converge}}$$

a) $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \int_{-\infty}^{0} e^{-x^2}\,dx + \int_{0}^{+\infty} e^{-x^2}\,dx$

$f(x) = e^{-x^2}$

$f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$

$f(x) = f(-x) \Rightarrow f$ es par

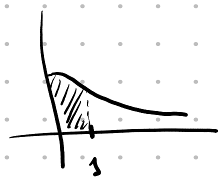


Como f es par: $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx$ converge $\Leftrightarrow \displaystyle\int_{0}^{+\infty} e^{-x^2}\,dx$ converge

vamos a estudiar $\displaystyle\int_{0}^{+\infty} e^{-x^2}\,dx = \int_{0}^{+\infty} \frac{1}{e^{x^2}}\,dx$

vamos a comparar con $\dfrac{1}{x^2}$

$\displaystyle\int_{1}^{+\infty} \frac{1}{x^2}\,dx$



$$\int_{0}^{+\infty} \frac{1}{e^{x^2}}\,dx = \underbrace{\int_{0}^{1} \frac{1}{e^{x^2}}\,dx}_{\in \mathbb{R}} + \int_{1}^{+\infty} \frac{1}{e^{x^2}}\,dx$$

veamos que $\displaystyle\int_{1}^{+\infty} \frac{1}{e^{x^2}}\,dx$ converge

$$\frac{1}{e^{x^2}} \overset{?}{\leq} \frac{1}{x^2} \Leftrightarrow 1 \overset{?}{\leq} \frac{e^{x^2}}{x^2}$$

como $\displaystyle\lim_{x\to+\infty} \frac{e^{x^2}}{x^2} = +\infty$, a partir de algún momento $1 \leq \dfrac{e^{x^2}}{x^2}$

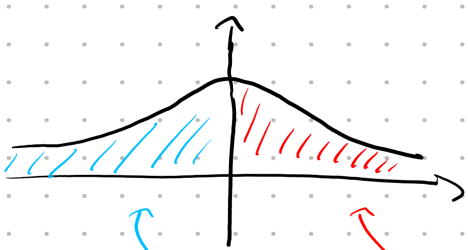
entonces $\dfrac{1}{e^{x^2}} \leq \dfrac{1}{x^2}$ a partir de algún momento entonces $\int_{1}^{+\infty} \frac{1}{x^2}\,dx$ converge $\Rightarrow \int_{1}^{+\infty} \frac{1}{e^{x^2}}\,dx$ converge

$\Rightarrow \int_{0}^{+\infty} \frac{1}{e^{x^2}}\,dx$ converge

$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{e^{x^2}}\,dx$ converge

$\int_{-\infty}^{+\infty} f(x)\,dx$ con f función par

f función par: $f(-x) = f(x)$



$$\int_{-\infty}^{+\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{+\infty} f(t)\,dt$$

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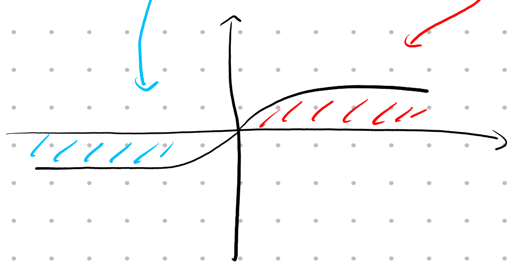
f impar: $f(-x) = -f(x)$

$$\int_{-\infty}^{+\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{+\infty} f(t)\,dt$$



g) $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\cosh(x)}\,dx$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$f(x) = \frac{x}{\frac{e^{x}+e^{-x}}{2}} = \frac{2x}{e^{x}+e^{-x}} \quad \text{es impar}$$

$$f(-x) = \frac{-2x}{e^{-x}+e^{x}} = -f(x)$$

$\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\cosh(x)}\,dx$ converg $\Leftrightarrow$ $\displaystyle\int_{0}^{+\infty} \frac{x}{\cosh(x)}\,dx$ converge