Physics-Informed Neural Networks: A Deep Learning Framework for Solving Forward and Inverse Problems Involving Nonlinear Partial Differential Equations

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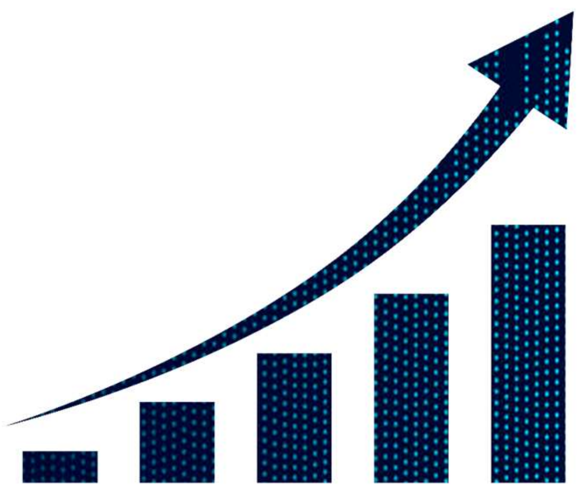
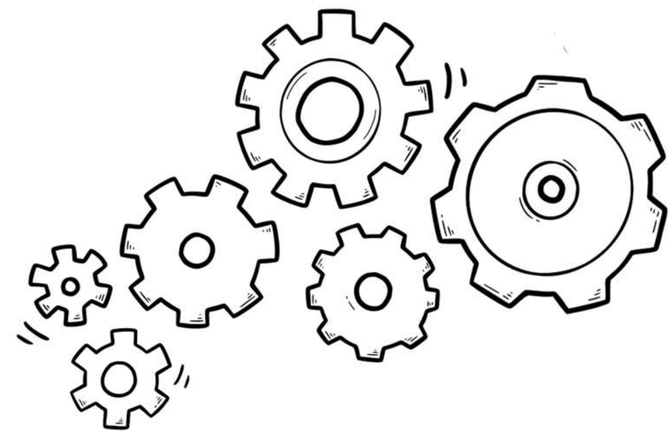
- Introducción
- Planteo del problema
- Solución a EDP en tiempo continuo
- Solución a EDP en tiempo discreto
- Descubrimiento de parámetros de EDP en tiempo continuo

Introducción

Redes neuronales impulsadas por el crecimiento de la cantidad de datos disponibles y la capacidad computacional.





Introducción

Muchos datos de entrenamiento en varios campos excepto en física e ingeniería.

Otros campos

Física e ingeniería

Introducción

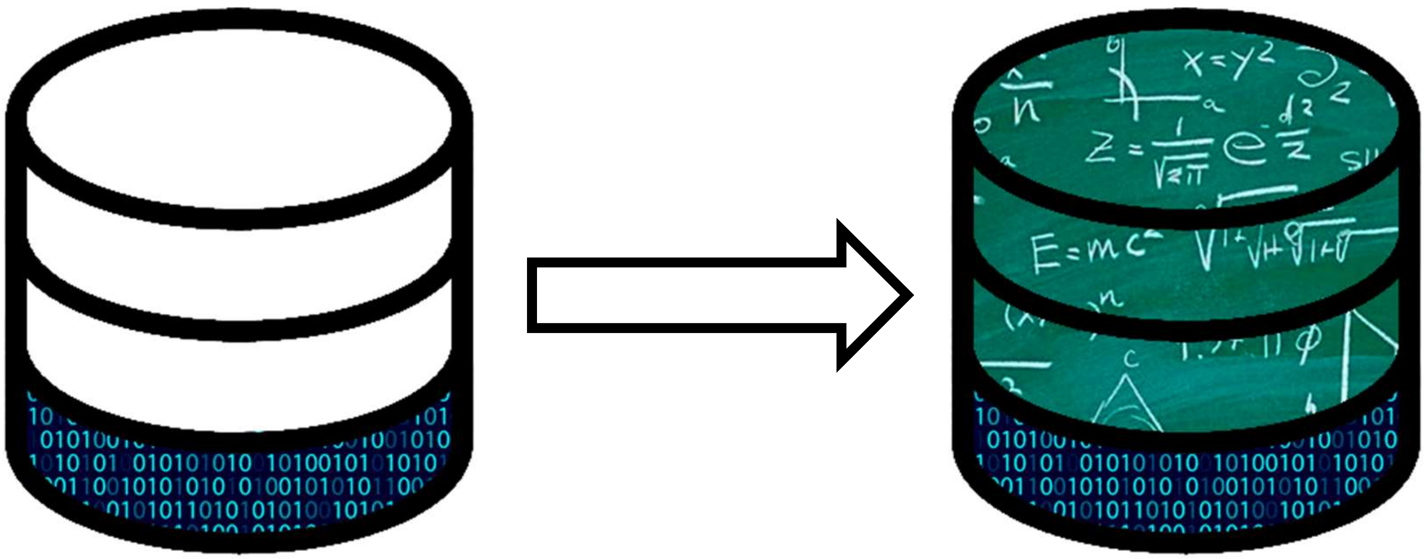
- Las soluciones estaban basadas en pocos datos y no se aprovechaba todo el conocimiento teórico adquirido hasta el momento.

- Al no utilizar esta información las soluciones pueden converger en algo que podría violar las leyes físicas.

Introducción

"…en esta casa obedecemos las leyes de la termodinámica."

Introducción



Introducción

- La física actúa como una regularización que limita el espacio de soluciones admisibles.

- Este nuevo paradigma se conoce como Redes Neuronales Informadas por Física.

Planteo del problema

Redes Neuronales Informadas por Física:

1. Solución a ecuaciones diferenciales en derivadas parciales basadas en datos.

2. Descubrimiento de parámetros de ecuaciones diferenciales en derivadas parciales basadas en datos.

Planteo del problema

Redes Neuronales Informadas por Física:

1. -
$$u_t + \mathcal{N}[u] = 0,\ x \in \Omega,\ t \in [0, T],$$

2. -
$$u_t + \mathcal{N}[u; \lambda] = 0,\ x \in \Omega,\ t \in [0, T],$$

Planteo del problema

Redes Neuronales Informadas por Física:

1. -

$$u_t - 0.0001u_{xx} + 5u^3 - 5u = 0$$

2. -

$$u_t + \lambda_1 u u_x + \lambda_2 u_{xxx} = 0$$

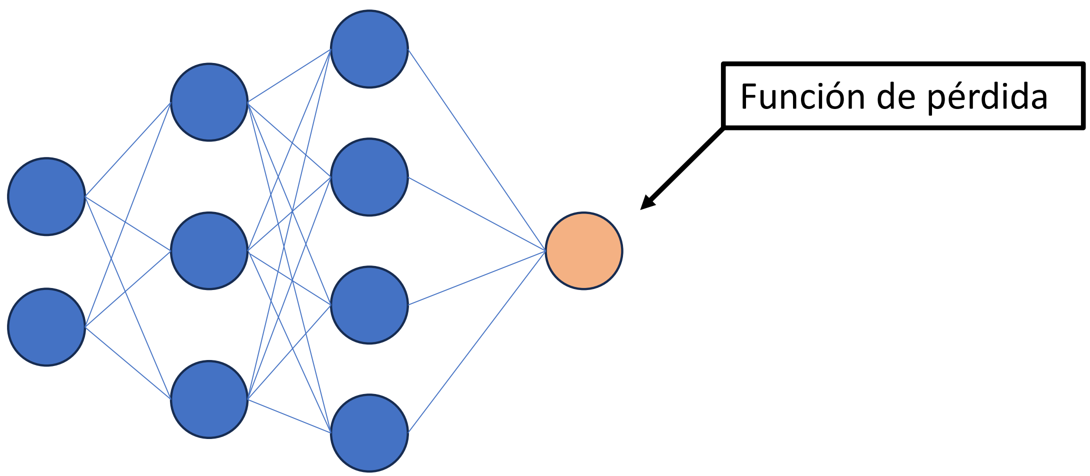
Solución a EDP en tiempo continuo

Ecuación diferencial en derivadas parciales:

$$u_t + \mathcal{N}[u] = 0,\ x \in \Omega,\ t \in [0, T],$$

Solución a EDP en tiempo continuo

Red neuronal profunda:



Solución a EDP en tiempo continuo

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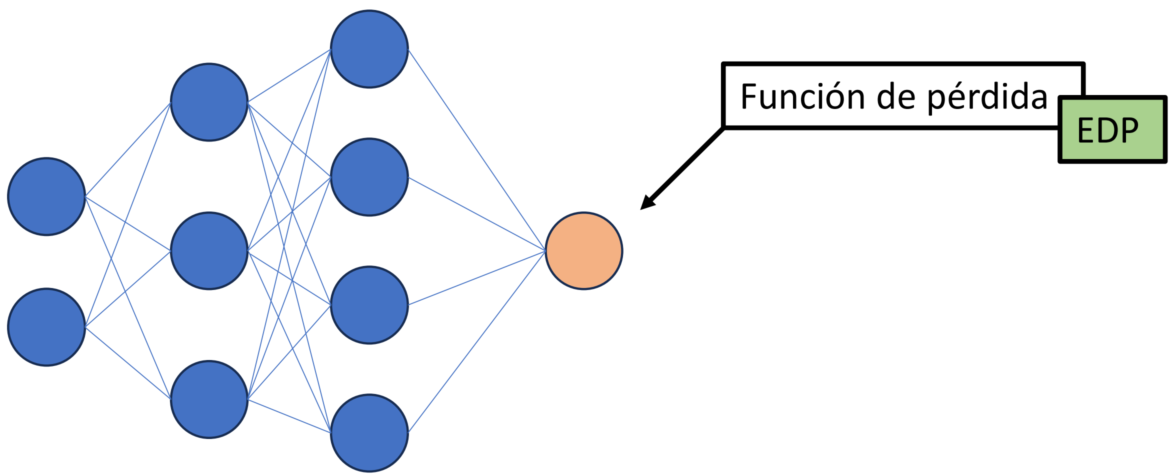

Solución a EDP en tiempo continuo

Ecuación diferencial en derivadas parciales:

$$f := u_t + \mathcal{N}[u],$$

Solución a EDP en tiempo continuo

Función de pérdida:

$$MSE = MSE_u + MSE_f,$$

Solución a EDP en tiempo continuo

Función de pérdida:

$$MSE = MSE_u + MSE_f,$$

$$MSE_u = \frac{1}{N_u}\sum_{i=1}^{N_u}|u(t_u^i, x_u^i) - u^i|^2,$$

$$MSE_f = \frac{1}{N_f}\sum_{i=1}^{N_f}|f(t_f^i, x_f^i)|^2.$$

Solución a EDP en tiempo continuo

Ejemplo:

Ecuación de Burgers con las condiciones de borde de Dirichlet

$$
\begin{aligned}
&u_t + uu_x - (0.01/\pi)u_{xx} = 0, \quad x \in [-1, 1], \quad t \in [0, 1], \\
&u(0, x) = -\sin(\pi x), \\
&u(t, -1) = u(t, 1) = 0.
\end{aligned}
$$

Solución a EDP en tiempo continuo

Ejemplo:

Ecuación de Burgers con las condiciones de borde de Dirichlet

$$MSE = MSE_u + MSE_f,$$

$$MSE_u = \frac{1}{N_u}\sum_{i=1}^{N_u} |u(t_u^i, x_u^i) - u^i|^2, \qquad MSE_f = \frac{1}{N_f}\sum_{i=1}^{N_f} |f(t_f^i, x_f^i)|^2.$$

Solución a EDP en tiempo continuo

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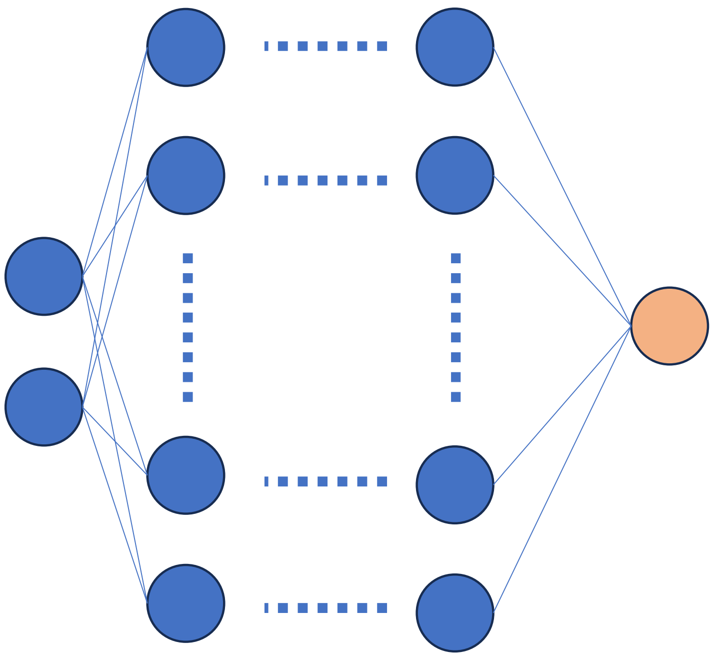
Solución a EDP en tiempo continuo

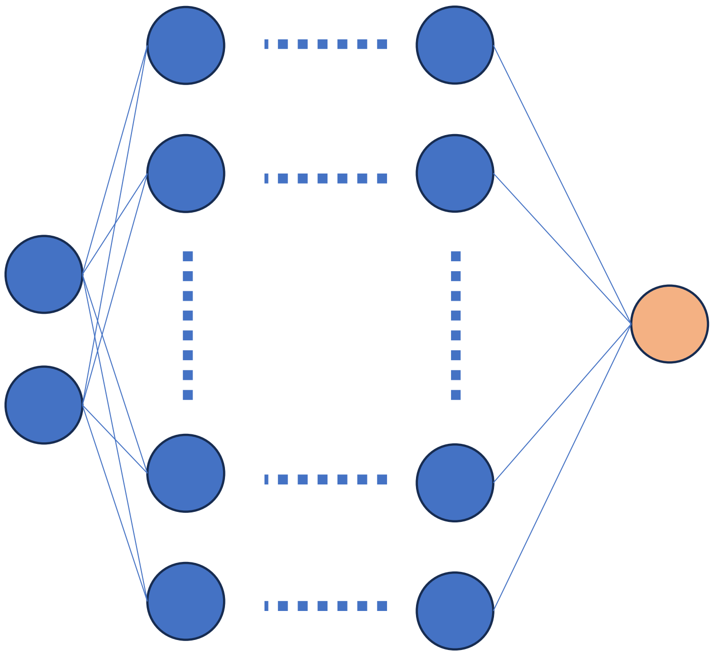
Configuración de la red neuronal:



- 2 entradas t , x

Solución a EDP en tiempo continuo

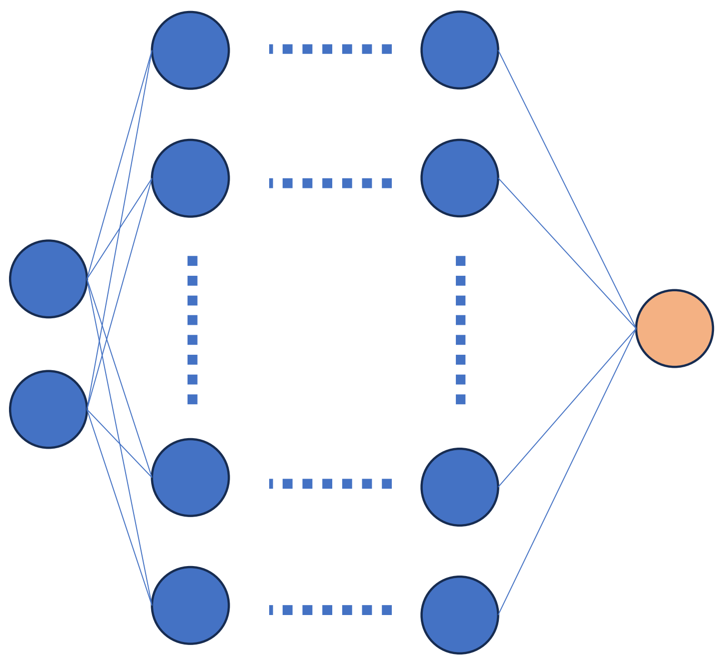
Configuración de la red neuronal:



- 2 entradas t , x
- 1 salida $u(t,x)$

Solución a EDP en tiempo continuo

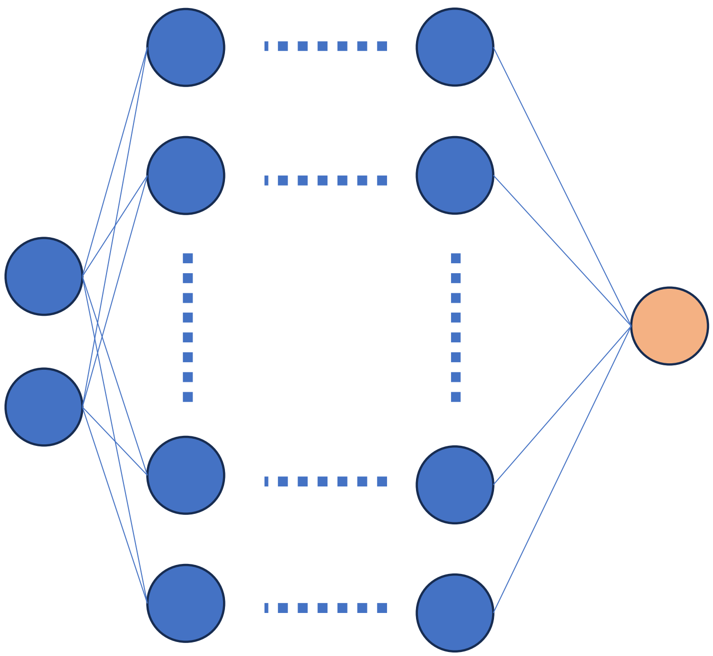
Configuración de la red neuronal:



- 2 entradas t , x
- 1 salida $u(t,x)$
- 8 capas ocultas

Solución a EDP en tiempo continuo

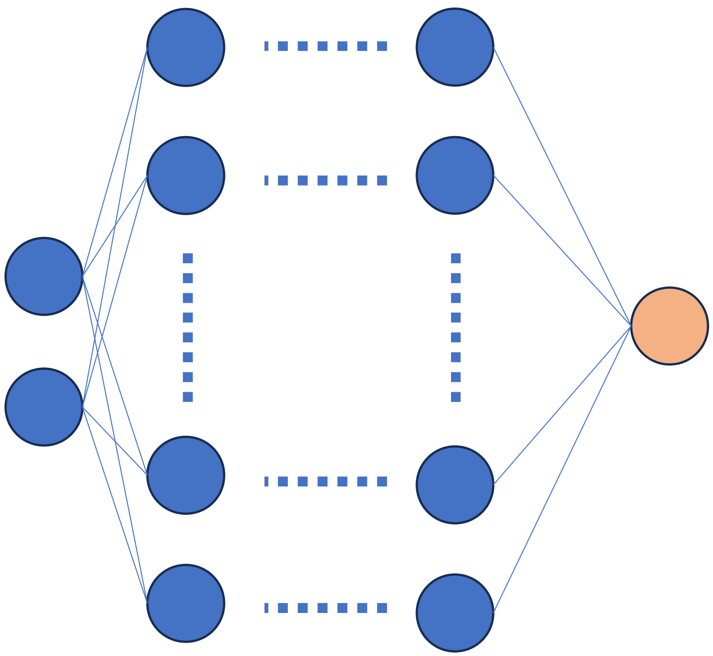
Configuración de la red neuronal:



- 2 entradas *t* , *x*
- 1 salida *u(t,x)*
- 8 capas ocultas
- 20 neuronas cada capa oculta

Solución a EDP en tiempo continuo

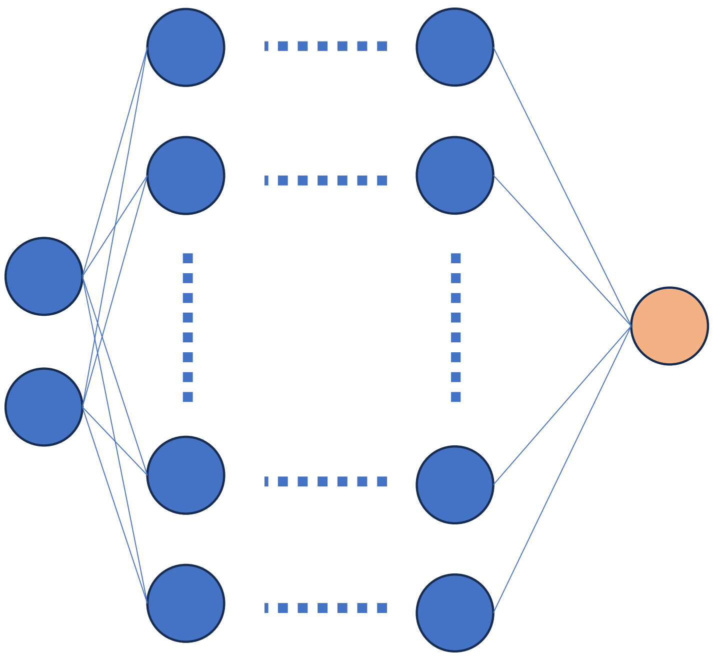
Configuración de la red neuronal:



- 2 entradas t , x
- 1 salida $u(t,x)$
- 8 capas ocultas
- 20 neuronas cada capa oculta
- tangente hiperbólica

Solución a EDP en tiempo continuo

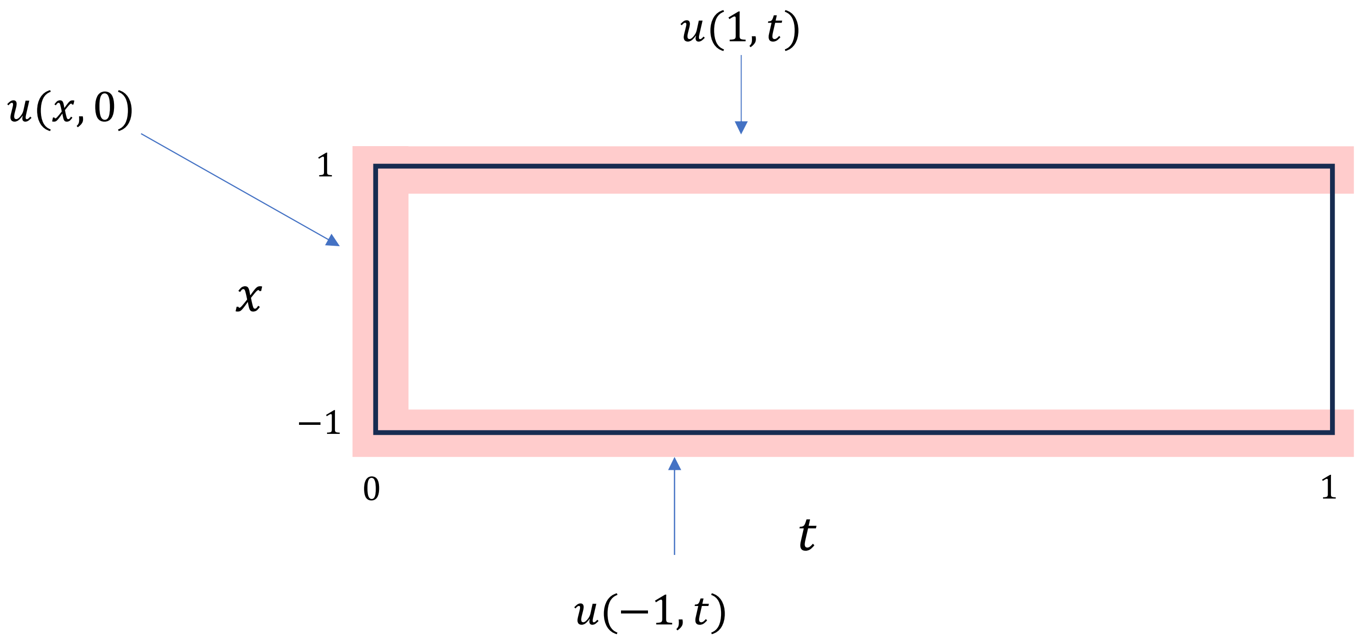
Configuración de la red neuronal:



- 2 entradas *t* , *x*
- 1 salida *u(t,x)*
- 8 capas ocultas
- 20 neuronas cada capa oculta
- tangente hiperbólica
- L-BFGS

Solución a EDP en tiempo continuo

Condiciones de borde:

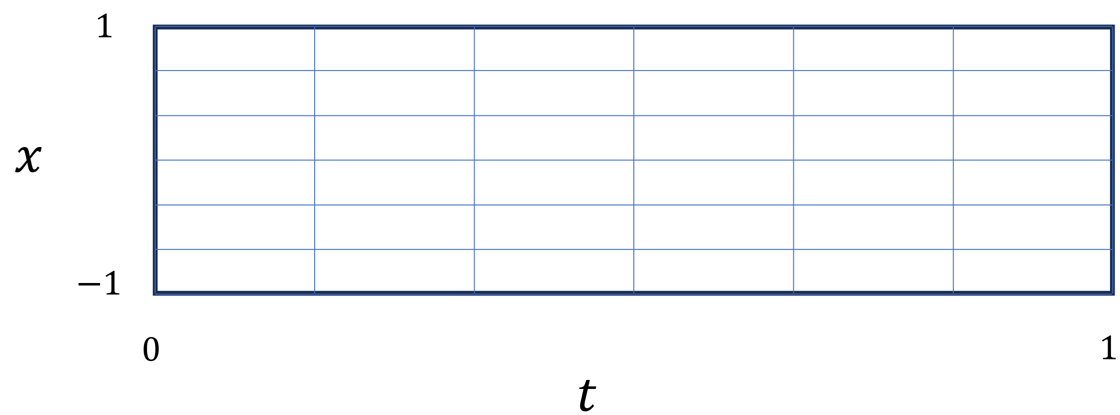


100 puntos al azar

Solución a EDP en tiempo continuo

Puntos de colocación:

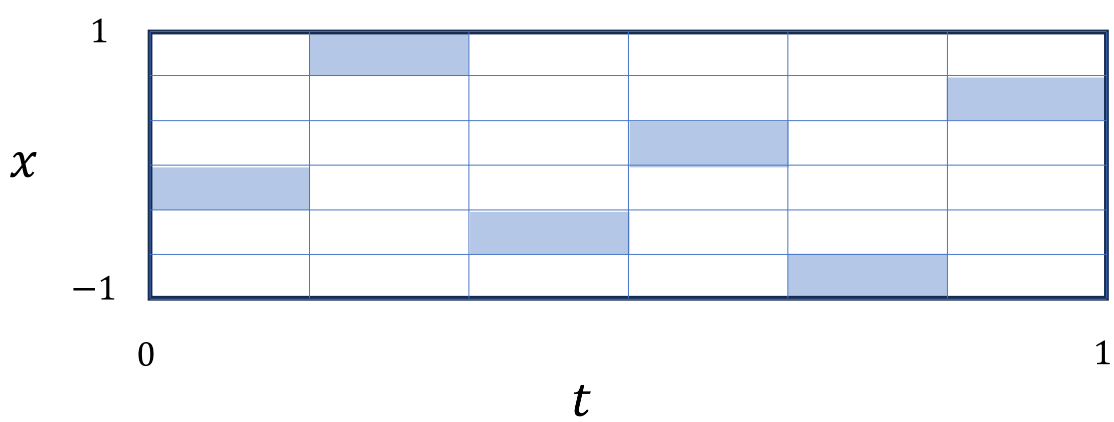
Latin hypercube sampling



Solución a EDP en tiempo continuo

Configuración de la red neuronal:

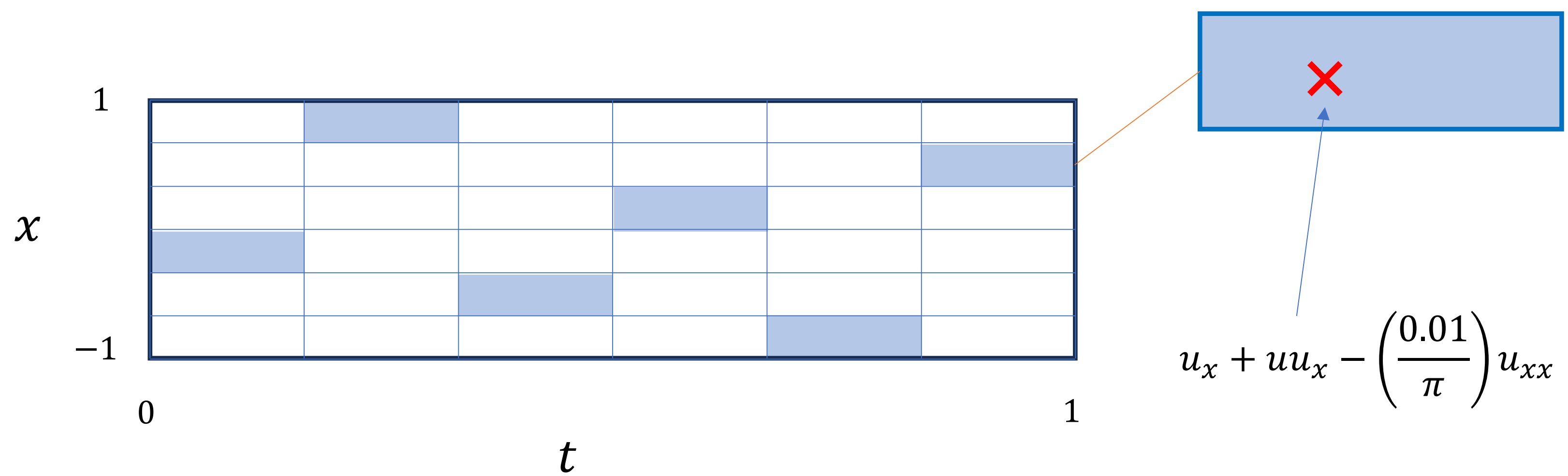
Latin hypercube sampling



Solución a EDP en tiempo continuo

Configuración de la red neuronal:

Latin hypercube sampling

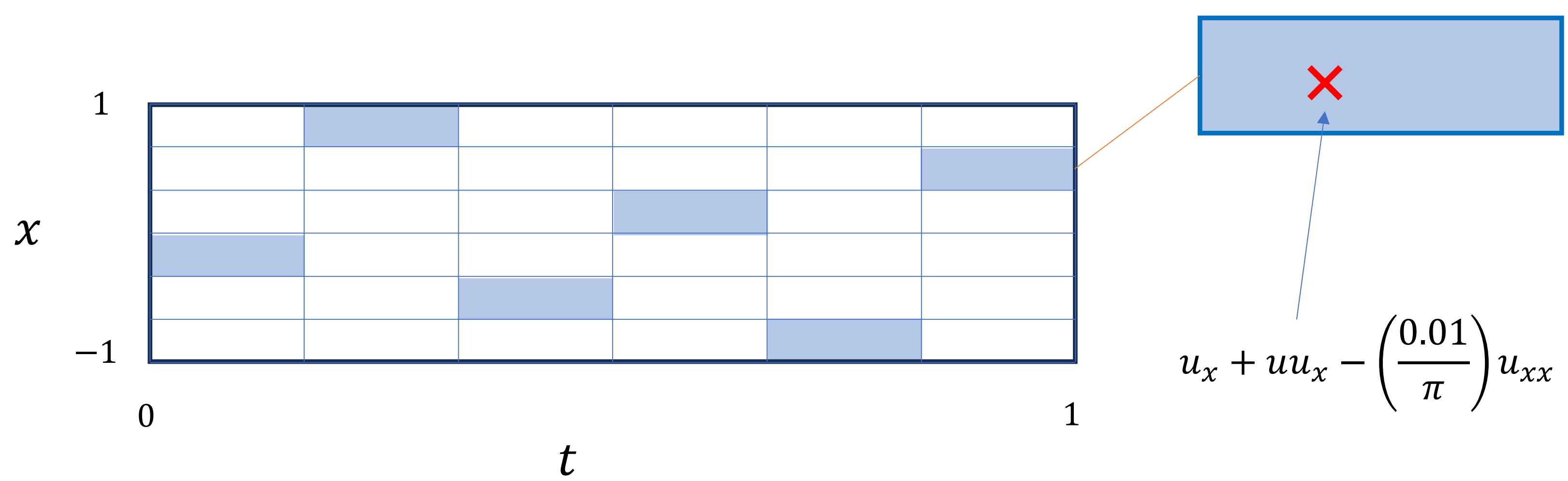


Solución a EDP en tiempo continuo

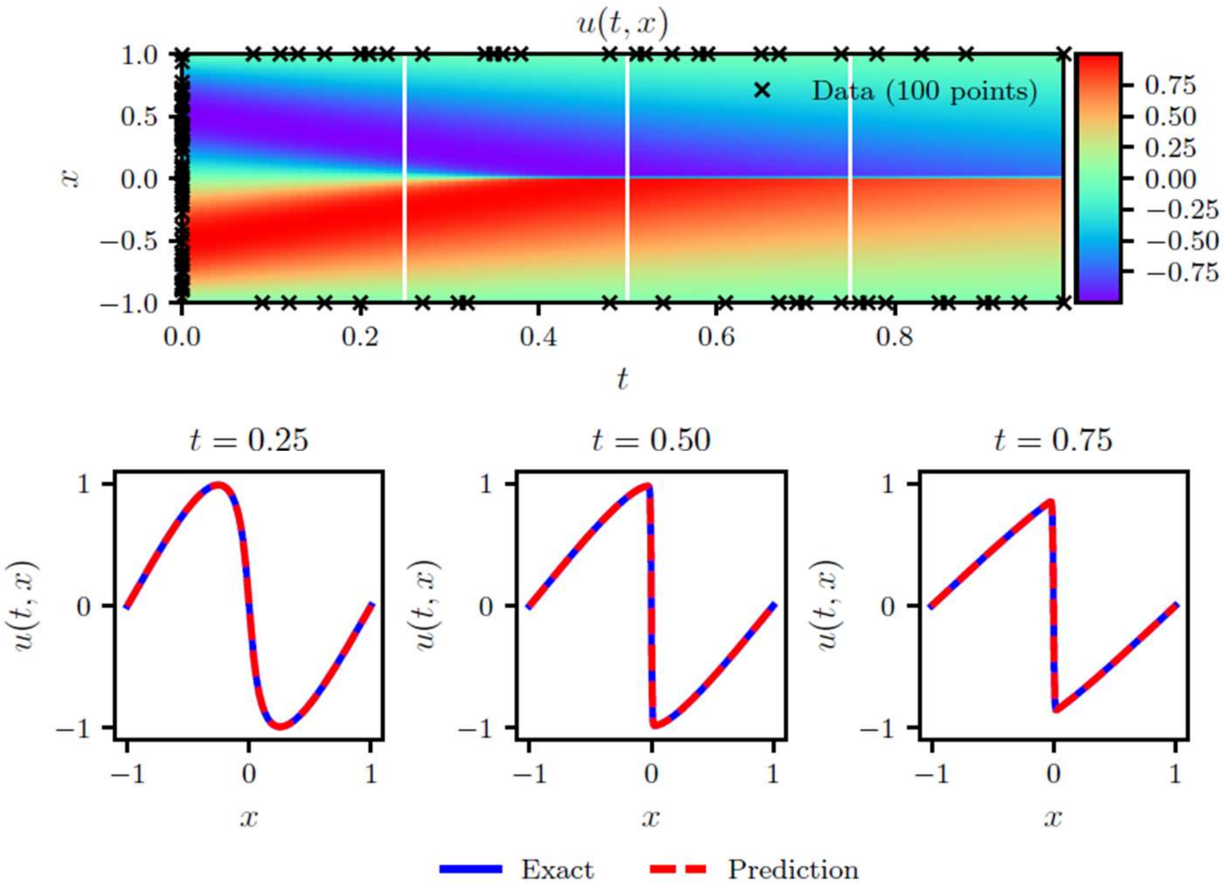
Configuración de la red neuronal:

Latin hypercube sampling

10.000 puntos



Solución a EDP en tiempo continuo



Solución a EDP en tiempo discreto

Runge-Kutta

Familia de métodos iterativos para aproximar la solución a la EDP en tiempo discreto.

Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

$$u^{n+1} = u^n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$t^{n+1} = t^n + h$$

$$k_1 = f(t^n, u^n)$$
$$k_2 = f\left(t^n + \frac{h}{2}, u^n + h\frac{k_1}{2}\right)$$
$$k_3 = f\left(t^n + \frac{h}{2}, u^n + h\frac{k_2}{2}\right)$$
$$k_4 = f(t^n + h, u^n + hk_3)$$

Solución a EDP en tiempo discreto

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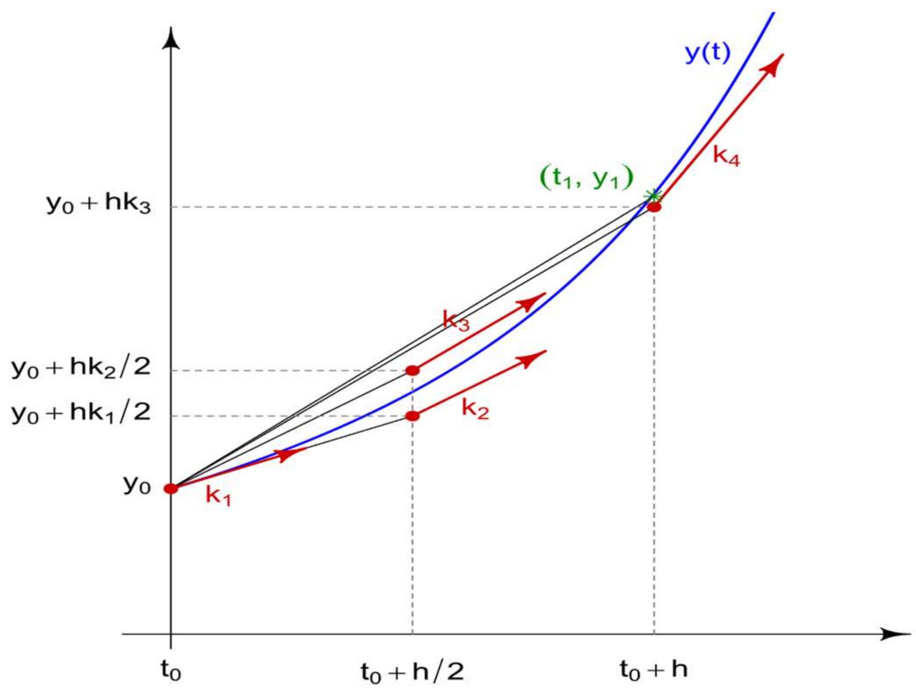

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Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

$$u^{n+1} = u^n + h\left(\frac{1}{6}k_1 + \frac{2}{6}k_2 + \frac{2}{6}k_3 + \frac{1}{6}k_4\right)$$

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Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

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$$t^{n+1} = t^n + \boxed{h}$$

$$k_1 = f(t^n, u^n)$$

$$k_2 = f\left(t^n + \frac{\boxed{h}}{2}, u^n + \boxed{h}\frac{k_1}{2}\right)$$

$$k_3 = f\left(t^n + \frac{\boxed{h}}{2}, u^n + \boxed{h}\frac{k_2}{2}\right)$$

$$k_4 = f(t^n + \boxed{h}, u^n + \boxed{h}k_3)$$

Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

$$u^{n+1} = u^n + \Delta t\left(\frac{1}{6}k_1 + \frac{2}{6}k_2 + \frac{2}{6}k_3 + \frac{1}{6}k_4\right)$$

$$t^{n+1} = t^n + \Delta t$$

$$k_1 = f(t^n, u^n)$$

$$k_2 = f\left(t^n + \frac{\Delta t}{2}, u^n + \Delta t\frac{k_1}{2}\right)$$

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Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

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Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

$$u^{n+1} = u^n + \Delta t\left(\frac{1}{6}O(u^{t+c_1}) + \frac{2}{6}O(u^{t+c_2}) + \frac{2}{6}O(u^{t+c_3}) + \frac{1}{6}O(u^{t+4})\right)$$

$$t^{n+1} = t^n + \Delta t$$

$$O(u^{t+c_1}) = f(t^n, u^n)$$

$$O(u^{t+c_2}) = f\left(t^n + \frac{\Delta t}{2}, u^n + \frac{\Delta t}{2}O(u^{t+c_1})\right)$$

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$$O(u^{t+c_4}) = f(t^n + c_4\Delta t, u^n + \Delta t\ a_{43}O(u^{t+c_3}))$$

Solución a EDP en tiempo discreto

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Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

$$u^{n+1} = u^n + \Delta t \left(\boxed{\frac{1}{6}} O(u^{t+c_1}) + \boxed{\frac{2}{6}} O(u^{t+c_2}) + \boxed{\frac{2}{6}} O(u^{t+c_3}) + \boxed{\frac{1}{6}} O(u^{t+4}) \right)$$

$$O(u^{t+c_1}) = f(t^n + c_1 \Delta t, u^n)$$
$$O(u^{t+c_2}) = f(t^n + c_2 \Delta t, u^n + \Delta t \; a_{21} O(u^{t+c_1}))$$
$$O(u^{t+c_3}) = f(t^n + c_3 \Delta t, u^n + \Delta t \; a_{32} O(u^{t+c_2}))$$
$$O(u^{t+c_4}) = f(t^n + c_4 \Delta t, u^n + \Delta t \; a_{43} O(u^{t+c_3}))$$

Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

$$u^{n+1} = u^n + \Delta t(b_1 O(u^{t+c_1}) + b_2 O(u^{t+c_2}) + b_3 O(u^{t+c_3}) + b_4 O(u^{t+4}))$$

$$O(u^{t+c_1}) = f(t^n + c_1 \Delta t, u^n)$$
$$O(u^{t+c_2}) = f(t^n + c_2 \Delta t, u^n + \Delta t\ a_{21} O(u^{t+c_1}))$$
$$O(u^{t+c_3}) = f(t^n + c_3 \Delta t, u^n + \Delta t\ a_{32} O(u^{t+c_2}))$$
$$O(u^{t+c_4}) = f(t^n + c_4 \Delta t, u^n + \Delta t\ a_{43} O(u^{t+c_3}))$$

Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:

$$\boxed{u^{n+1} = u^n + \Delta t\left(b_1 O\left(u^{t+c_1}\right) + b_2 O\left(u^{t+c_2}\right) + b_3 O\left(u^{t+c_3}\right) + b_4 O\left(u^{t+4}\right)\right)}$$

$$O\left(u^{t+c_1}\right) = f\left(t^n + c_1 \Delta t, u^n\right)$$
$$O\left(u^{t+c_2}\right) = f\left(t^n + c_2 \Delta t, u^n + \Delta t\ a_{21} O\left(u^{t+c_1}\right)\right)$$
$$O\left(u^{t+c_3}\right) = f\left(t^n + c_3 \Delta t, u^n + \Delta t\ a_{32} O\left(u^{t+c_2}\right)\right)$$
$$O\left(u^{t+c_4}\right) = f\left(t^n + c_4 \Delta t, u^n + \Delta t\ a_{43} O\left(u^{t+c_3}\right)\right)$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$\begin{gathered} O(u^{t+c_1}) = f(t^n + c_1 \Delta t, u^n) \\ O(u^{t+c_2}) = f(t^n + c_2 \Delta t, u^n + \Delta t \; a_{21} O(u^{t+c_1})) \\ O(u^{t+c_3}) = f(t^n + c_3 \Delta t, u^n + \Delta t \; a_{32} O(u^{t+c_2})) \\ O(u^{t+c_4}) = f(t^n + c_4 \Delta t, u^n + \Delta t \; a_{43} O(u^{t+c_3})) \end{gathered}$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$O(u^{t+c_1}) = f(t^n + c_1 \Delta t, u^n)$$
$$\boxed{O(u^{t+c_2}) = f(t^n + c_2 \Delta t, u^n + \Delta t \; a_{21} O(u^{t+c_1}))}$$
$$O(u^{t+c_3}) = f(t^n + c_3 \Delta t, u^n + \Delta t \; a_{32} O(u^{t+c_2}))$$
$$O(u^{t+c_4}) = f(t^n + c_4 \Delta t, u^n + \Delta t \; a_{43} O(u^{t+c_3}))$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$O(u^{t+c_2}) = f(t^n + c_2 \Delta t, u^n + \Delta t \; a_{21} O(u^{t+c_1}))$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$O(\boxed{u^{t+c_2}}) = f(t^n + c_2 \Delta t, u^n + \Delta t \; a_{21} O(u^{t+c_1}))$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^{n} + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$O(\boxed{u^{t+c_2}}) = f(t^{n} + c_2 \Delta t, \boxed{u^{n} + \Delta t \; a_{21} O(u^{t+c_1})})$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$O(u^{t+c_2}) = f(t^n + c_2 \Delta t, u^n + \Delta t \; a_{21} O(u^{t+c_1}))$$

$$u^{t+c_2} = u^n + \Delta t \; a_{21} O(u^{t+c_1})$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$\boxed{O(u^{t+c_2}) = f(t^n + c_2\Delta t, u^n + \Delta t \; a_{21} O(u^{t+c_1}))}$$

$$u^{t+c_2} = u^n + \Delta t \; a_{21} O(u^{t+c_1})$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j\, O(\, u^{t+c_j})$$

$$u^{t+c_2} = u^n + \Delta t\; a_{21} O(u^{t+c_1})$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j\, O(\, u^{t+c_j})$$

$$u^{t+c_2} = u^n + \Delta t\, \boxed{a_{21} O(u^{t+c_1})}$$

Método explícito depende de los estados anteriores

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u^{n+c_2} = u^n + \Delta t \sum_{j=1}^{q} a_{2j} \, O(\, u^{t+c_j})$$

Método implícito
depende de todos los estados

Solución a EDP en tiempo discreto

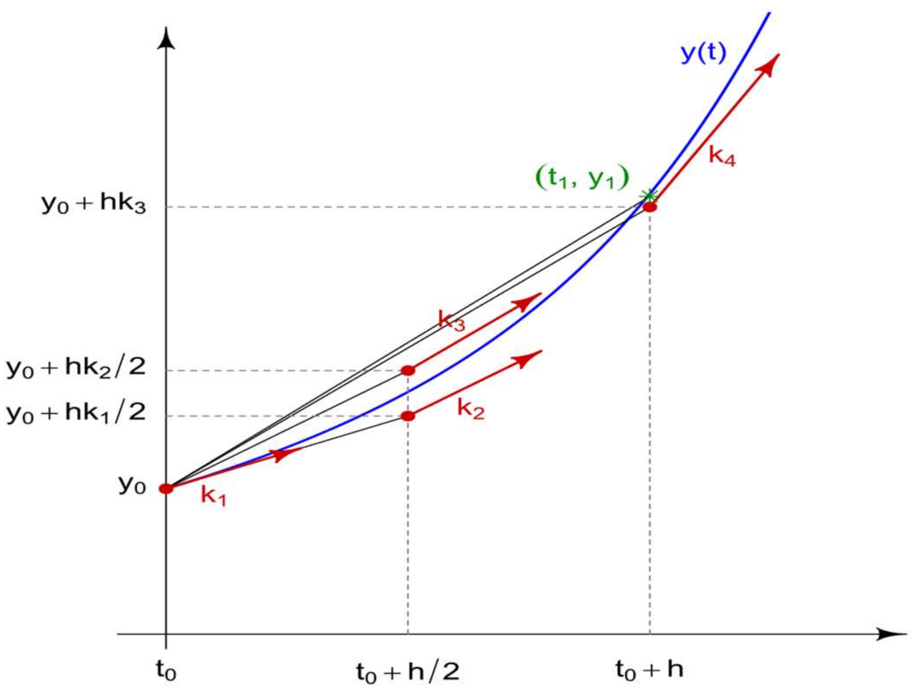
Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u^{n+c_i} = u^n + \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \ldots, q$$

Solución a EDP en tiempo discreto

Runge-Kutta de 4to orden:



Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u^{n+c_i} = u^n + \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n \boxed{+} \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u^{n+c_i} = u^n \boxed{+} \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n - \Delta t \sum_{j=1}^{q} b_j\, O(\, u^{t+c_j})$$

$$u^{n+c_i} = u^n - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n - \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u^{n+c_i} = u^n - \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

$$[u^{n+c_1}(x), \dots, u^{n+c_q}(x), u^{n+1}(x)]$$

Salida de la red

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^{n+1} = u^n - \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u^{n+c_i} = u^n - \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \ldots, q$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u^n = u^{n+1} + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u^n = u^{n+c_i} + \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u_{q+1}^{n} \coloneqq u^{n} = u^{n+1} + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u_{i}^{n} \coloneqq u^{n} = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

Solución a EDP en tiempo discreto

Runge-Kutta:

$$u_{q+1}^{n} \coloneqq u^{n} = u^{n+1} + \Delta t \sum_{j=1}^{q} b_j \, O(\, u^{t+c_j})$$

$$u_{i}^{n} \coloneqq u^{n} = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij} \, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

$$[u_1^n, \dots, u_q^n, u_{q+1}^n]$$

Solución a EDP en tiempo discreto

Función de pérdida:

$$SSE = SSE_n + SSE_b$$

Solución a EDP en tiempo discreto

Función de pérdida:

$$SSE = SSE_n + SSE_b$$

$$SSE_n = \sum_{j=1}^{q} \sum_{i=1}^{N_n} \left| u_j^n \left(x^{n,i} \right) - u^{n,i} \right|^2$$

SSE_b depende de las condiciones de borde

Solución a EDP en tiempo discreto

Ejemplo:

Allen-Cahn Equation

$$u_t - 0.0001u_{xx} + 5u^3 - 5u = 0, \qquad x \in [-1,1], \qquad t \in [0,1]$$
$$u(0,x) = x^2 \cos(\pi x)$$
$$u(t,-1) = u(t,1)$$
$$u_x(t,-1) = u_x(t,1)$$

Solución a EDP en tiempo discreto

Ejemplo:

Allen-Cahn Equation

$$O(u^{n+c_j}) = -0.0001u_{xx}^{n+c_j} + 5(u^{n+c_j})^3 - 5u^{n+c_j}$$

Solución a EDP en tiempo discreto

Función de pérdida:

$$SSE = SSE_n + SSE_b$$

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n\left(x^{n,i}\right) - u^{n,i} \right|^2$$

$$SSE_b = \sum_{i=1}^{q} |u^{n+c_i}(-1) - u^{n+c_i}(1)|^2 + |u^{n+1}(-1) - u^{n+1}(1)|$$
$$+ \sum_{i=1}^{q} \left| u_x^{n+c_i}(-1) - u_x^{n+c_i}(1) \right|^2 + |u_x^{n+1}(-1) - u_x^{n+1}(1)|$$

Solución a EDP en tiempo discreto

Función de pérdida:

$$SSE = SSE_n + SSE_b$$

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n\left(x^{n,i}\right) - u^{n,i} \right|^2$$

$$SSE_b = \sum_{i=1}^{q} |\boxed{u^{n+c_i}(-1)} - \boxed{u^{n+c_i}(1)}|^2 + |\boxed{u^{n+1}(-1)} - \boxed{u^{n+1}(1)}|$$

$$+ \sum_{i=1}^{q} \left| u_x^{n+c_i}(-1) - u_x^{n+c_i}(1) \right|^2 + |u_x^{n+1}(-1) - u_x^{n+1}(1)|$$

Solución a EDP en tiempo discreto

Función de pérdida:

$$SSE = SSE_n + SSE_b$$

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| \boxed{u_j^n} \left(x^{n,i} \right) - u^{n,i} \right|^2$$

$$SSE_b = \sum_{i=1}^{q} |u^{n+c_i}(-1) - u^{n+c_i}(1)|^2 + |u^{n+1}(-1) - u^{n+1}(1)|$$
$$+ \sum_{i=1}^{q} \left| u_x^{n+c_i}(-1) - u_x^{n+c_i}(1) \right|^2 + |u_x^{n+1}(-1) - u_x^{n+1}(1)|$$

Solución a EDP en tiempo discreto

Función de pérdida:

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n\left(x^{n,i}\right) - u^{n,i} \right|^2$$

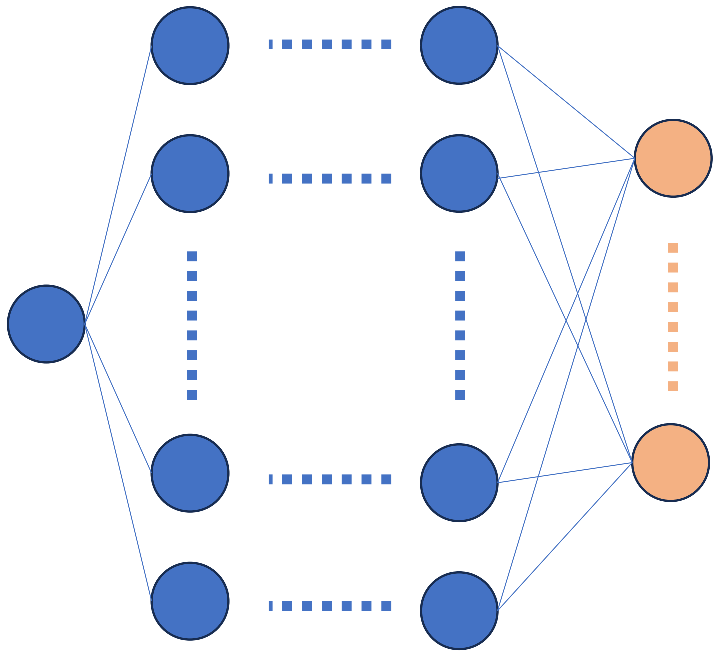
$$u_i^n \coloneqq u^n = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$

$$u_{q+1}^n \coloneqq \; u^n = u^{n+1} + \Delta t \sum_{j=1}^{q} b_j\, O(\, u^{t+c_j})$$

$$O(u^{n+c_j}) = -0.0001 u_{xx}^{n+c_j} + 5(u^{n+c_j})^3 - 5u^{n+c_j}$$

Solución a EDP en tiempo discreto

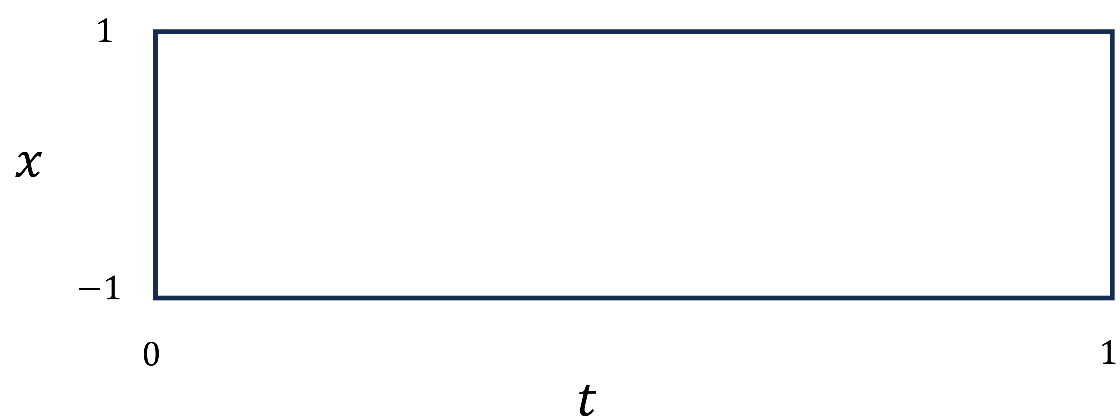
Configuración del problema:



- 1 entrada (x)
- q+1 salidas (estados de Runge-Kutta)
- 4 capas ocultas
- 200 neuronas cada capa oculta

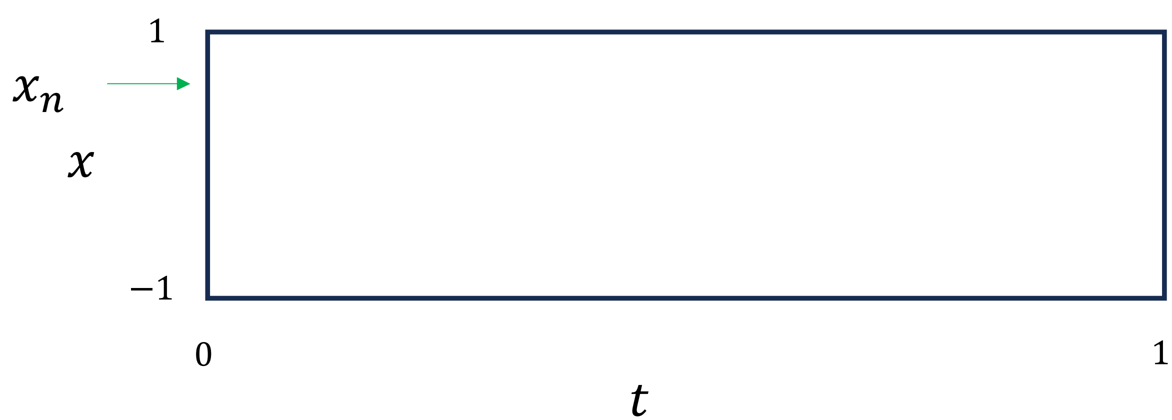
Solución a EDP en tiempo discreto

Configuración del problema:



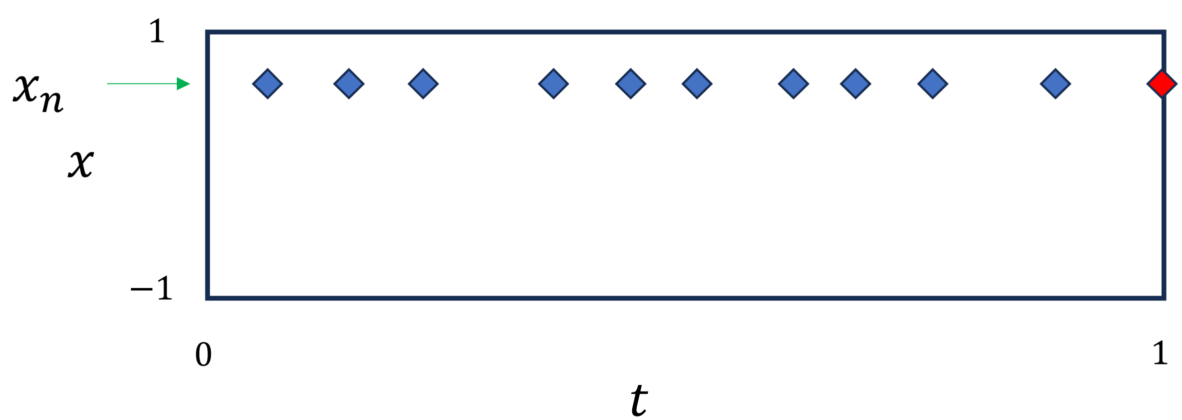
Solución a EDP en tiempo discreto

Configuración del problema:



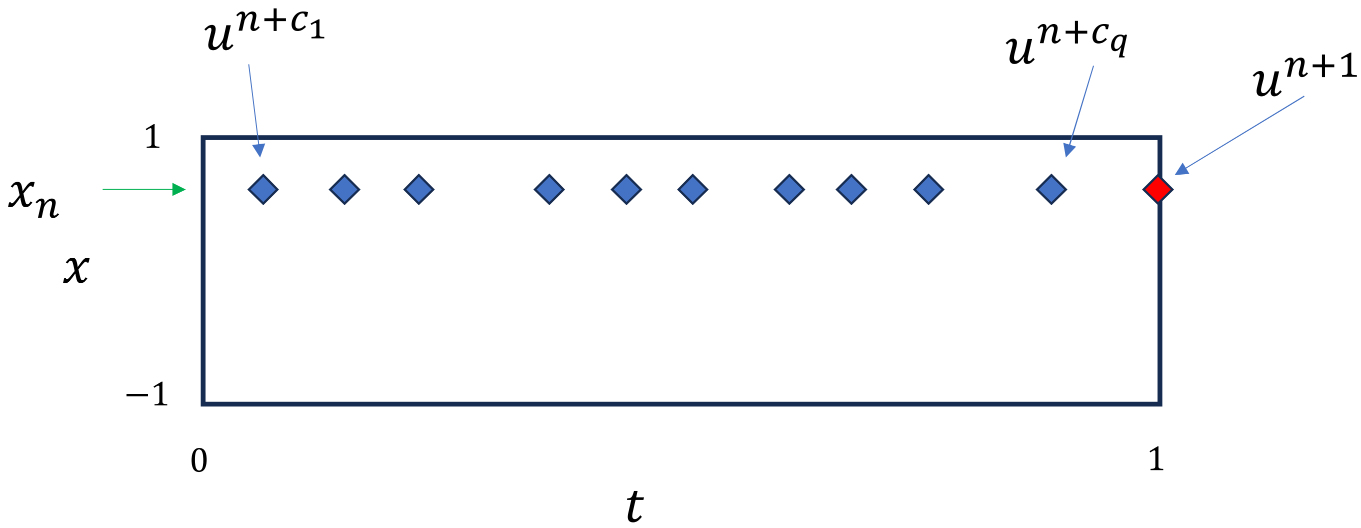
Solución a EDP en tiempo discreto

Configuración del problema:



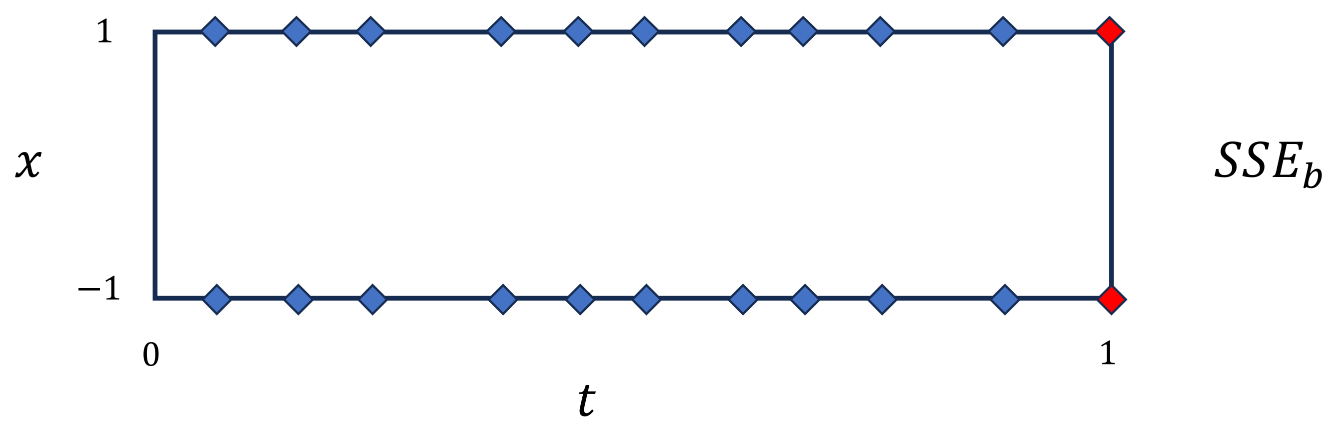
Solución a EDP en tiempo discreto

Configuración del problema:



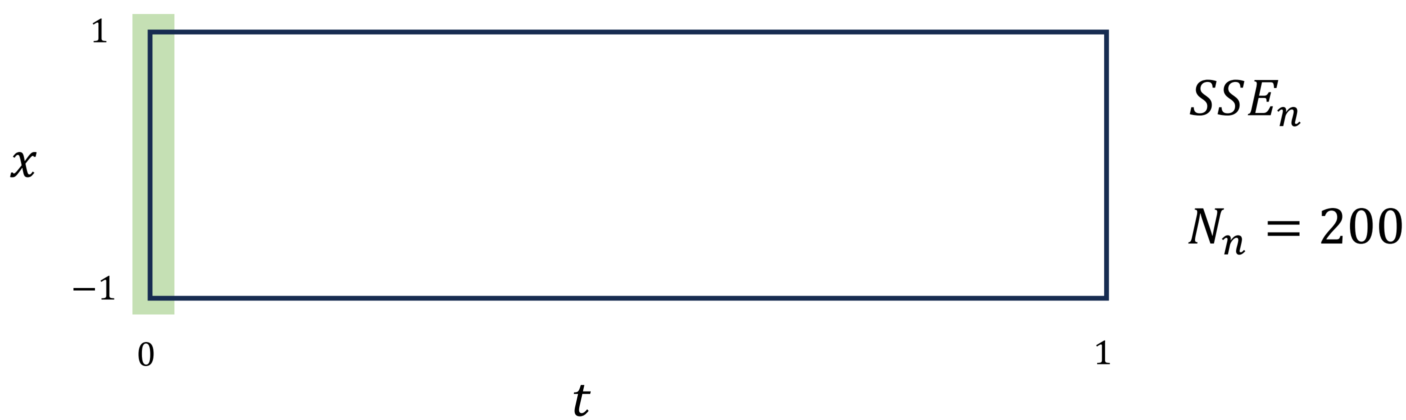
Solución a EDP en tiempo discreto

Configuración del problema:



Solución a EDP en tiempo discreto

Configuración del problema:

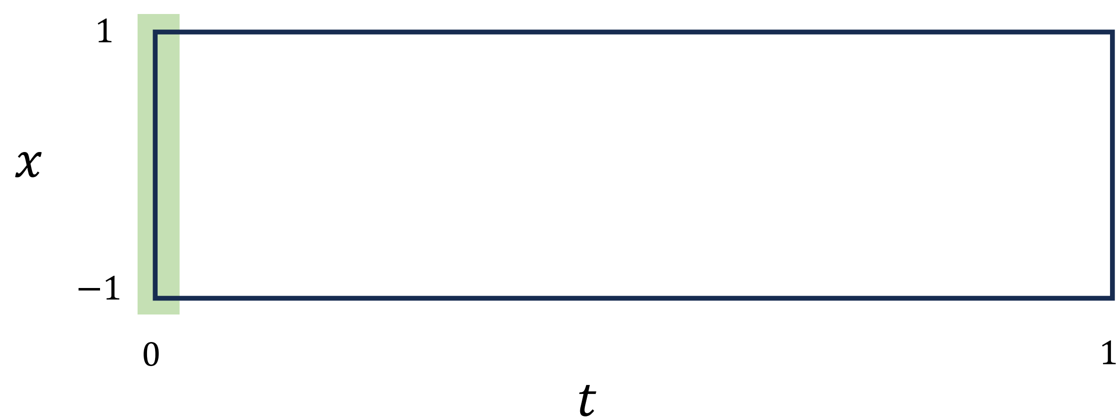


Solución a EDP en tiempo discreto

Configuración del problema:

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n\left(x^{n,i}\right) - u^{n,i} \right|^2$$

$$u_i^n = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$



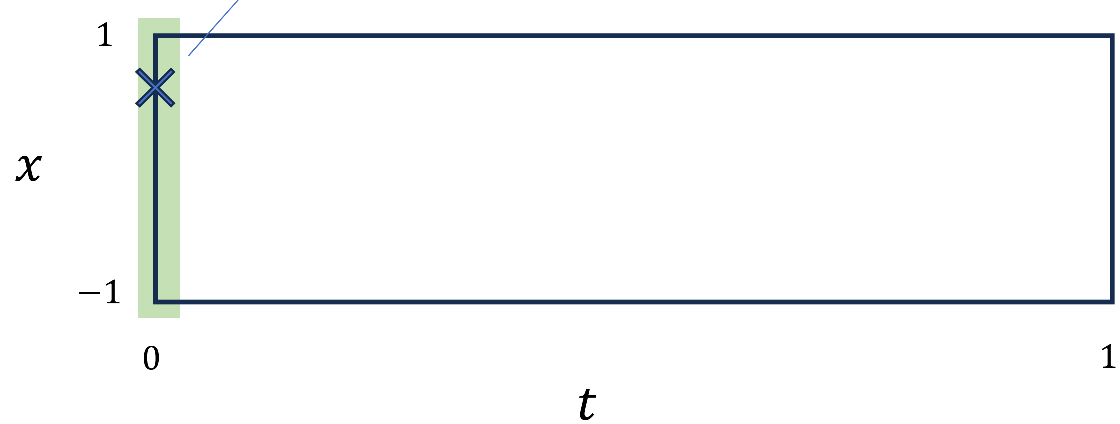
SSE_n

$N_n = 200$

Solución a EDP en tiempo discreto

Configuración del problema:

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n(x^{n,i}) - u^{n,i} \right|^2 \qquad u_i^n = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$



SSE_n

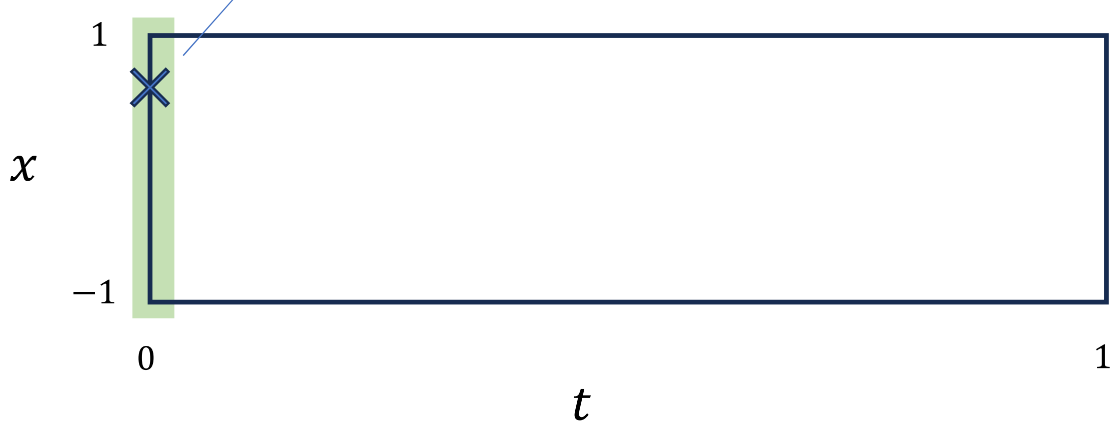
$N_n = 200$

Solución a EDP en tiempo discreto

Configuración del problema:

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n\left(x^{n,i}\right) - u^{n,i} \right|^2$$

$$u_i^n = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$



SSE_n

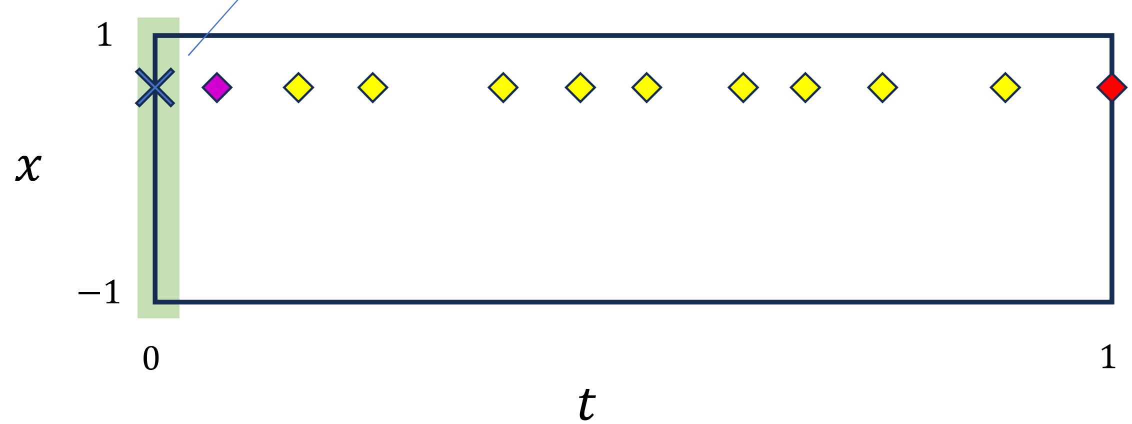
$N_n = 200$

Solución a EDP en tiempo discreto

Configuración del problema:

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n(x^{n,i}) - u^{n,i} \right|^2$$

$$u_i^n = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$



SSE_n

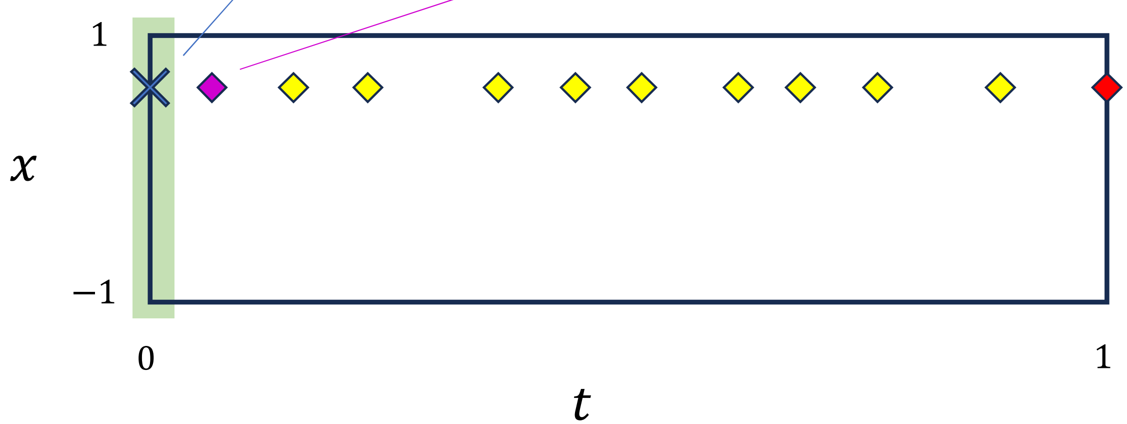
$N_n = 200$

Solución a EDP en tiempo discreto

Configuración del problema:

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n\left(x^{n,i}\right) - u^{n,i} \right|^2$$

$$u_i^n = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \dots, q$$



SSE_n

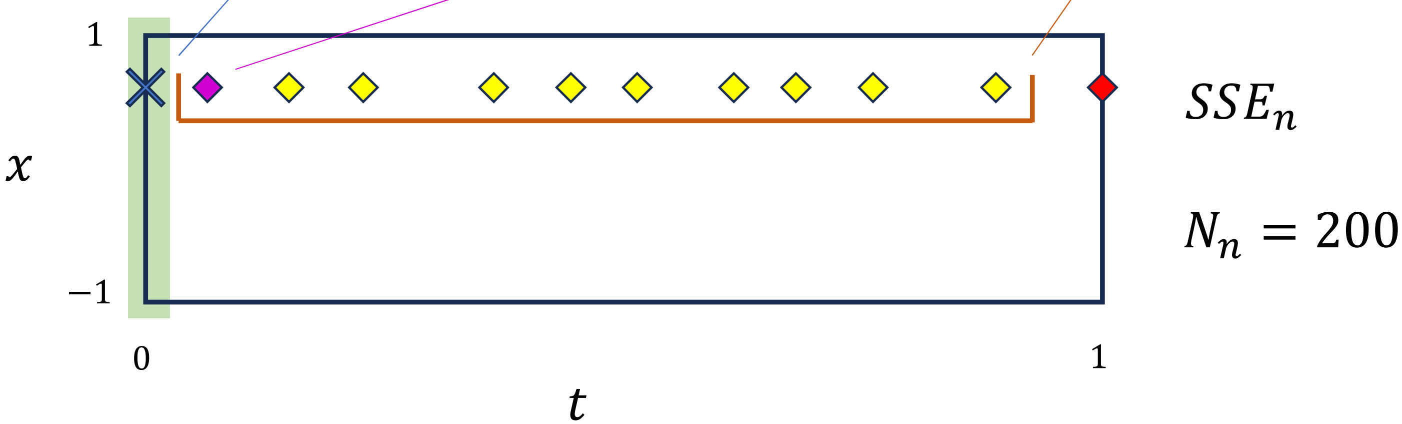
$N_n = 200$

Solución a EDP en tiempo discreto

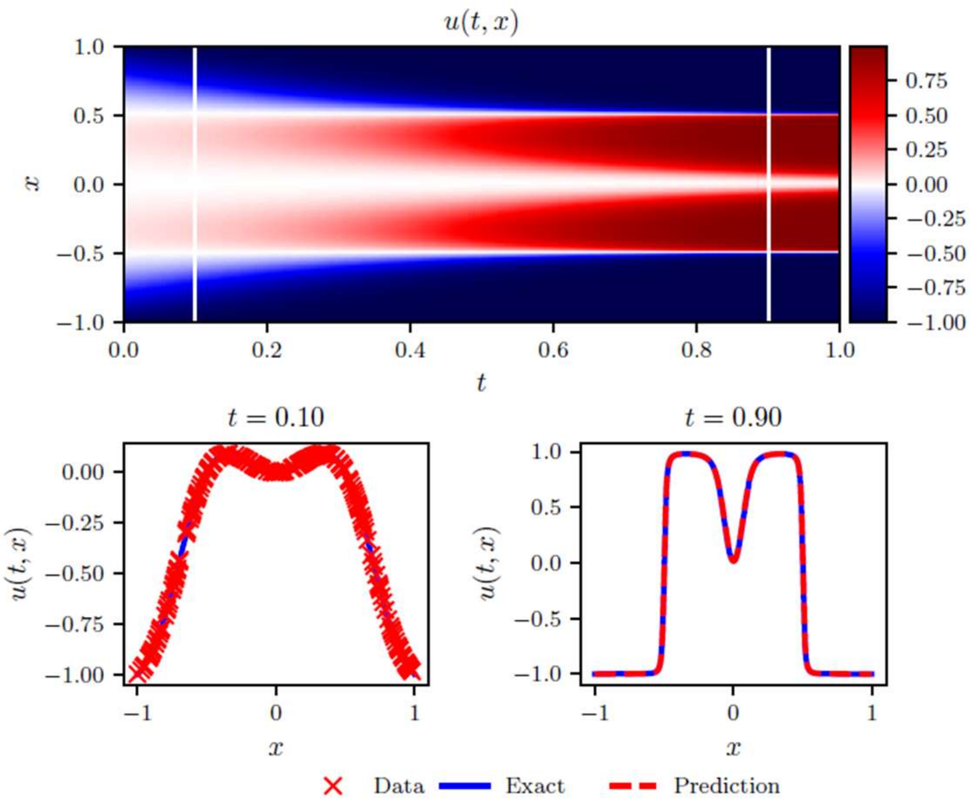
Configuración del problema:

$$SSE_n = \sum_{j=1}^{q+1} \sum_{i=1}^{N_n} \left| u_j^n\left(x^{n,i}\right) - u^{n,i} \right|^2$$

$$u_i^n = u^{n+c_i} - \Delta t \sum_{j=1}^{q} a_{ij}\, O(\, u^{t+c_j}), \qquad i = 1, \ldots, q$$



Solución a EDP en tiempo discreto



Descubrimiento de parámetros de EDP en tiempo continuo

Ecuación diferencial en derivadas parciales con parámetros:

$$u_t + O(u, \lambda) = 0, \qquad x \in \Omega, \qquad t \in [0, T]$$

Descubrimiento de parámetros de EDP en tiempo continuo

Ecuación de Burgers con las condiciones de borde de Dirichlet

$$u_t + \lambda_1 u u_x + \lambda_2 u_{xx} = 0, \qquad x \in [-1,1], \qquad t \in [0,1]$$
$$u(0,x) = -\sin(\pi x)$$
$$u(t,-1) = u(t,1) = 0$$

Descubrimiento de parámetros de EDP en tiempo continuo

Ecuación de Burgers con las condiciones de borde de Dirichlet

$$u_t + \lambda_1 u u_x + \lambda_2 u_{xx} = 0, \qquad x \in [-1,1], \qquad t \in [0,1]$$
$$u(0,x) = -\sin(\pi x)$$
$$u(t,-1) = u(t,1) = 0$$

$$f \coloneqq u_t + \lambda_1 u u_x + \lambda_2 u_{xx}$$

Descubrimiento de parámetros de EDP en tiempo continuo

Función de pérdida

$$MSE = MSE_u + MSE_f,$$

$$MSE_u = \frac{1}{N_u}\sum_{i=1}^{N_u} |u(t_u^i, x_u^i) - u^i|^2,$$

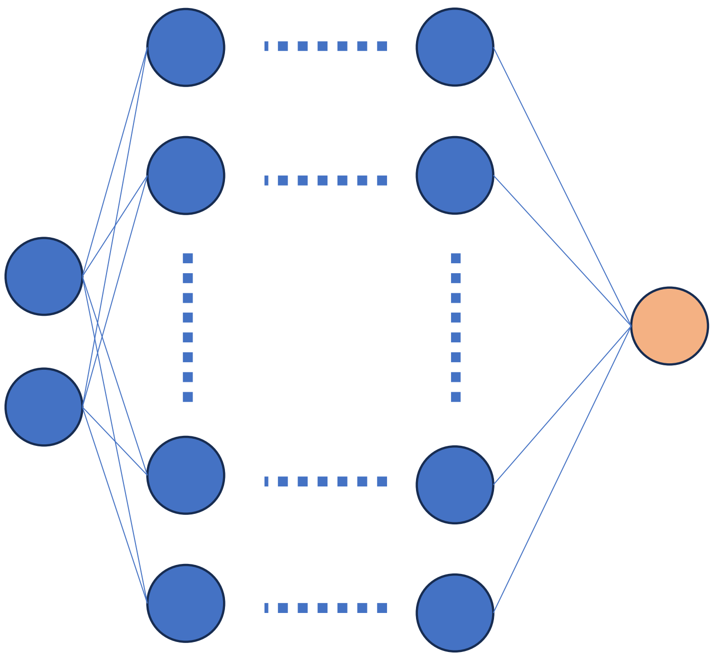
$$MSE_f = \frac{1}{N_f}\sum_{i=1}^{N_f} |f(t_f^i, x_f^i)|^2.$$

$$u(0,x) = -\sin(\pi x),$$
$$u(t,-1) = u(t,1) = 0.$$

$$f := u_t + \lambda_1 u u_x - \lambda_2 u_{xx}$$

Descubrimiento de parámetros de EDP en tiempo continuo

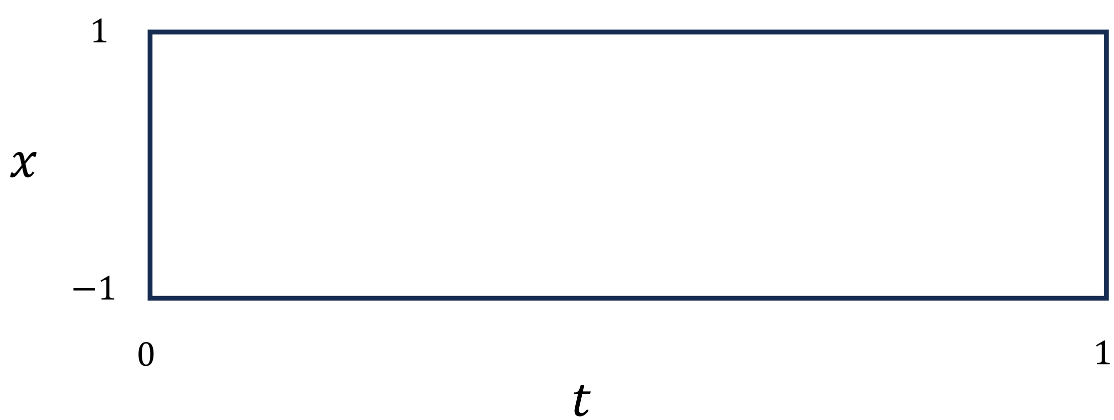
Configuración de la red neuronal:



- 2 entradas t , x
- 1 salida $u(t,x)$
- 8 capas ocultas
- 20 neuronas cada capa oculta
- tangente hiperbólica
- L-BFGS

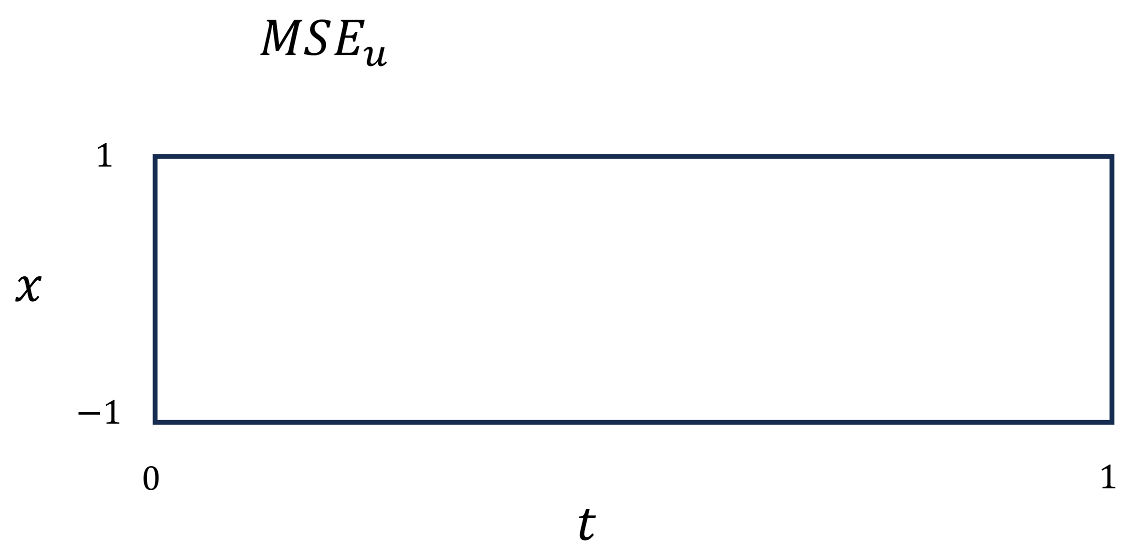
Descubrimiento de parámetros de EDP en tiempo continuo

Función de pérdida:



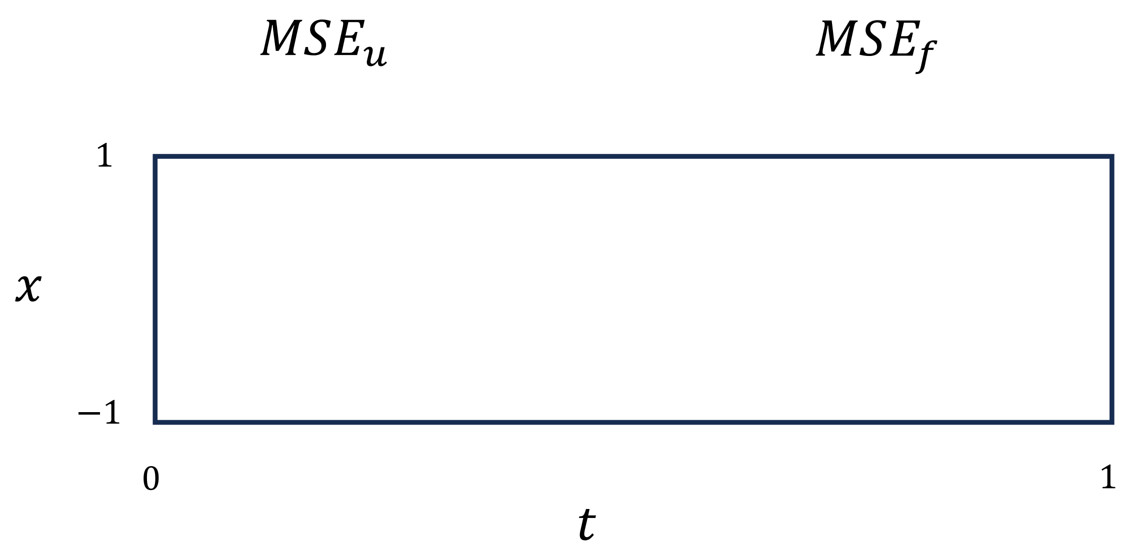
Descubrimiento de parámetros de EDP en tiempo continuo

Función de pérdida:



Descubrimiento de parámetros de EDP en tiempo continuo

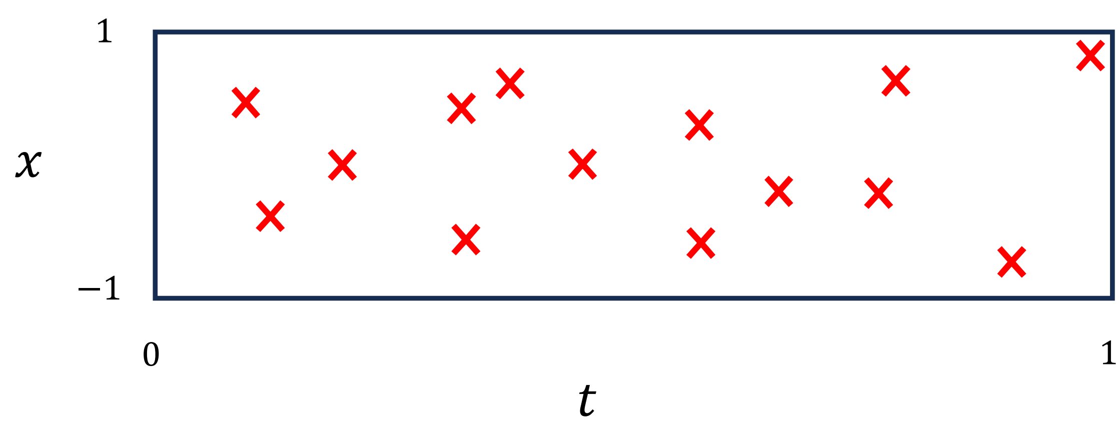
Función de pérdida:



Descubrimiento de parámetros de EDP en tiempo continuo

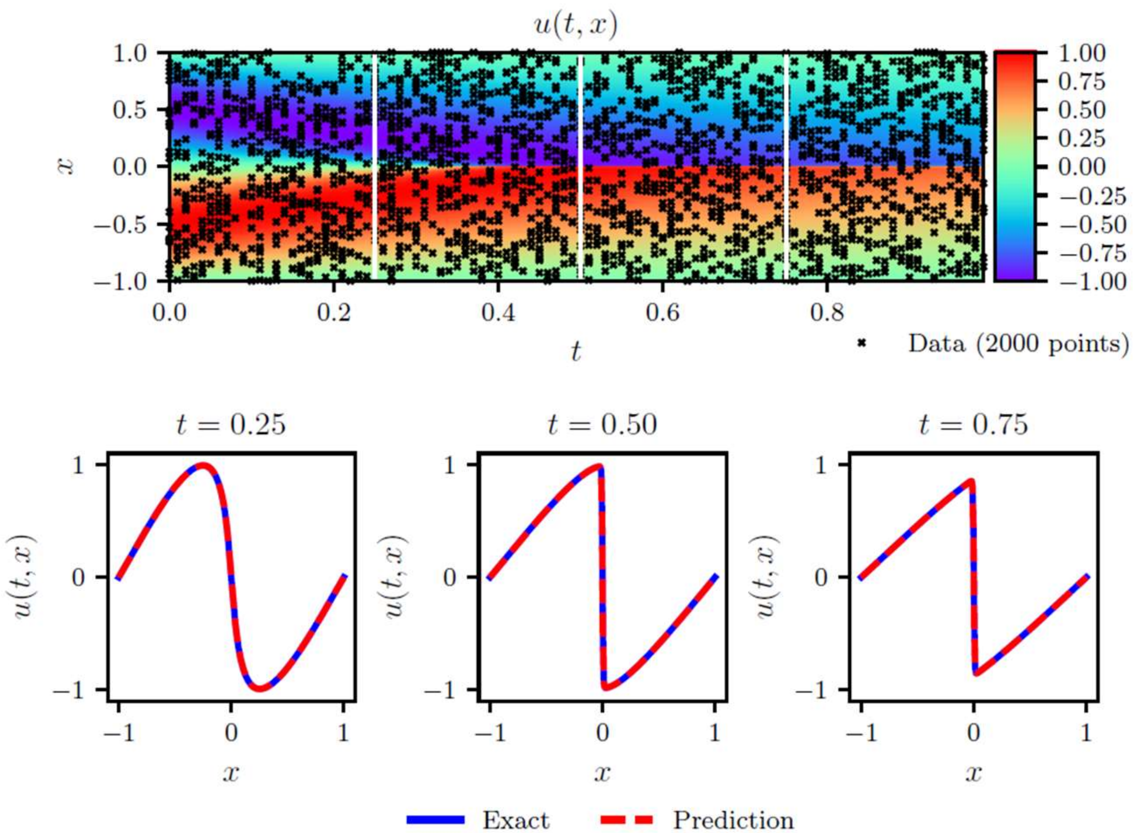
Función de pérdida:

MSE_u $\quad$ MSE_f

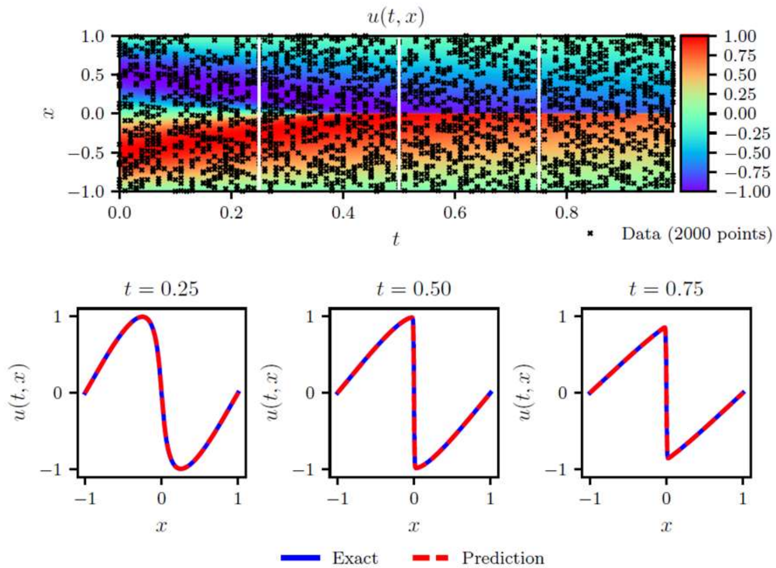


2000 puntos al azar

Descubrimiento de parámetros de EDP en tiempo continuo



Descubrimiento de parámetros de EDP en tiempo continuo



$$\lambda_1 = 1.0$$

$$\lambda_2 = 0.01/\pi$$

Correct PDE	$u_t + uu_x - 0.0031831 u_{xx} = 0$
Identified PDE (clean data)	$u_t + 0.99915 uu_x - 0.0031794 u_{xx} = 0$
Identified PDE (1% noise)	$u_t + 1.00042 uu_x - 0.0032098 u_{xx} = 0$

Gracias

FACULTAD DE INGENIERÍA 175 AÑOS

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