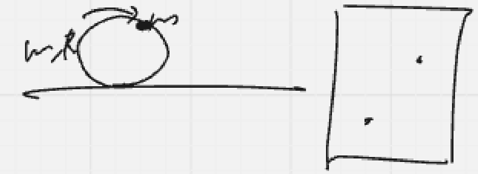
Práctico 8: Rígido en el espacio



Reposo:

- 1ª Cardinal: $M\bar{a}_G = \bar{R}^{EXT}$
- 2ª Cardinal: $M(G-Q)\times \bar{a}_Q + \dfrac{d(\mathbb{I}_Q\bar{\omega})}{dt} = \bar{M}_Q^{EXT}$

Misma base

- Energía: $T = \frac{1}{2} M v_Q^2 + \frac{1}{2}\bar{\omega}\cdot\mathbb{I}_Q\bar{\omega} + M\bar{v}_Q\cdot[\bar{\omega}\times(\bar{r}_G-\bar{r}_Q)]$

$$P = \bar{R}^{EXT}\cdot\bar{v}_Q + \bar{M}_Q^{EXT}\cdot\bar{\omega}$$ (Potencia)

- Momento angular: $\bar{L}_Q = M(\bar{r}_G-\bar{r}_Q)\times\bar{v}_Q + \mathbb{I}_Q\bar{\omega}$

$$\frac{d\bar{L}_Q}{dt} = M\bar{v}_G\times\dot{\bar{r}}_Q + \bar{M}_Q^{EXT}$$ ← 2ª Card

- Cambio de punto de aplicación de momentos:

$\bar{p} = M\bar{v}_G$

$$\begin{cases}\bar{L}_{Q_1} = \bar{L}_{Q_2} + \bar{p}\times(\bar{r}_{Q_1}-\bar{r}_{Q_2})\\ \bar{M}^{EXT}_{Q_1} = \bar{M}^{EXT}_{Q_2} + \bar{R}^{EXT}\times(\bar{r}_{Q_1}-\bar{r}_{Q_2})\end{cases}$$

- Distr de $\bar{v}$ y $\bar{a}$:

$$\begin{cases}\bar{v}_P = \bar{v}_Q + \bar{\omega}\times(\bar{r}_P-\bar{r}_Q)\\ \bar{a}_P = \bar{a}_Q + \dot{\bar{\omega}}(\bar{r}_P-\bar{r}_Q) + \bar{\omega}\times(\bar{\omega}\times(\bar{r}_P-\bar{r}_Q))\end{cases}$$

- Suma de vel ang: $\bar{\omega}_{31} = \bar{\omega}_{32} + \bar{\omega}_{21}$
- Calcular y trasladar tensores de inercia!!

↓ Simetrías

↳ Steiner: $\mathbb{I}_Q = \mathbb{I}_G + \mathbb{J}^{GQ}$

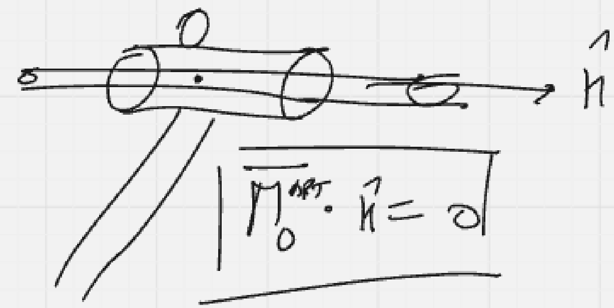
$$\left(\mathbb{J}^{GQ}\right)_{\alpha\beta} = M(G-Q)^2\delta_{\alpha\beta} - M(G-Q)_\alpha(G-Q)_\beta$$

$$\bar{M}^F = \bar{r} \times \bar{F}$$

- Momentos: → Línea de acción

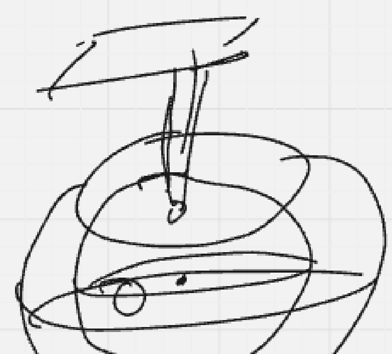
→ Reducción de sist de fuerzas

→ Articulaciones lisas: - Cilíndricas



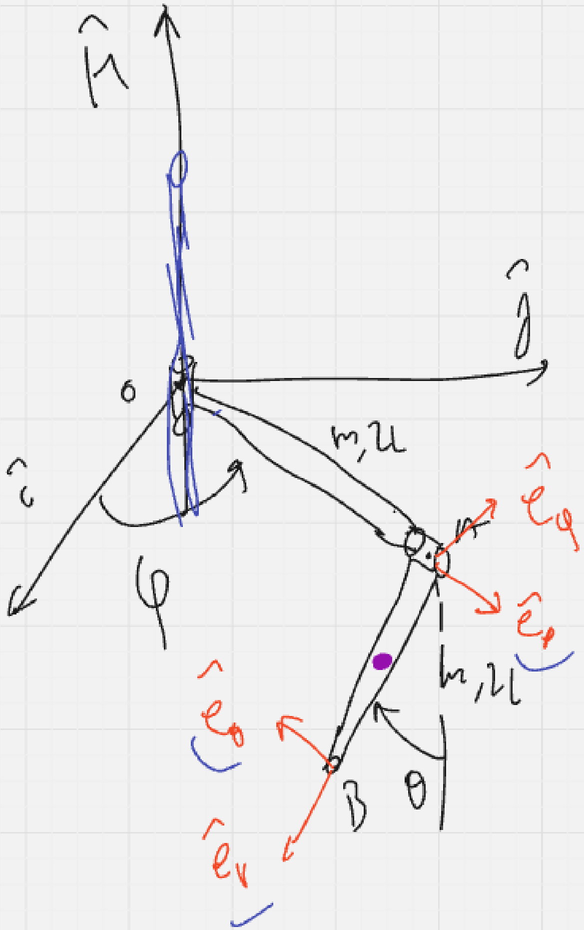
$$\boxed{\bar{M}_0^{ART.} \cdot \hat{n} = 0}$$

- Esféricas



$$\boxed{\bar{M}_0 = 0}$$

Ejercicio 9:



a) ¿$\overline{L_0} \cdot \hat{n}$ se conserva?

$$\frac{d}{dt}(\overline{L_0} \cdot \hat{k}) = \dot{\overline{L_0}} \cdot \hat{n} + \overline{L_0} \cdot \cancel{\dot{\hat{n}}}^{\,0}$$

$$= \frac{d\overline{L_0}}{dt} \cdot \hat{k}$$

2° Card: $\frac{d\overline{L_0}}{dt} = M\overline{v_G} \times \cancel{\overline{v_0}}^{\,0} + \overline{M_0^{EXT}}$

$= \overline{M_0^{EXT}} = \overline{M^{OA}_{PESO}} + \overline{M^{AB}_{PESO}} + \cancel{\overline{M^{ART(A)}}} + \overline{M^{ART(O)}}$

$\overline{M^{OA}_{PESO}} \perp \hat{n}$, $\overline{M^{AB}_{PESO}} \perp \hat{n}$, $\overline{M^{ART(O)}} \perp \hat{n}$

Si considero el sist de las 2 barras esto se cancela.

$\overline{F_{OA \to AB}} = -\overline{F_{AB \to OA}}$

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Todos los momentos son perp a $\hat{k}$.

$\Rightarrow \frac{d\overline{L_0}}{dt} \cdot \hat{n} = 0 \longrightarrow \frac{d}{dt}(\overline{L_0} \cdot \hat{n}) = 0 \Rightarrow \overline{L_0} \cdot \hat{n}$ se conserva!

También se conserva la energía: $\overline{F^{react}}$ actúan en O (solo el peso)

pero $\overline{v} = 0 \Longrightarrow P^{react} = 0$.

b) Hay que calcular $\overline{L_O} = \overline{L_O^{OA}} + \overline{L_O^{AB}}$

$$\overline{L_O^{OA}} = m(G-O)\times \cancel{\overline{v_O}}^{0} + \mathbb{I}_O\,\overline{\omega} = \mathbb{I}_O\,\overline{\omega^{OA}}$$

$$\overline{\omega^{OA}} = \dot\varphi\hat{k} = \begin{pmatrix}0\\0\\\dot\varphi\end{pmatrix}$$

$$\mathbb{I}_O^{\{\hat e_\rho \hat e_\varphi \hat k\}} = \begin{pmatrix}0 & 0 & 0\\ 0 & 4/3 & 0\\ 0 & 0 & 4/3\end{pmatrix} mL^2$$

$$\boxed{\overline{L_O^{OA}} = \frac{4}{3}mL^2\dot\varphi\hat{k}}$$

$$\overrightarrow{L_O^{AB}} = \underbrace{\overline{L_A^{AB}}}_{} + \overline{p^{AB}}\times(O-A)$$

$$\overline{\omega^{AB}} = -\dot\theta\hat e_\rho + \dot\varphi\hat k \qquad \vec{k} = -\cos\theta\,\hat e_r + \sin\theta\,\hat e_\theta$$

$$= -\dot\theta\hat e_\rho + \dot\varphi\sin\theta\,\hat e_\theta - \dot\varphi\cos\theta\,\hat e_r$$

$$\overline{L_A^{AB}} = \underbrace{m(G^{AB}-A)\times\overline{v_A}}_{2} + \underbrace{\mathbb{I}_A\,\overline{\omega^{AB}}}_{1}$$

$$= \begin{pmatrix}-\dot\varphi\cos\theta\\ -\dot\theta\\ \dot\varphi\sin\theta\end{pmatrix}$$

1 $\quad \mathbb{I}_A\,\overline{\omega^{AB}} = \begin{pmatrix}0\\ -\dot\theta\frac{4}{3}mL^2\\ \dot\varphi\sin\theta\frac{4}{3}mL^2\end{pmatrix} = -\frac{4}{3}mL^2\dot\theta\,\hat e_\rho + \frac{4}{3}mL^2\dot\varphi\sin\theta\,\hat e_\theta$

$$\mathbb{I}^{\{\hat e_\rho \hat e_\theta \hat e_r\}} = \begin{pmatrix}0 & 0 & 0\\ \ldots\end{pmatrix}$$

$$\overline{L}_A^{AB} = \underbrace{m(G^{AB}-A)\times\overline{v}_A}_{②} + \underbrace{\mathbb{I}_A\,\overline{\omega}^{AB}}_{①}$$

$$= -\dot{\theta}\hat{e}_\rho + \dot{\varphi}\sin\theta\,\hat{e}_\theta - \dot{\varphi}\cos\theta\,\hat{e}_r = \begin{pmatrix} -\dot{\varphi}\cos\theta \\ -\dot{\theta} \\ \dot{\varphi}\sin\theta \end{pmatrix}$$

① $$\mathbb{I}_A\,\overline{\omega}^{AB} = \begin{pmatrix} 0 \\ -\dot{\theta}\frac{4}{3}mL^2 \\ \dot{\varphi}\sin\theta\frac{4}{3}mL^2 \end{pmatrix} = -\frac{4}{3}mL^2\dot{\theta}\,\hat{e}_\rho + \frac{4}{3}mL^2\dot{\varphi}\sin\theta\,\hat{e}_\theta$$

$$\mathbb{I}_A^{\{\hat{e}_r,\hat{e}_\rho,\hat{e}_\theta\}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{pmatrix} mL^2$$

② $(G^{AB}-A) = L\hat{e}_r$

$\overline{r}_A = 2L\hat{e}_\rho$

$\overline{v}_A = 2L\dot{\varphi}\hat{e}_\varphi$ $\qquad \hat{e}_r\times\hat{e}_\varphi = \cos\theta\,\hat{e}_\rho$

$$m\,2L^2\dot{\varphi}\,\hat{e}_r\times\hat{e}_\varphi = 2mL^2\dot{\varphi}\cos\theta\,\hat{e}_\rho$$

$$\overline{L}_A^{AB} = \left(2mL^2\dot{\varphi}\cos\theta - \frac{4}{3}mL^2\dot{\theta}^2\right)\hat{e}_\rho + \frac{4}{3}mL^2\dot{\varphi}\sin\theta\,\hat{e}_\theta$$

Pero preciso $\overline{L}_O^{AB} \longrightarrow \overline{L}_O^{AB} = \underbrace{\overline{L}_A^{AB}}_{\checkmark} + \overline{p}^{AB}\times(O-A)$

$O - A = -2L\hat{e}_\rho$

$\overline{p}^{AB} = m\overline{v}_G^{AB}$

$\overline{r}_G^{AB} = 2L\hat{e}_\rho + L\hat{e}_r$

$\dot{\hat{e}}_r = (\dot{\varphi}\hat{n} - \dot{\theta}\hat{e}_\rho)\times\hat{e}_r$

$\overline{v}_G^{AB} = 2L\dot{\varphi}\hat{e}_\varphi + L(\dot{\theta}\hat{e}_\theta + \sin\theta\dot{\varphi}\hat{e}_\rho)$

Pero preciso $\overline{L}_O^{AB} \longrightarrow \overrightarrow{L}_O^{AB} = \underbrace{\overline{L}_A^{AB}}_{\checkmark} + \overline{p}^{AB} \times (O - A)$

$\overline{v}_G^{AB} = 2L\dot{\hat{e}}_\rho + L\dot{\hat{e}}_r$

$\dot{\hat{e}}_r = (\dot{\varphi}\hat{k} - \dot{\theta}\hat{e}_\rho) \times \hat{e}_r$

$\overline{v}_G^{AB} = 2L\dot{\varphi}\hat{e}_\varphi + L(\dot{\theta}\hat{e}_\theta + \sin\theta\dot{\varphi}\hat{e}_\rho)$

$$\overline{p}^{AB} \times (O - A) = \left[m2L\dot{\varphi}\hat{e}_\varphi + mL(\dot{\theta}\hat{e}_\theta + \sin\theta\dot{\varphi}\hat{e}_\rho)\right] \times (-2L\hat{e}_\rho)$$

$$= +4ml^2\dot{\varphi}\hat{k} + 2ml^2\dot{\theta}\hat{e}_r$$

$$\overline{L}_O^{AB} = \left(2ml^2\dot{\varphi}\cos\theta - \frac{4}{3}ml^2\dot{\theta}^2\right)\hat{e}_\rho + \frac{4}{3}ml^2\dot{\varphi}\sin\theta\hat{e}_\theta + 4ml^2\dot{\varphi}\hat{k} + 2ml^2\dot{\theta}\hat{e}_r$$

$$\overline{L}_O^{OA} = \frac{4}{3}ml^2\dot{\varphi}\hat{k}$$

$$\overrightarrow{L}_O \cdot \hat{k} = \frac{4}{3}ml^2\dot{\varphi} + \frac{4}{3}ml^2\dot{\varphi}\sin^2\theta + 4ml^2\dot{\varphi} - 2ml^2\dot{\theta}\cos\theta$$

$$\boxed{\overline{L}_O \cdot \hat{k} = \frac{16}{3}ml^2\dot{\varphi} + \frac{4}{3}ml^2\dot{\varphi}\sin^2\theta - 2ml^2\dot{\theta}\cos\theta}$$

Energía cinética: $T = T^{OA} + T^{AB}$

$$T^{OA} = \frac{2mL^2\dot{\varphi}^2}{3}$$

$$\bar{\omega}^{AB} = \begin{pmatrix} -\dot{\varphi}\cos\theta \\ -\dot{\theta} \\ \dot{\varphi}\sin\theta \end{pmatrix}$$

$$T^{AB} = \frac{1}{2}m\bar{v}_A^2 + \frac{1}{2}\bar{\omega}^{AB}\cdot\mathbb{I}_A\bar{\omega}^{AB} + m\bar{v}_A\cdot\left[\bar{\omega}^{AB}\times(\bar{r}_G-\bar{r}_A)\right]$$

$$= 2L^2\dot{\varphi}^2 m + \frac{2}{3}mL^2\dot{\theta}^2 + \frac{2}{3}mL^2\dot{\varphi}^2\sin^2\theta - 2mL^2\dot{\theta}\dot{\varphi}\cos\theta$$

$$\mathbb{I}_A^{(\hat{e}_r,\hat{e}_\varphi,\hat{e}_\theta)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4/3 & 0 \\ 0 & 0 & 4/3 \end{pmatrix} mL^2$$

$$\bar{v}_A = 2L\dot{\varphi}\hat{e}_\varphi$$

$$\mathbb{I}_A\bar{\omega}^{AB} = \begin{pmatrix} 0 \\ -\frac{4}{3}mL^2\dot{\theta} \\ \frac{4}{3}mL^2\dot{\varphi}\sin\theta \end{pmatrix}$$

$$\bar{r}_{G_{AB}} - \bar{r}_A = L\hat{e}_r$$

$$\bar{\omega}^{AB}\times(\bar{r}_G-\bar{r}_A) = -\dot{\theta}L\underbrace{\hat{e}_\varphi\times\hat{e}_r}_{-\hat{e}_\theta} + \dot{\varphi}L\sin\theta\underbrace{\hat{e}_\theta\times\hat{e}_r}_{\hat{e}_\varphi}$$

$$m\bar{v}_A\cdot\left[\bar{\omega}^{AB}\times(\bar{r}_G-\bar{r}_A)\right] = -2mL^2\dot{\theta}\dot{\varphi}\cos\theta$$

$$T = \frac{2mL^2\dot{\varphi}^2}{3} + 2L^2\dot{\varphi}^2 m + \frac{2}{3}mL^2\dot{\theta}^2 + \frac{2}{3}mL^2\dot{\varphi}^2\sin^2\theta - 2mL^2\dot{\theta}\dot{\varphi}\cos\theta$$

$$T = \frac{8}{3} m l^2 \dot{\varphi}^2 + \frac{2}{3} m l^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta \right) - 2 m l^2 \dot{\theta} \dot{\varphi} \cos\theta$$

c) $\dot{\varphi}(\theta = 0)$ $\qquad \varphi_0 = \pi/2 \qquad \dot{\varphi}_0 = 0$

$\dot{\theta}_0 = 0$

$$U = -mgl\cos\theta$$

$$E = T + U = E(0) = 0$$

$$L_0 \cdot \hat{k} = L_0 \cdot \hat{k}(0) = 0$$

$$\rightarrow \begin{cases} \frac{8}{3} m l^2 \dot{\varphi}^2 + \frac{2}{3} m l^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta \right) - 2 m l^2 \dot{\theta} \dot{\varphi} \cos\theta - mgl\cos\theta = 0 \\ \frac{16}{3} m l^2 \dot{\varphi} + \frac{4}{3} m l^2 \dot{\varphi} \sin^2\theta - 2 m l^2 \dot{\theta} \cos\theta = 0 \end{cases}$$