Series

a_n sucesión real

* S_n sucesión de sumas parcial

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = \sum_{i=1}^{n} a_i$

* serie de término general a_n

$\sum_{n=1}^{\infty} a_n$

$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n$

cuando S_n converge $\sum_{n=1}^{\infty} a_n = \lim_{n} S_n = S \in \mathbb{R}$

serie geométrica

serie de termino general $a_n = q^n$ con $q \in \mathbb{R}$

$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + q^3 + \dots$ $\qquad \sum_{i=0}^{n} q^i = \frac{1 - q^{n+1}}{1-q}$

→ si $|q| < 1$, la serie converge y $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$

→ si $|q| \geq 1$, la serie no converge

1. Indicar si las siguientes series son convergentes o no, hallando sus suma en caso de serlo.

a) $\sum_{n=0}^{+\infty} \left(\frac{1}{3}\right)^n$ *b)* $\sum_{n=1}^{+\infty} \left(\frac{1}{\sqrt{3}}\right)^{n+3}$ *c)* $\sum_{n=1}^{+\infty} 5^{n+1}$ *d)* $\sum_{n=1}^{+\infty} \frac{3}{n(n+3)}$

e) $\sum_{n=1}^{+\infty} \log\left(\frac{n^2+2n+1}{n^2}\right)$ *f)* $\sum_{n=1}^{+\infty} \frac{n}{(n+1)(n+2)(n+3)}$ *g)* $\sum_{n=1}^{\infty} \frac{n \operatorname{arc\,tg}(n+1) - (n+1) \operatorname{arc\,tg}(n)}{n(n+1)}$

a) $\sum_{n=0}^{+\infty} \left(\frac{1}{3}\right)^n$

serie geométrica con $q = \frac{1}{3}$

$|q| < 1 \Rightarrow$ la serie converge

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1 - 1/3} = \frac{1}{2/3} = \frac{3}{2}$$

b) $$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^{n+3} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^4 \left(\frac{1}{\sqrt{3}}\right)^{n-1}$$

$$= \left(\frac{1}{\sqrt{3}}\right)^4 \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^{n-1} \quad i = n-1$$

$$= \left(\frac{1}{\sqrt{3}}\right)^4 \sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^i$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^n$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^4 \left(\frac{1}{\sqrt{3}}\right)^{n-1} = \underbrace{\left(\frac{1}{\sqrt{3}}\right)^4}\left(\frac{1}{\sqrt{3}}\right)^0 + \underbrace{\left(\frac{1}{\sqrt{3}}\right)^4}\left(\frac{1}{\sqrt{3}}\right)^1 + \underbrace{\left(\frac{1}{\sqrt{3}}\right)^4}\left(\quad\right)$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^{n+3} = \left(\frac{1}{\sqrt{3}}\right)^4 \underbrace{\sum_{i=0}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^i}$$

geométrica con $q = \frac{1}{\sqrt{3}}$

$|q| < 1 \Rightarrow$ la serie converge

$$= \left(\frac{1}{\sqrt{3}}\right)^4 \frac{1}{1 - 1/\sqrt{3}}$$

$$= \frac{1}{9} \frac{1}{1 - 1/\sqrt{3}}$$

series telescópicas

ejemplo

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+1)} \right)$$

termino general $a_n = \frac{1}{2n} - \frac{1}{2(n+1)}$

sea $b_n = \frac{1}{2n}$

$$\Rightarrow a_n = \underbrace{\frac{1}{2n}}_{b_n} - \underbrace{\frac{1}{2(n+1)}}_{b_{n+1}} = b_n - b_{n+1}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+1)} \right) = \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \right) + \left(\cancel{\frac{1}{6}} - \frac{1}{8} \right) + \ldots$$

$$s_3 = \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \right) + \left(\cancel{\frac{1}{6}} - \frac{1}{8} \right)$$

$$= \underbrace{\frac{1}{2}}_{b_1} - \underbrace{\frac{1}{8}}_{b_{3+1}}$$

$$s_4 = \left(\frac{1}{2} - \cancel{\frac{1}{4}} \right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} \right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{8}} \right) + \left(\cancel{\frac{1}{8}} - \frac{1}{10} \right)$$

$$= \underbrace{\frac{1}{2}}_{b_1} - \underbrace{\frac{1}{10}}_{b_{4+1}}$$

$$s_n = \underbrace{\frac{1}{2}}_{b_1} - \underbrace{\frac{1}{2(n+1)}}_{b_{n+1}}$$

$$\lim s_n = \frac{1}{2} - \lim \frac{1}{2(n+1)}$$

$$= b_1 - \lim_{n\to\infty} b_{n+1} \qquad \lim_{n\to\infty} b_{n+1} = \lim_{n\to\infty} b_n$$

$$= b_1 - \underbrace{\lim b_n}_{=0}$$

$$= \frac{1}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{1}{2(n+1)}$$ converge y la suma es $\frac{1}{2}$

<u>otro ejemplo</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$$

$$a_n = \frac{1}{n} - \frac{1}{n+3}$$

sea $b_n = \frac{1}{n}$

$$\Rightarrow a_n = b_n - b_{n+3}$$

$$\sum_{n=1}^{\infty} (b_n - b_{n+3}) = (b_1 - \cancel{b_4}) + (b_2 - \cancel{b_5}) + (b_3 - \cancel{b_6})$$
$$+ (\cancel{b_4} - b_7) + (\cancel{b_5} - b_8) + (\cancel{b_6} - b_9) +$$

$$S_6 = b_1 + b_2 + b_3 - b_7 - b_8 - b_9$$

$$S_7 = b_1 + b_2 + b_3 - b_8 - b_9 - b_{10}$$

$$S_n = b_1 + b_2 + b_3 - b_{n+1} - b_{n+2} - b_{n+3}$$

$$\lim s_n = b_1 + b_2 + b_3 - \underbrace{\lim b_{n+1}}_{= \lim b_n} - \underbrace{\lim b_{n+2}}_{= \lim b_n} - \underbrace{\lim b_{n+3}}_{= \lim b_n}$$

$$= b_1 + b_2 + b_3 - 3 \lim_{n \to \infty} b_n$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+3}\right) = b_1 + b_2 + b_3 - 3\underbrace{\lim b_n}_{=0 \text{ porque } b_n = \frac{1}{n}}$$

$$= 1 + \frac{1}{2} + \frac{1}{3}$$

$$a_n = \frac{1}{n} - \frac{1}{n+3}$$

tomamos $b_n = \frac{1}{n}$

$$\Rightarrow a_n = b_n - b_{n+3}$$

1. Indicar si las siguientes series son convergentes o no, hallando sus suma en caso de serlo.

a) $\sum_{n=0}^{+\infty}\left(\frac{1}{3}\right)^n$ b) $\sum_{n=1}^{+\infty}\left(\frac{1}{\sqrt{3}}\right)^{n+3}$ c) $\sum_{n=1}^{+\infty}5^{n+1}$ d) $\sum_{n=1}^{+\infty}\frac{3}{n(n+3)}$

e) $\sum_{n=1}^{+\infty}\log\left(\frac{n^2+2n+1}{n^2}\right)$ f) $\sum_{n=1}^{+\infty}\frac{n}{(n+1)(n+2)(n+3)}$ g) $\sum_{n=1}^{\infty}\frac{n \operatorname{arc\,tg}(n+1) - (n+1)\operatorname{arc\,tg}(n)}{n(n+1)}$

g) $\displaystyle\sum_{n=1}^{\infty}\frac{n\,\mathrm{arctg}(n+1) - (n+1)\,\mathrm{arctg}(n)}{n(n+1)}$

$$\frac{n\,\mathrm{arctg}(n+1) - (n+1)\,\mathrm{arctg}(n)}{n(n+1)} = \frac{\cancel{n}\,\mathrm{arctg}(n+1)}{\cancel{n}(n+1)} - \frac{\cancel{(n+1)}\,\mathrm{arctg}(n)}{n\cancel{(n+1)}}$$

$$= \frac{\mathrm{arctg}(n+1)}{n+1} - \frac{\mathrm{arctg}(n)}{n} = b_n$$

$$= b_{n+1} - b_n$$

$$\sum_{n=1}^{\infty}(b_{n+1}-b_n) = (\cancel{b_2}-b_1) + (\cancel{b_3}-\cancel{b_2}) + (\cancel{b_4}-\cancel{b_3}) + (b_5-\cancel{b_4}) + \dots$$

$$S_4 = -b_1 + b_5$$

$$S_5 = -b_1 + b_6$$

$$S_n = -b_1 + b_{n+1}$$

$$\lim_{n\to\infty} S_n = -b_1 + \lim_{n\to\infty} b_{n+1}$$

$$= -b_1 + \lim_{n\to\infty} b_n$$

$$\sum \frac{n\operatorname{arctg}(n+1) - (n+1)\operatorname{arctg}(n)}{n(n+1)} = \sum \left(\overbrace{\frac{\operatorname{arctg}(n+1)}{n+1}}^{b_{n+1}} - \overbrace{\frac{\operatorname{arctg}(n)}{n}}^{b_n} \right)$$

$$= \sum (b_{n+1} - b_n)$$

$$= -b_1 + \lim b_n$$

$$= -\frac{\operatorname{arctg}(1)}{1} + \lim_{n\to\infty} \frac{\operatorname{arctg}(n)}{n}$$

$$= -\frac{\pi}{4}$$

e) $\displaystyle\sum_{n=1}^{\infty} \log\left(\frac{n^2+2n+1}{n^2}\right)$

$$\log\left(\frac{n^2+2n+1}{n^2}\right) = \log\left(\frac{(n+1)^2}{n^2}\right)$$

$$= \log((n+1)^2) - \log(n^2)$$

$$= 2\log(n+1) - 2\log(n)$$

$$\sum_{n=1}^{\infty} \log\left(\frac{n^2+2n+1}{n^2}\right) = \sum_{n=1}^{\infty} 2\log(n+1) - 2\log(n)$$

$$= 2\sum_{n=1}^{\infty} (\log(n+1) - \log(n))$$

vamos a estudiar $\sum \log(n+1) - \log(n)$

$$\sum_{n=1}^{\infty} \log(n+1) - \log(n) = \sum_{n=1}^{\infty} b_{n+1} - b_n = -b_1 + \lim_{n\to\infty} b_n$$

con $b_n = \log(n)$

$$= -\log(1) + \overbrace{\lim_{n\to\infty} \log(n)}^{=\infty}$$

entonces $\sum_{n=1}^{\infty} \log(n+1) - \log(n)$ diverge

$$\Rightarrow \sum_{n=1}^{\infty} \log\left(\frac{n^2+2n+1}{n^2}\right) \text{ diverge}$$

$$\lim \log\left(\frac{n^2+2n+1}{n^2}\right) = 0$$

<u>Condición necesaria de convergencia</u>

$$\sum a_n \text{ converge} \Rightarrow \lim a_n = 0$$

no es condición suficiente

esta condición sirve para probar que una serie no converge

$$\lim a_n \neq 0 \Rightarrow \sum a_n \text{ no converge}$$

a_n termino general de la serie

$$\sum a_n = \lim s_n$$

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = \sum_{i=1}^{n} a_i$$