Numeros complejos

Un numero complejo es un par ordenado de numeros reales

$$z = \underbrace{(a}_{\text{parte real de } z},\ \underbrace{b)}_{\text{parte imaginaria de } z} \in \mathbb{C}$$

* suma: $(a,b) + (c,d) = (a+c,\ b+d)$
* producto por un real: $\underset{\mathbb{R}}{\lambda}(a,b) = (\lambda a, \lambda b)$
* producto de complejos: $(a,b)\cdot(c,d) = (ac - bd,\ ad + bc)$

Notación binómica

$$z = (a,b) = (a,0) + (0,b) = a\underbrace{(1,0)}_{1} + b\underbrace{(0,1)}_{i} = \underset{\text{parte real}}{a} + \underset{\text{parte imaginaria}}{b}i$$

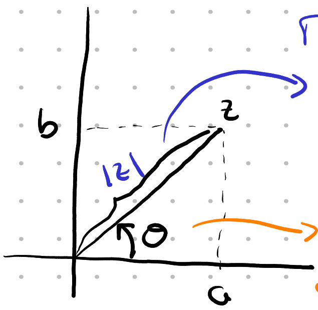
$$i^2 = (0,1)\cdot(0,1) = (-1,0) = -1 \leadsto i^2 = -1$$

$$(a+ib)(c+id) = ac + iad + ibc - bd = ac - bd + i(ad+bc)$$

Notación polar

$z \in \mathbb{C}$

$z = (a,b) = a + ib$



el modulo de z

longitud de este segmento es: $\sqrt{a^2+b^2} = |z|$

angulo θ: $\tan\theta = \dfrac{b}{a}$

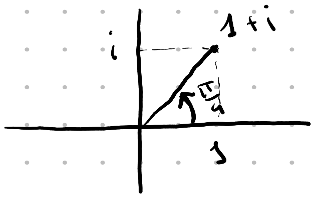
argumento de z

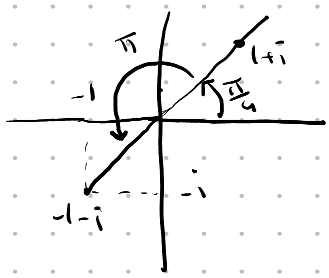
$\arg(z)$

$\theta = \arctan\left(\dfrac{b}{a}\right) \Leftarrow$ cuidado con esta fórmula!

modulo de z : $|z| = \sqrt{a^2 + b^2}$

argumento de z : $\arg(z) = \arctan\left(\frac{b}{a}\right)$ ← cuidado!





$z_1 = 1+i$

$|z_1| = \sqrt{1^2+1^2} = \sqrt{2}$

$\arg(z_1) = \arctan\left(\frac{1}{1}\right)$

$= \arctan(1)$

$= \frac{\pi}{4}$

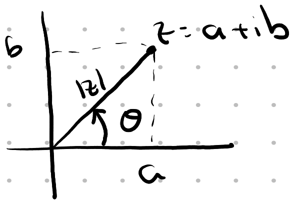
$z_2 = -1-i$

$|z_2| = \sqrt{2}$

$\arg(z_2) = \arctan\left(\frac{-1}{-1}\right) = \arctan(1)$

$\arg(z_2) = \frac{\pi}{4} + \pi$

relación con la notación binómica



$a = |z|\cos(\theta)$

$b = |z|\operatorname{sen}(\theta)$

$a + ib = |z|\cos(\theta) + i|z|\operatorname{sen}(\theta)$

$= |z|(\cos\theta + i\operatorname{sen}(\theta))$

$= |z|e^{i\theta}$ ← notación polar de z

exponencial compleja:

$$e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\operatorname{sen}(y))$$

$z_1 = |z_1| e^{i\theta_1} \qquad \arg(z_1) = \theta_1$

$z_2 = |z_2| e^{i\theta_2} \qquad \arg(z_2) = \theta_2$

$$z_1 z_2 = |z_1| e^{i\theta_1} \, |z_2| e^{i\theta_2}$$

$$= |z_1||z_2| \, e^{i\theta_1} e^{i\theta_2}$$

$$= |z_1||z_2| \, e^{i\theta_1 + i\theta_2}$$

$$= |z_1||z_2| \, e^{i\underbrace{(\theta_1 + \theta_2)}_{\arg(z_1 z_2)}}$$

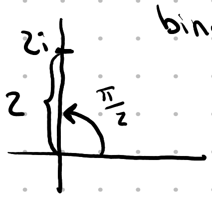
$|z_1 z_2| = |z_1||z_2|$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Ejercicio 2 expresar en notación binómica y polar

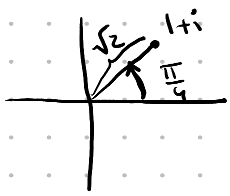
a) $(1+i)^2 = (1+i)(1+i) = 1 + i + i - 1 = 2i = 2e^{i\frac{\pi}{2}}$

$\uparrow$ binómica $\qquad \uparrow$ polar



otra forma

$(1+i)^2 = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^2 = 2\left(e^{i\frac{\pi}{4}}\right)^2 = 2e^{i\frac{\pi}{4}\cdot 2} = 2e^{i\frac{\pi}{2}}$ ← notación polar



$|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$

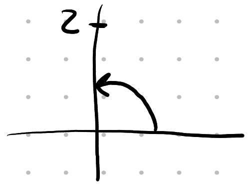
$\mathrm{Re}(1+i) = 1$

$\mathrm{Im}(1+i) = 1$

$|a+bi| = \sqrt{a^2 + b^2}$

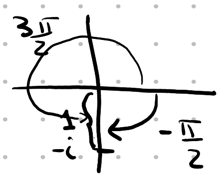
$(x^n)^m = x^{nm}$

$(1+i)^2 = 2e^{i\frac{\pi}{2}} = 2i$



$$2e^{i\frac{\pi}{2}} = 2(\underbrace{\cos(\tfrac{\pi}{2})}_{0} + i\underbrace{\text{sen}(\tfrac{\pi}{2})}_{1}) = 2i$$

b) $\frac{1}{i} \cdot \frac{i}{i} = \frac{i}{-1} = -i$ ← notación binómica



$-i = 1e^{-i\frac{\pi}{2}}$ ← notación polar

c) $\frac{1}{1+i} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$ ← notación binómica

↑ multiplicamos y dividimos por el conjugado de $1+i$

$z = a+bi$

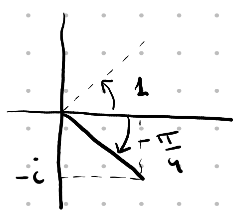
el conjugado de z es: $\overline{z} = a-bi$

$(1+i)(1-i) = 1^2 - i^2 = 1-(-1) = 2$

$\frac{1}{2} - \frac{1}{2}i = \frac{1}{2}(1-i) = \frac{1}{2}\sqrt{2}\, e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{4}}$

$= \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}} e^{-i\frac{\pi}{4}}$

$= \frac{1}{\sqrt{2}} e^{-i\frac{\pi}{4}}$

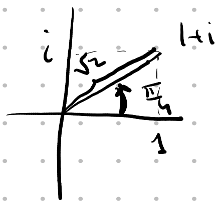


$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\arg(1-i) = -\frac{\pi}{4}$

Ejercicio 3 expresar en notación binómica

$$d)\ (1+i)^{100} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{100}$$



$$= \left(\sqrt{2}\right)^{100} \left(e^{i\frac{\pi}{4}}\right)^{100}$$

$$= \left(\left(\sqrt{2}\right)^{2}\right)^{50} e^{i\frac{\pi}{4}\cdot 100}$$

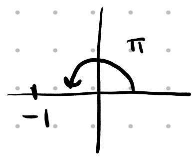
$$1+i = \sqrt{2}\, e^{i\frac{\pi}{4}} \qquad = 2^{50}\, e^{i\,25\pi}$$

$$= 2^{50}\, e^{i(24\pi + \pi)}$$

$$= 2^{50}\, e^{i\,24\pi}\, e^{i\pi}$$

$$= 2^{50}\, e^{i\cdot 0}\, e^{i\pi}$$

$$= 2^{50} \underbrace{e^{i\pi}}_{=-1}$$



$$= -2^{50}$$

$1e^{i\pi}$ → un complejo con modulo 1 y argumento π

argumento

modulo