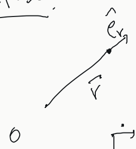
Práctico 4: Movimiento Central

Repaso:



- $\vec{F} = f\hat{e}_r$
- Momento angular: $\vec{L}_o = \vec{r}\times\vec{p}$ ← Producto vectorial

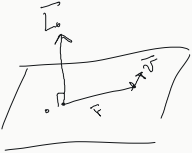
$$= m\vec{r}\times\vec{v}$$

$$\dot{\vec{L}}_o = \overbrace{m\dot{\vec{r}}\times\vec{v}}^{0} + m\vec{r}\times\dot{\vec{v}} = \vec{r}\times(\overbrace{m\vec{a}}^{\vec{F}}) = 0$$

$$\Longrightarrow \boxed{\vec{L}_o = \vec{cte}}$$

→ ① El movimiento ocurre en el plano.

→ ② $|\vec{L}_o| = \ell_o = cte.$



- Al ser un movimiento en el plano: Coord polares.

$$\vec{r} = r\hat{e}_r \to \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\to \vec{a} = \ddot{r}\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta$$

$$\Longrightarrow \boxed{\begin{cases} f = m(\ddot{r} - r\dot{\theta}^2) \\ 0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} \end{cases}} \xrightarrow{\times r} r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0$$

$$\Longleftrightarrow \frac{d(r^2\dot{\theta})}{dt} = 0 \Longleftrightarrow mr^2\dot{\theta} = cte$$

$|\bar{L}_0| = \ell_0 = m r^2 \dot{\theta}$, $\bar{L}_0 = m \bar{r} \times \bar{v} = m r \hat{e}_r \times (\dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta)$

$= m r^2 \dot{\theta} \hat{k}$

Fuerzas isotrópicas: $\underline{\bar{F} = f(r) \hat{e}_r}$

$\Longrightarrow$ ¡Son conservativas! $\longleftrightarrow$ $\underline{\bar{F} = -\nabla U}$

$T + U = E = cte \Longrightarrow \frac{1}{2} m v^2 + U = cte$

$\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (r \dot{\theta})^2 + U = cte$ $\qquad \ell = m r^2 \dot{\theta} \iff \dot{\theta} = \frac{\ell}{m r^2}$

$$\iff \boxed{\frac{1}{2} m \dot{r}^2 + \boxed{\frac{\ell^2}{2 m r^2} + U} = cte = E_0}$$

$\rightarrow$ Potencial efectivo: $U_{eff} = \frac{\ell^2}{2 m r^2} + U$.

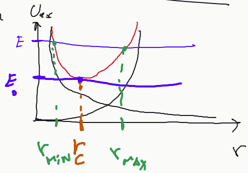
Soluciones interesantes: $\boxed{\dot{r} = 0}$ $\rightarrow$ r_{min} y r_{max}

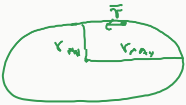
$\rightarrow$ Órbitas circulares

Son las soluciones o los puntos con $\boxed{E = U_{eff}}$

· F elástica: $\bar{F} = -k r \hat{e}_r \rightarrow U = \frac{k r^2}{2}$

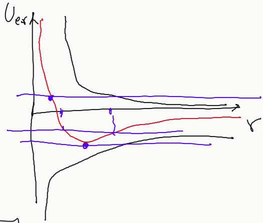
$U_{ef} = \frac{\ell^2}{2 m r^2} + \frac{k r^2}{2}$





· F grav: $\left(\bar{F} = -\frac{k}{r^2} \hat{e}_r \rightarrow U = -k/r\right)$

$$U_{ef} = \frac{l^2}{2mr^2} - \frac{k}{r}$$



$$\longrightarrow \dot r = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E_0 - U_{ef}\right)}$$

$$\longrightarrow \int dt = \boxed{\Delta t = \int_{r_{MIN}}^{r_{max}} \frac{dr}{\sqrt{\frac{2}{m}\left(E_0 - \frac{l^2}{2mr^2} - U\right)}}}$$

Binet: (Cambio de variable A VECES útil)

$$\begin{cases} F = m(\ddot r - r\dot\theta^2) \\ 0 = m(r\ddot\theta + 2\dot r\dot\theta) \end{cases}$$

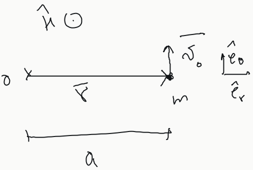
$$\hookrightarrow \boxed{u(\theta) = \frac{1}{r(\theta)}}$$

$$\dot r = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \overset{-1/u^2}{\overset{\shortparallel}{\frac{dr}{du}}}\ \overset{u'}{\overset{\shortparallel}{\frac{du}{d\theta}}}\ \overset{\dot\theta}{\overset{\shortparallel}{\frac{d\theta}{dt}}} = -r^2u'\dot\theta = -\frac{l}{m}u'$$

$$\ddot r = -\frac{l^2}{m^2}u''u^2 \longrightarrow \boxed{\frac{F}{m} = -\frac{l^2}{m^2}u^2\left[u + u''\right]}$$

Ejercicio 6:

$$F = -k\frac{\vec r}{r^4} = -\frac{k}{r^3}\hat e_r$$



$\bar r = r\hat e_r$

$\bar v = \dot r\hat e_r + r\dot\theta\hat e_\theta$

$l = mr^2\dot\theta = mav_0$

$\dot\theta^2 = \frac{l^2}{m^2r^4}$

$\bar L_0(t=0) = ma\hat e_r \times v_0\hat e_\theta$

$= mav_0\hat k$

a) $\boxed{U = -\frac{k}{2r^2}}$

b) $T+U = E_0$

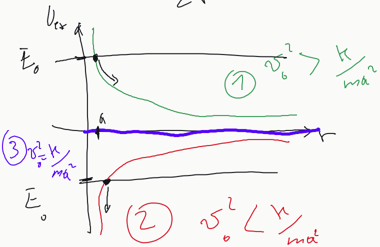
$\frac{1}{2}mv^2 - \frac{k}{2r^2} = E_0$

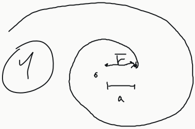
$\frac{1}{2}m\dot r^2 + \frac{l^2}{2mr^2} - \frac{k}{2r^2} = E_0$

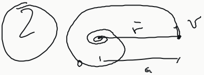
$\frac{1}{2}m\dot r^2 + \frac{ma^2v_0^2}{2r^2} - \frac{k}{2r^2} = E_0 = \frac{mv_0^2}{2} - \frac{k}{2a^2}$

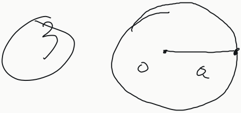
$\Rightarrow \boxed{\frac{1}{2}m\dot r^2 + \underbrace{\frac{ma^2v_0^2-k}{2r^2}} = \frac{mv_0^2a^2-k}{2a^2}}$

$U_{ef} = \frac{ma^2v_0^2-k}{2r^2}$









$\frac{1}{2}m\dot r^2 + \frac{ma^2v_0^2-k}{2r^2} = \boxed{\frac{mv_0^2a^2-k}{2a^2}}$

c) $m(\ddot{r} - r\dot{\theta}^2) = -k/r^3$

Circular $\Longleftrightarrow$ $\left.\begin{matrix}\dot{r} = 0 \\ \ddot{r} = 0\end{matrix}\right\}$ $\rightarrow m r \dot{\theta}^2 = k/r^3$

$\downarrow$

$r = a$

$\Longleftrightarrow \dot{\theta}^2 = \frac{k}{mr^4}$

$l = mr^2\dot{\theta}$
$= mav_0$ $\Longleftrightarrow$ $\dot{\theta}^2 = \frac{a^2 v_0^2}{r^4}$ $\rightarrow$ $\frac{v_0^2 a^2}{\cancel{r^4}} = \frac{k}{m\cancel{r^4}}$

$\Longleftrightarrow \boxed{v_0^2 = \frac{k}{ma^2}}$

d) ② $v_0^2 < \frac{k}{ma^2}$

Binet: $u\underbrace{\left[1 - \frac{k}{ma^2 v_0^2}\right]}_{Ⓒ} = -u''$

$\longrightarrow u = Ae^{\sqrt{C}\theta} + Be^{-\sqrt{C}\theta}$

$\left.\begin{matrix} u(\theta=0) = 1/r(0) = 1/a = A + B \\ u'(\theta=0) = 0 = \sqrt{C}(A - B)\end{matrix}\right\}$ $A = \frac{1}{2a} = B$

$A = B$

$\dot{r} = -\frac{l}{m} u'$