Series alternadas

$\sum a_n$ donde a_n no es de signo constante

* Convergencia absoluta

definición: $\sum a_n$ converge absolutamente si $\sum |a_n|$ converge

criterio: $\sum a_n$ converge absolutamente $\Rightarrow$ $\sum a_n$ converge

* Criterio de Leibniz

a_n sucesión monótona decreciente y que tiende a 0

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n$ converge

ejemplo:

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

* converge absolutamente? NO

$$\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverge}$$

* converge? SI

$a_n = \frac{1}{n}$ es monótona decreciente y $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converge por el criterio de Leibniz

7. Estudiar la convergencia de las siguientes series alternadas. En caso de que sean convergentes, estudiar si también lo son absolutamente.

$$a)\ \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{3^n} \qquad b)\ \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} n}{n^2+1}$$

a) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{3^n}$

* converge absolutamente?

$$\sum_{n=1}^{+\infty} \left|(-1)^{n+1} \frac{1}{3^n}\right| = \sum_{n=1}^{+\infty} \frac{1}{3^n} = \underbrace{\sum_{n=1}^{+\infty} \left(\frac{1}{3}\right)^n}_{\text{geométrica}} \text{ converge}$$

$$\sum_{n=0}^{\infty} q^n \quad \begin{cases} |q|<1 & \text{converge} \\ |q|\geq 1 & \text{diverge} \end{cases}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n} \text{ converge absolutamente}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3^n} \text{ converge}$$

$$\left[\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\cdot\frac{1}{3^{n-1}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = \frac{1}{3} \sum_{i=0}^{\infty} \frac{1}{3^i} \right]$$

$i=n-1$

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

$$3^n = 3\cdot 3^{n-1}$$

b) $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{n}{n^2+1}$ $\quad a_n$

* converge absolutamente? NO

$$\sum_{n=1}^{+\infty} \left|(-1)^{n+1} \frac{n}{n^2+1}\right| = \sum_{n=1}^{+\infty} \frac{n}{n^2+1}$$

$$\frac{n}{n^2+1} \sim \frac{n}{n^2} \sim \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverge}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \left|(-1)^{n+1} \frac{n}{n^2+1}\right| \text{ diverge}$$

* Converge?

veamos si podemos aplicar el criterio de Leibniz

$$a_n = \frac{n}{n^2+1}$$

→ $\lim_{n\to\infty} a_n = 0$

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

→ a_n es monótona decreciente

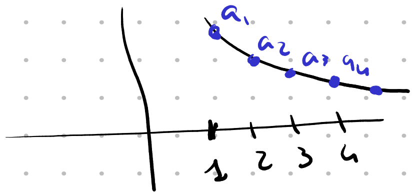
<u>forma 1</u>: $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

$$f'(x) \leq 0 \iff 1-x^2 \leq 0$$

$$\iff 1 \leq x^2$$

$$\iff \underbrace{x \geq 1}\ \text{o}\ x \leq -1$$



$$\Rightarrow a_{n+1} \leq a_n$$

$$\Rightarrow a_n \text{ es monótona decreciente}$$

<u>forma 2</u>:

$$a_{n+1} \overset{?}{\leq} a_n \iff \frac{n+1}{(n+1)^2+1} \overset{?}{\leq} \frac{n}{n^2+1}$$

$$\iff \frac{n+1}{n^2+2n+2} \overset{?}{\leq} \frac{n}{n^2+1}$$

$$\iff (n+1)(n^2+1) \overset{?}{\leq} n(n^2+2n+2)$$

$$\iff \cancel{n^3}+\cancel{n}+\cancel{n^2}+1 \overset{?}{\leq} \cancel{n^3}+\cancel{2}n^2+\cancel{2}n$$

$$\iff 1 \leq n^2+n \quad \text{cierto para todo } n \geq 1$$

entonces $a_{n+1} \leq a_n$ para todo n

$\Rightarrow a_n$ es monótona decreciente

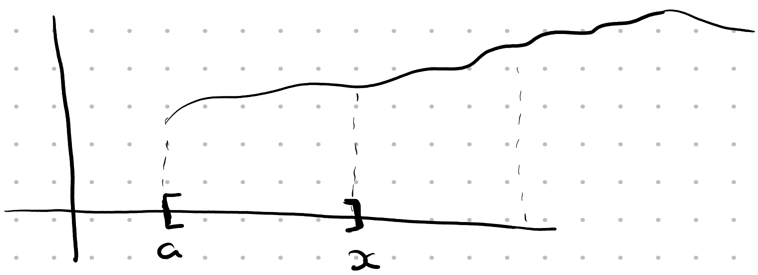
entonces por el criterio de Leibniz

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1} \text{ converge}$$

Integrales impropias

Primera especie

$f: [a, +\infty)$ integrable y acotada en cualquier intervalo de la forma $[a, x)$



$$\underbrace{\int_a^{+\infty} f(t)\,dt}_{\text{"serie"}} = \lim_{x \to \infty} \underbrace{\int_a^{x} f(t)\,dt}_{\text{"reducida } S_n\text{"}}$$

a_n

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots$

$S_n = \sum_{i=1}^{n} a_i$

$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$

$\int_a^{+\infty} f(t)\,dt$ converge o no?

2. Clasificar y hallar la integral en caso de convergencia

$$a) \int_2^{+\infty} \frac{1}{x \log^{\alpha}(x)}\,dx \qquad b) \int_0^{+\infty} x^3 e^{-x^2}\,dx$$

$$\log^{\alpha}(x) = \log(x)^{\alpha}$$

a) $\displaystyle\int_2^{+\infty} \frac{1}{x \log(x)^{\alpha}}\, dx$ ← parámetro

$$\int_2^{x} \frac{1}{t \log(t)^{\alpha}}\, dt = \int_{\log(2)}^{\log(x)} \frac{1}{u^{\alpha}}\, du$$

$u = \log(t)$

$du = \frac{1}{t}\, dt$

<u>CASO 1</u> : $\alpha = 1$ no converge

$$\int_2^{x} \frac{1}{t \log(t)}\, dt = \int_{\log(2)}^{\log(x)} \frac{1}{u}\, du$$

$$= \log(u) \Big|_{\log(2)}^{\log(x)}$$

$$= \log(\log(x)) - \log(\log(2))$$

$$\int_2^{+\infty} \frac{1}{t \log(t)}\, dt = \lim_{x \to \infty} \int_2^{x} \frac{1}{t \log(t)}\, dt$$

$$= \lim_{x \to \infty} \log(\log(x)) - \log(\log(2))$$

$$= +\infty$$

<u>CASO 2</u>: $\alpha \neq 1$

$$\int_2^{x} \frac{1}{t \log(t)^{\alpha}}\, dt = \int_{\log(2)}^{\log(x)} \frac{1}{u^{\alpha}}\, du$$

$$= \int_{\log(2)}^{\log(x)} u^{-\alpha}\, du$$

$$u^{-\alpha+1} = \frac{1}{u^{-(-\alpha+1)}} \qquad = \frac{u^{-\alpha+1}}{-\alpha+1}\Bigg|_{\log(2)}^{\log(x)}$$

$$u = \frac{1}{u^{-1}} \qquad = \frac{1}{1-\alpha}\cdot\frac{1}{u^{\alpha-1}}\Bigg|_{\log(2)}^{\log(x)}$$

$$= \frac{1}{1-\alpha}\left(\frac{1}{\log(x)^{\alpha-1}} - \frac{1}{\log(2)^{\alpha-1}}\right)$$

$$\lim_{x\to\infty} \frac{1}{1-\alpha}\left(\frac{1}{\log(x)^{\alpha-1}} - \frac{1}{\log(2)^{\alpha-1}}\right)$$

$$\lim_{x\to\infty} \frac{1}{\log(x)^4} = 0 \qquad \lim_{x\to\infty} \frac{1}{\log(x)^{-4}} = \lim_{x\to\infty} \log(x)^4 = +\infty$$

<u>CASO 2.1</u> : $\alpha - 1 > 0 \iff \alpha > 1$

$$\int_2^{+\infty} \frac{1}{t\log(t)^\alpha}\,dt = \lim_{x\to+\infty}\int_2^x \frac{1}{t\log(t)^\alpha}\,dt$$

$$= \lim_{x\to+\infty} \frac{1}{1-\alpha}\left(\underbrace{\frac{1}{\log(x)^{\alpha-1}}}_{\to 0} - \frac{1}{\log(2)^{\alpha-1}}\right)$$

$$= -\frac{1}{1-\alpha}\,\frac{1}{\log(2)^{\alpha-1}}$$

La integral impropia converge

<u>CASO 2.2</u> $\alpha - 1 < 0 \iff \alpha < 1$

$$\int_2^{+\infty} \frac{1}{t\log(t)^\alpha}\,dt = \lim_{x\to+\infty}\int_2^x \frac{1}{t\log(t)^\alpha}\,dt$$

$$= \lim_{x\to+\infty} \frac{1}{1-\alpha}\left(\frac{1}{\log(x)^{\underset{\text{negativo}}{\alpha-1}}} - \frac{1}{\log(2)^{\alpha-1}}\right)$$

$$= \lim_{x \to +\infty} \frac{1}{1-\alpha} \left(\log(x)^{1-\alpha} - \frac{1}{\log(2)^{\alpha-1}} \right)$$ positivo

$$= +\infty$$

Conclusión

$$\int_2^{+\infty} \frac{1}{t \log(t)^\alpha} \, dt$$

→ converge si $\alpha > 1$

→ diverge si $\alpha \leq 1$