Práctico 5 (parte C)

Repaso: Sist. de part.

$\longrightarrow M = \sum_{i=1}^{N} m_i$

$\longrightarrow \bar{r}_G = \frac{1}{M}\sum_i m_i \bar{r}_i \xrightarrow{\text{cont}} \frac{1}{M}\int \bar{r}\rho\, dV$ (σ dA, λ dx)

$\longrightarrow \bar{P} = M\bar{v}_G \longrightarrow M\bar{a}_G = \bar{R}^{EXT}$ (resultante)

$\longrightarrow \bar{L}_Q = \sum_{i=1}^{N}(P_i - Q)\times m\bar{v}_i$

Rígidos: $\bar{v}_{P_1} = \bar{v}_{P_2} + \bar{\omega}\times(P_1 - P_2)$ (Dist. de velocidades)

$$\Longrightarrow \boxed{\bar{L}_Q = M(G-Q)\times\bar{v}_Q + \mathbb{I}_Q\bar{\omega}}$$

$\mathbb{I}_Q$ → Tensor de inercia → Matriz

Casos part.: Si $\bar{v}_Q = 0$ o si $Q = G \Rightarrow \bar{L}_Q = \mathbb{I}_Q\bar{\omega}$

Energía cinética del rígido:

$$\boxed{T = \frac{1}{2}M\bar{v}_Q^2 + \frac{1}{2}\bar{\omega}\cdot\mathbb{I}_Q\bar{\omega} + M\bar{v}_Q\cdot[\bar{\omega}\times(G-Q)]}$$

$\mathbb{I}_Q$ y $\bar{\omega}$ tienen que estar en la misma BASE

Si $\bar{v}_Q = 0 \Rightarrow T = \frac{1}{2}\bar{\omega}\cdot\mathbb{I}_Q\bar{\omega}$

Si $G = Q \rightarrow T = \frac{1}{2}M\bar{v}_G^2 + \frac{1}{2}\bar{\omega}\cdot\mathbb{I}_G\bar{\omega}$

¿Qué es $\mathbb{I}$? ⟶ Matriz 3x3 simétrica

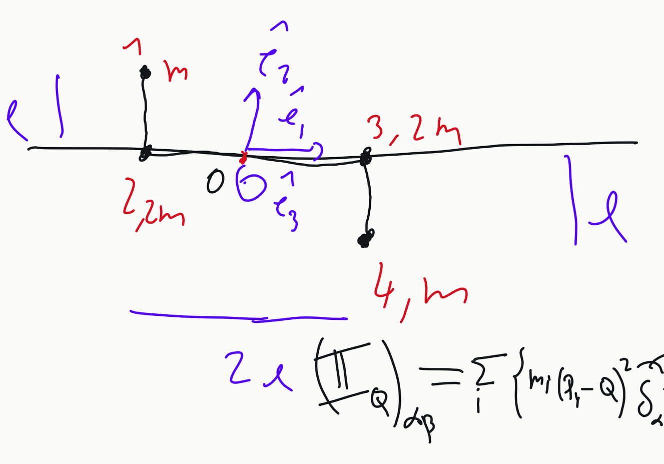
$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33}\end{pmatrix}$$

$$\boxed{(\mathbb{I}_Q)_{\alpha\beta} = \sum_i \left\{ m_i (P_i - Q)^2\delta_{\alpha\beta} - m_i(P_i - Q)_\alpha (P_i - Q)_\beta \right\}}$$

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{si } \alpha = \beta \\ 0 & \text{si } \alpha \neq \beta \end{cases}$$

Ejemplo discreto:

1: $m(-\ell, \ell, 0)$
2: $2m(-\ell, 0, 0)$
3: $2m(\ell, 0, 0)$
4: $m(\ell, -\ell, 0)$



$(\mathbb{I}_Q)_{\alpha\beta} = \sum_i \left\{ m_i (P_i - Q)^2\delta_{\alpha\beta} - m_i(P_i - Q)_\alpha (P_i - Q)_\beta \right\}$ ($\delta_{\alpha\beta}$ = 1 si α=β, = 0 si no)

$\mathbb{I}_{11} = m(2\ell^2 - \ell^2) + 2m(\ell^2 - \ell^2) + m(2\ell^2 - \ell^2) = 2m\ell^2$

$\mathbb{I}_{22} = m(2\ell^2 - \ell^2) + 2m\ell^2 + 2m\ell^2 + m(2\ell^2 - \ell^2) = 6m\ell^2$

$\mathbb{I}_{33} = m2\ell^2 + 2m\ell^2 + 2m\ell^2 + m2\ell^2 = 8m\ell^2$

$\mathbb{I}_{12} = 2m\ell^2$ / $\mathbb{I}_{13} = \mathbb{I}_{23} = 0$

$$\longrightarrow \boxed{\mathbb{I}_O^{(\hat{e}_1,\hat{e}_2,\hat{e}_3)} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4\end{pmatrix} 2m\ell^2}$$

Caso continuo:

$$\mathbb{I}_Q^{(\hat{x},\hat{y},\hat{z})} = \begin{pmatrix} \int dV\rho(y^2+z^2) & -\int dV\rho xy & -\int dV\rho xz \\ -\int dV\rho xy & \int dV\rho(x^2+z^2) & -\int dV\rho yz \\ -\int dV\rho xz & -\int dV\rho yz & \int dV\rho(x^2+y^2)\end{pmatrix}$$

→ Caso rígido plano es particularmente fácil.

→ Simetrías facilitan las cuentas.

→ Las entradas dependen de la base elegida.

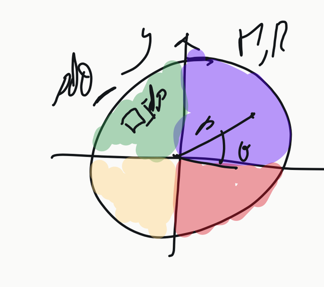
→ Si diagonalizo la matriz obtengo los mom. de inercia en torno a los ejes principales.

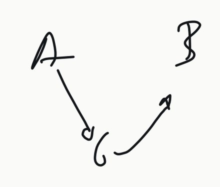
→ Puedo trasladarla ⟶ Teorema de Steiner

$$\mathbb{I}_Q = \mathbb{I}_G + \mathbb{J}_Q^{M,G}$$

$$\boxed{(\mathbb{J}^{M,G})_{\alpha\beta} = M(G-Q)^2\delta_{\alpha\beta} - M(G-Q)_\alpha(G-Q)_\beta}$$

Ej:





$\hat{\jmath}$, $\hat{\imath}$, $O\hat{n}$

$$\begin{pmatrix} \int dV\rho(y^2+z^2) & -\int dV\rho xy & -\int dV\rho xz \;(0) \\ -\int dV\rho xy & \int dV\rho(x^2+z^2) & -\int dV\rho yz \;(0) \\ -\int dV\rho xz & -\int dV\rho yz & \int dV\rho(x^2+y^2)\end{pmatrix}$$

$$\begin{pmatrix} \int dV\rho y^2 & -\int dV\rho xy & 0 \\ -\int dV\rho xy & \int dV\rho x^2 & 0 \\ 0 & 0 & \int dV\rho(x^2+y^2)\end{pmatrix}$$

$\rho = \frac{M}{R^2\pi}$

$\int dA\rho xy = \frac{M}{R^2}\int dA xy = \frac{M}{R^2\pi}\int dx dy\, xy = 0$

$\int dA\rho(x^2+y^2) = \frac{M}{R^2\pi}\int(\rho^2\cos^2\theta + \rho^2\sin^2\theta)\rho\, d\rho\, d\theta$

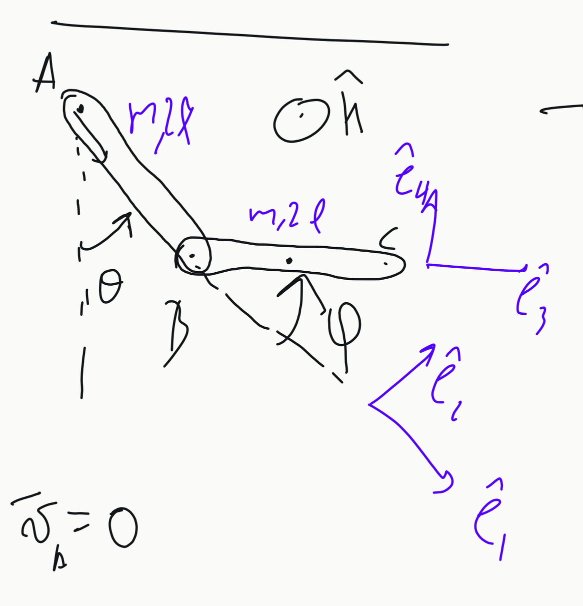
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \quad dA = \rho\, d\rho\, d\theta$$

$= \frac{M}{R^2\pi}\int_0^R \rho^3 d\rho \int_0^{2\pi} d\theta = \frac{MR^4}{4R^2\pi}2\pi$

$= \frac{MR^2}{2}$.

$$\boxed{\begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2}\end{pmatrix}}$$

Ejercicio 15: a) $T = T^1 + T^2$



$T = \frac{1}{2}M\bar{v}_Q^2 + \frac{1}{2}\bar{\omega}\mathbb{I}_Q\bar{\omega} + M\bar{v}_Q\cdot[\bar{\omega}\times(G-Q)]$ → vale para cada barra por separado

$\boxed{T^1_A = \frac{1}{2}\bar{\omega}_{AB}\mathbb{I}_A\bar{\omega}_{AB}}$ | $\bar{\omega}_{AB} = \dot{\theta}\hat{k} = \begin{pmatrix}0\\0\\\dot{\theta}\end{pmatrix}$

$\mathbb{I}_A^{(\hat{e}_1,\hat{e}_2,\hat{n})} = \begin{pmatrix} \text{No me importa} & & 0 \\ & & 0 \\ 0 & 0 & \cdot \end{pmatrix}$

$\bar{v}_A = 0$

$(\mathbb{I}^{(\hat{e}_1,\hat{e}_2,\hat{n})})_{33} = \int dV(x^2+y^2)\rho = \int \frac{dx}{dV}\rho x^2 = \frac{M}{2\ell}\int x^2 dx = \frac{Mx^3}{6\ell}\Big|_0^{2\ell} = \frac{8M\ell^2}{6} = \frac{4M\ell^2}{3}$

$\mathbb{I}_A\bar{\omega} = \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ 0 & 0 & \frac{4M\ell^2}{3}\end{pmatrix}\begin{pmatrix}0\\0\\\dot{\theta}\end{pmatrix} = \begin{pmatrix}0\\0\\\frac{4M\ell^2}{3}\dot{\theta}\end{pmatrix} = \frac{4M\ell^2}{3}\dot{\theta}\hat{n}$

$T^1 = \frac{1}{2}\bar{\omega}\mathbb{I}_A\bar{\omega} = \frac{1}{2}\dot{\theta}\hat{n}\cdot\frac{4\ell^2}{3}\dot{\theta}\hat{n} = \boxed{\frac{2M\ell^2\dot{\theta}^2}{3}}$

$\boxed{T = \frac{1}{2}M\bar{v}_Q^2 + \frac{1}{2}\bar{\omega}\mathbb{I}_Q\bar{\omega} + M\bar{v}_Q\cdot[\bar{\omega}\times(G-Q)]}$

$T^2_B = \underbrace{\frac{1}{2}M\bar{v}_B^2}_{(1)} + \underbrace{\frac{1}{2}\bar{\omega}_{BC}\mathbb{I}_B^{(\cdot)}\bar{\omega}_{BC}}_{(2)} + \underbrace{M\bar{v}_B\cdot[\bar{\omega}\times(G-B)]}_{(3)}$

$\bar{\omega}_{BC} = (\dot{\theta}+\dot{\varphi})\hat{n} = \begin{pmatrix}0\\0\\\dot{\theta}+\dot{\varphi}\end{pmatrix}$ | $\bar{v}_B = 2\ell\dot{\theta}\hat{e}_2$ | $(\dot{\theta}+\dot{\varphi})\hat{n}\times\ell\hat{e}_3$

$G - B = \ell\hat{e}_3$ | $\ell(\dot{\theta}+\dot{\varphi})\hat{e}_4$

$\mathbb{I}_B^{(\hat{e}_3,\hat{e}_4,\hat{n})} = \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ 0 & 0 & \frac{4M\ell^2}{3}\end{pmatrix}$

(1) $= \frac{1}{2}M4\ell^2\dot{\theta}^2$

(2) $= \frac{2M\ell^2}{3}(\dot{\theta}+\dot{\varphi})^2$

(3) $= 2M\ell^2\dot{\theta}(\dot{\theta}+\dot{\varphi})\cos\varphi$

$\Rightarrow T^2 = 2M\ell^2\dot{\theta}^2 + \frac{2M\ell^2}{3}(\dot{\theta}+\dot{\varphi})^2 + 2M\ell^2\dot{\theta}(\dot{\theta}+\dot{\varphi})\cos\varphi$

$T_1 = \frac{2M\ell^2\dot{\theta}^2}{3}$

$$\longrightarrow \boxed{T = \frac{8}{3}M\ell^2\dot{\theta}^2 + \frac{2M\ell^2}{3}(\dot{\theta}+\dot{\varphi})^2 + 2M\ell^2\dot{\theta}(\dot{\theta}+\dot{\varphi})\cos\varphi}$$