8

Introduction to Network Design and Preanalysis

CHAPTER MENU

Objectives
After studying this chapter, you should be able to:

1) Discuss the problems of network design.
2) Explain different design variables and how they relate to each other, including their uses and importance.
3) Perform simple preanalysis of survey observations.
4) Perform network design and simulation involving one-dimensional, two-dimensional, and three-dimensional cases.

8.1 Introduction

Network design is about selecting observables to measure, measurement procedures, instrumentation, etc. for a project in order to achieve the specific goals of the project. Network design methods have evolved over time to solve

Understanding Least Squares Estimation and Geomatics Data Analysis, First Edition.
John Olusegun Ogundare.

Companion website: www.wiley.com/go/ogundare/Understanding-lse-and-gda

different cases of network design problems, which can be summarized as follows (Grafarend 1974; Vanicek and Krakiwsky 1986; Cooper 1987):

1) *Zero-order design (ZOD) problem* or *datum problem*. In this problem, a suitable reference frame or a coordinate system is selected (or fixed) for the yet to be determined unknown coordinate parameters and their covariance matrices; this is about what points or lines to fix for the network adjustment. This is important in parametric model and general model least squares adjustment, where the network points fixed for datum definition also constitute the *zero-variance referenc*e base for adjustment.
2) *First-order design (FOD) problem* or *configuration problem*. In this problem, a suitable geometric layout or configuration (first design matrix or A-matrix) of network observables to be measured and the locations of network points must be determined based on the given covariance matrices of the unknown parameters and the measurements. This configuration must be selected with a consideration for intervisibility between network points and the nature of the topography of the project site.
3) *Second-order design (SOD) problem* or *generalized weight problem*. This problem requires determining the covariance matrix (C_ℓ) of measurements of the observables selected in the FOD solution (based on the given configuration or first design matrix (A-matrix) and the covariance matrix (C_x) of the unknown parameters). Since precisions of measurements are related to instruments and observation procedures (including the number of repetitions of a measurement), the SOD problem can be seen as a problem of selecting suitable instrument and observation procedures.
4) *Third-order design (ThOD) problem* or *densification problem*. This problem involves selecting observables, measurements, and weights for the purpose of improving an existing network. It can be seen as a problem of selecting how to best connect or integrate a new survey to an existing one (e.g. a national survey), which may involve considering the estimated coordinates of the existing survey and their covariances as a priori values in an adjustment.

Another case of design problem is described by Grafarend et al. (1979) as *combined design* (COMD) problem, where optimal solution to FOD and SOD problems is determined simultaneously, i.e. the network configuration (A-matrix) and the covariance matrix (C_ℓ) of measurements are determined, given the covariance matrix (C_x) of the unknown parameters. When a design is performed with consideration for the available instruments, most economic field survey, intervisibility of survey points, ability of network to allow identification and elimination of gross errors in observations, and the effects of undetected gross errors in observations on the network, in achieving a minimum value of a set objective function, the design is said to be *optimized*. An objective function to be maximized or minimized within some constraints could be reliability and sensitivity; in this case, the optimized design will produce the most reliable and sensitive network possible under the constraints of instrumentation, time, cost, etc. The *optimization of a design* is to achieve the best design in an analytical way (e.g. Grafarend 1974; Cross 1985; Kuang 1991). In his analytical approach, Kuang (1991) used the so-called multiobjective optimization method to solve all the cases of network design problems in a single mathematical way. Since the analytical approach is difficult to understand and implement, it is common to design networks that are just acceptable in terms of precision, reliability, cost, etc., but not necessarily the best.

As discussed above, the main variables of a design are reference frame, configuration, and observation precisions (or weights). Design and preanalysis allow one to experiment with these variables in the process of trying to meet a given accuracy specification for a project. According to Vanicek and Krakiwsky (1986), the main objective of preanalysis is to come up with a set of guidelines on what observables to measure and the acceptable accuracy of those measurements, given the expected tolerance limits of the unknown parameters. In this case, preanalysis will determine optimal accuracy (or precisions) of measurements by solving FOD, SOD, or COMD problems.

The variables selected in a survey network design usually depend on the type of network involved: *simple* or *complex*. Simple networks may have observation precisions as the only variables, while complex networks, which usually require least squares adjustment, may have variables that include reference frame, configuration, and observation precisions. After the initial selection of values for these variables (initial design), these variables are changed in the process of achieving an optimal results by the procedure of *preanalysis* or *simulation*. Simple network (survey) design will require simple preanalysis, while complex network design will require complex preanalysis. In this chapter, simple survey design and preanalysis will be referred to as simply *preanalysis of survey observations*; complex network design (or network design) and preanalysis will be treated under *network design* with the preanalysis aspect treated under *network simulation*.

8.2 Preanalysis of Survey Observations

Preanalysis of survey observations discussed in this section is about analyzing simple survey observations for a project before the project is actually started. A simple network design involved in this section is the SOD type with the variables being the precisions of observations. This means that preanalysis of survey is done in order to determine precisions of observations that satisfy a specified tolerance limit for the unknown quantities to be determined. At the end of a preanalysis, it may be concluded by the surveyor that the requirements for the accuracies of measurements are within or beyond what can be achieved. If the requirements are beyond the capabilities of the surveyor, the client must be told that the survey tolerance limits specified are beyond what can be satisfied.

8.2.1 Survey Tolerance Limits

Survey tolerance limits are the intervals within which the maximum allowable error of observation must fall. Based on the concepts of interval estimation discussed in Section 7.3.2, Equation (7.17) or Equation (7.18) can be used to determine the survey tolerance. Usually, at the preanalysis stage, Equation (7.20) should be used, such that the survey tolerance will be given as $z_{1-\alpha/2}$(SE). The most commonly used uncertainty for survey tolerance is at a probability of 99.7% or $\alpha = 0.003$. Using Equation (7.20) with $z_{1-0.003/2} = 3.0$, the survey tolerance or maximum error acceptable can be given as 3(SE) or three times the standard error (SE) of measurement. Note that if the error of a single measurement is of interest, the SE will be taken as the standard deviation of the single measurement. For example, a measurement is said to meet its specified tolerance if its standard deviation times three (3σ) is less than the given tolerance. For example, if the tolerance of ±15 mm is allowed in a measurement, the allowable standard deviation should be ±5 mm.

8.2.2 Models for Preanalysis of Survey Observations

The mathematical models for performing preanalysis of survey observations relate with the laws of random error (variance–covariance) propagation in reversed form. In this case the mathematical model relating the unknown parameter (x) is first formulated as a function of the expected measurements (ℓ) in the form of $x = f(\ell)$. The usual variance–covariance propagation laws discussed in Chapter 2 are then applied to the functional model with the covariance matrix of observations considered unknown to be solved for. A simple example can be used to illustrate the model with reference to variance–covariance propagation laws in Chapter 2. For example, consider a simple case where the total random error expected in the measurement of a 500 m distance is to be 16 mm. The expected random error in each of the 50 m tape measurement can be determined when the tape is used to measure the 500 m distance as follows. This is simply an error propagation problem in reversed form. By using 50 m tape, the total distance $D = 500$ m will have to be measured in 10 segments (with each segment $d_i = 50$ m). This can be expressed mathematically as follows:

$$D = d_1 + d_2 + \cdots + d_{10} \tag{8.1}$$

Equation (8.1) is a form of a model $x = f(\ell)$ given in Equation (2.1), where in this problem, $x = D$ is the parameter whose error propagation is to be made and the vector of observations is $\ell = [d_1 \ d_2 \ \cdots \ d_{10}]^T$. Applying the variance–covariance propagation rules (referring to Equation (2.43) or (2.44)) to Equation (8.1) and assuming zero correlation between them,

$$\sigma_{\mathrm{D}}^2 = \left(\frac{\partial D}{\partial d_1}\right)^2 \sigma_{d_1}^2 + \left(\frac{\partial D}{\partial d_2}\right)^2 \sigma_{d_2}^2 + \cdots + \left(\frac{\partial D}{\partial d_{10}}\right)^2 \sigma_{d_{10}}^2 \tag{8.2}$$

where, for example, $\frac{\partial D}{\partial d_1}$ is the partial derivative of the parameter D with respect to the observation d_1. There is an additional assumption necessary in order to evaluate Equation (8.2), which is to assume that all observations will contribute the same amount of error to the overall error for the parameter (assuming *balanced accuracy of measurements*). Note that balanced accuracy of measurements is used to mean each term in the variance–covariance propagation for the unknown measurements will have equal contribution to the given variance of the unknown parameter. This can be expressed mathematically as

$$\left(\frac{\partial D}{\partial d_1}\right)^2 \sigma_{d_1}^2 = \left(\frac{\partial D}{\partial d_2}\right)^2 \sigma_{d_2}^2 = \cdots = \left(\frac{\partial D}{\partial d_{10}}\right)^2 \sigma_{d_{10}}^2 = \frac{\sigma_{\mathrm{D}}^2}{10} \tag{8.3}$$

As can be seen in Equation (8.3), each component in the variance–covariance propagation in Equation (8.2) is equated to the square of overall error divided by the number of observations (10) involved. All the partial derivatives of the Equation (8.1) are ones; substituting the partial derivatives and $\sigma_{\mathrm{D}} = 16$ mm into Equation (3.37b) gives

$$(1)^2 \sigma_{d_1}^2 = \frac{\sigma_{\mathrm{D}}^2}{10} \rightarrow \sigma_{d_1} = \sqrt{\frac{256}{10}}\,\mathrm{mm} \quad \text{or} \quad \pm 5\,\mathrm{mm} \tag{8.4}$$

which is the same for the remaining observations. The standard deviation of each 50 m tape measurement must be ±5 mm for the total random error of 16 mm to be achieved in the measurement of the 500 m distance. The following example is given to further illustrate how simple preanalysis of survey observations can be done.

Example 8.1 For visible and near-infrared radiation and neglecting the effects of water vapor pressure, the formula for computing the refractive index, n, in an EDM can be determined by

$$n - 1 = \frac{0.269\,578[n_0 - 1]}{273.15 + t} p$$

where n_0 is the constant refractive index set in the EDM, t is the temperature in °C, p is the pressure in mbar, and n is the realistic refractive index to be determined. The EDM has a set constant value, n_0 = 1.000 294 497; and the average temperature and pressure during the measurements are expected to be +30 °C and 950 mb, respectively. Assuming the standard deviation of measuring temperature, σ_t = 1.0 °C, what would be the largest value of σ_p so that the error in σ_n will not be more than 2 ppm?

Solution:

By variance–covariance propagation laws in Chapter 2,

$$J = \begin{bmatrix} \frac{\partial n}{\partial p} & \frac{\partial n}{\partial t} \end{bmatrix} \quad J = \begin{bmatrix} \frac{0.269578[n_0 - 1]}{273.15 + t} & -\frac{0.269578[n_0 - 1]p}{(273.15 + t)^2} \end{bmatrix}$$

$$\sigma_t = 1.0°\text{C},\ n_0 = 1.000\,294\,497;\ \ t = +30°\text{C};\ \ p = 950\ \text{mb},$$

or

$$J = [2.618\,833\text{E}-7 \quad -8.206\,798\,7\text{E}-7]$$

Covariance matrix (C) of measurements and variance–covariance propagation:

$$C = \begin{bmatrix} (1)^2 & \\ & \sigma_t^2 \end{bmatrix} \quad \sigma_n^2 = JCJ^T$$

or

$$\sigma_n^2 = \left(\frac{0.269\,578[n_0 - 1]}{273.15 + t}\right)^2 \sigma_p^2 + \left(\frac{0.269\,578[n_0 - 1]}{(273.15 + t)^2} P\right)^2 \sigma_t^2$$

$$(2.0\text{E}-6)^2 = (2.618\,832\,665\,87\text{E}-7\sigma_p)^2 + (8.206\,798\,722\,02\text{E}-7\sigma_t)^2$$

or

$$(2.0\text{E}-6)^2 = (8.206\,798\,722\,02\text{E}-7(1))^2 + (2.618\,832\,665\,87\text{E}-7\sigma_p)^2$$

$$(2.0\text{E}-6)^2 - (8.206\,798\,722\,02\text{E}-7(1))^2 = (2.618\,832\,665\,87\text{E}-7\sigma_p)^2$$

$$\sigma_p = \sqrt{\frac{3.326\,484\,547\,36\text{E}-12}{6.858\,284\,531\,83\text{E}-14}} = \sigma_p = \sqrt{48.503\,157\,5} \ \text{ or } \ 6.96\ \text{mb}$$

The largest value of the pressure so that the error in refractive index will not be more than 2 ppm is 6.96 mb.

8.2.3 Trigonometric Leveling Problems

The elevation difference when leveling between backsight point A and foresight point B with the total station instrument set at the midpoint based on leapfrog trigonometric leveling procedure (assuming the effect of earth curvature is negligible) can be given as

$$\Delta h = (S_B \cos Z_B - S_A \cos Z_A) - \frac{1}{2R}\left(k_B S_B^2 \sin^2 Z_B - k_A S_A^2 \sin^2 Z_A\right) \tag{8.5}$$

where S_A and S_B are the backsight and foresight slope distances to points A and B, respectively; Z_A and Z_B are the zenith angle measurements to points A and B, respectively; k_A and k_B are the coefficients of refraction to A and B, respectively; and $R = 6371$ km is the radius of the earth. Assume the average slope of the terrain (which is covered with the same material) is 10° (or zenith angle of 80°), $\sin Z_A = \sin Z_B$, $\cos Z_A = -\cos Z_B$, $\Delta k = k_B - k_A$, or $\Delta k = 0.3$; the average sight lengths (with $S_A = S_B$) are to be 250 m, and the deflection of the vertical at the two stations are assumed negligible. Simplify Equation (8.5) based on some of the assumptions given above, and determine the standard deviations of the zenith angle and the slope distance measurements and the standard deviation of the difference in the coefficient of refraction so that the standard deviation of the elevation difference at this setup will be less than 2 mm.

Given $S_A = S_B$, $\sin Z_A = \sin Z_B$, $\cos Z_A = -\cos Z_B$, $k_A - k_B = 0.3$, substitute them into Equation (8.5):

$$\Delta h = 2S\cos Z - \frac{(\Delta k)S^2 \sin^2 Z}{2R} \tag{8.6}$$

By error propagation law,

$$\sigma_{\Delta h}^2 = \left(\frac{\partial \Delta h}{\partial S}\right)^2 \sigma_S^2 + \left(\frac{\partial \Delta h}{\partial Z}\right)^2 \sigma_Z^2 + \left(\frac{\partial \Delta h}{\partial \Delta k}\right)^2 \sigma_{\Delta k}^2 \tag{8.7}$$

For $S = 250$ m, $Z = 80°$, $\sigma_S = ?$, $\sigma_Z = ?$

$$\frac{\partial \Delta h}{\partial S} = 2\cos Z - \frac{2(\Delta k)S\sin^2 Z}{2R}$$

$$= 0.347\,296\,355\,334 - 1.902\,853\,379\,36\text{E}-5 \quad (\text{or } 0.347\,277\,326\,8)$$

$$\frac{\partial \Delta h}{\partial Z} = -2S \times \sin Z - \frac{(\Delta k)S^2 \sin 2Z}{2R} -$$

$$= -492.403\,876\,506 - 8.388\,109\,777\,9\text{E}-4 \quad (\text{or } -492.404\,715\,317)$$

$$\frac{\partial \Delta h}{\partial \Delta k} = \frac{S^2 \sin^2 Z}{2R} = 4.757\,133\,448\,4\text{E}-3$$

If it is assumed that the measurements contribute equally to the overall accuracy, then

$$\left(\frac{\partial \Delta h}{\partial S}\right)^2 \sigma_S^2 = \left(\frac{\partial \Delta h}{\partial Z}\right)^2 \sigma_Z^2 = \left(\frac{\partial \Delta h}{\partial \Delta k}\right)^2 \sigma_{\Delta k}^2 = \frac{\sigma_{\Delta h}^2}{3}$$

$$\sigma_{\Delta h} = 2 \text{ mm}$$

$$(0.347\,277\,326\,8)^2 \times \sigma_S^2 = \frac{2^2}{3} \quad \rightarrow \quad \sigma_S = \pm 3.3 \text{mm}$$

$$(-492.404\,715\,317)^2 \times \sigma_Z^2 = \frac{2^2}{3} \quad \rightarrow \quad \sigma_Z = 5.499\,134\,4\text{E}-6 \text{ rad (or } 1.1''\text{)}$$

$$(4.757\,133\,448\,4\text{E}-3)^2 \times \sigma_{\Delta k}^2 = \frac{2^2}{3} \quad \rightarrow \quad \sigma_{\Delta k} = 242.730$$

Example 8.2 A slope distance D and a zenith angle Z must be measured in order to calculate an elevation difference h. What should be the accuracy of D and Z in order to obtain h with a standard deviation $\sigma_h < 5$ mm? Assume that $D = 500$ m and $Z = 60°$.

Solution:

Equation for calculating the elevation difference h:

$$h = D \cdot \cos Z \tag{8.8}$$

Applying the variance–covariance propagation rules to Equation (8.8) with h as a function of D and Z and assuming zero correlation between D and Z,

$$\sigma_h^2 = \left(\frac{\partial h}{\partial D}\right)^2 \sigma_D^2 + \left(\frac{\partial h}{\partial Z}\right)^2 \sigma_Z^2 \tag{8.9}$$

$$\sigma_h^2 = \left(\cos^2 Z\right)\sigma_D^2 + \left(D^2 \sin^2 Z\right)\sigma_Z^2 \tag{8.10}$$

Substituting $Z = 60°$ and $D = 500\,000$ mm so that the right-hand side result will be in the same unit (mm^2) as the left-hand side of the equal sign and assuming σ_z will be in radians, the following is obtained:

$$\sigma_h^2 = 0.25\sigma_D^2 + 1.875\text{E}^{11}\sigma_z^2 \ \left(\text{mm}^2\right) \tag{8.11}$$

where σ_D and σ_z are the standard deviations of the slope distance and zenith angle measurements, respectively (which are unknown to be determined). Note that the distance D is converted to millimeters and the random error in zenith is expressed in radians for the purpose of making the unit in the equation consistent. Assuming that the distance and the angular measurements contribute the same amount of error into the elevation difference calculation, each of the terms of the Equation (8.11) will contribute $\frac{\sigma_h^2}{2} = 12.5\,\text{mm}^2$, giving the following:

$$0.25\sigma_D^2 = 12.5 \quad \rightarrow \quad \sigma_D = \sqrt{\frac{12.5}{0.25}} < 7 \text{ mm}$$

$$1.875\text{E}^{11}\sigma_z^2 = 12.5 \quad \rightarrow \quad \sigma_z = \sqrt{\frac{12.5}{1.875\text{E}^{11}}} < 8.165\text{E}^{-6} \text{ rad (or } 1.7'')$$

The elevation difference h can be measured to a standard deviation of less than 5 mm if the distance and zenith angle can be measured with standard deviations of less than 7 mm and 1.7″, respectively.

Example 8.3 A total station is to be used to measure the elevation difference between a setup point and another point Q as shown in Figure 8.1. In the process, the slope distance d_S, the zenith angle z, the height of instrument (HI), and the height of reflector (HR) will be measured in order to determine V (the simple height difference from the total station to the reflector). Answer the following.

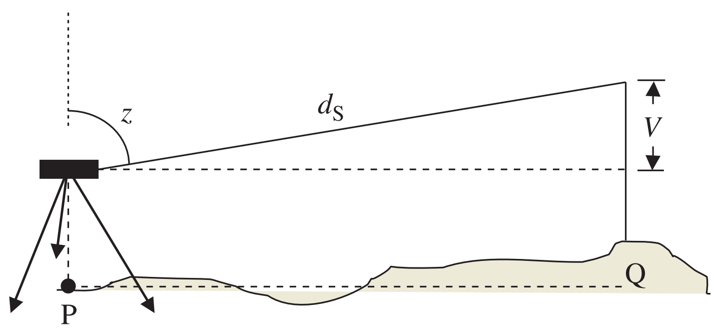


Figure 8.1 Illustration of the total station setup.

a) If the error in V (at 99.7% confidence level) is not to exceed ±15 mm, determine expected standard deviations in measuring the zenith angle (z), d_S, HI, and HR, assuming balanced accuracies with approximate values of these quantities as z = 100°, HI = 1.6 m, and d_S = 200.0 m.

Solution:

For the total station setup:
In one direction:

$$z = 100° \pm 5''; \ \text{HI} = 1.6\,\text{m} \pm 0.003; \ \ d_S = 200.0\,\text{m} \pm 0.003.$$

Elevation difference:

$$V = \text{HI} + d_S \cos z - \text{HR}$$

$$\sigma_V^2 = \sigma_{\text{HI}}^2 + (\cos z)^2 \sigma_{d_S}^2 + (d_S \sin z)^2 \sigma_z^2 + \sigma_{\text{HR}}^2 \quad (8.12)$$

The error in V at 99.7% confidence level is ±15 mm; the standard deviation can be derived from this using the relation that (SE) $z_{1-\alpha/2}$ = 15 mm; for $z_{1-0.003/2}$ = 3, the standard deviation (SE) can be given as

$$\sigma_{\mathrm{V}} = \frac{0.015}{3} = 0.005\ \mathrm{m}$$

Balancing the accuracies in Equation (8.12) gives

$$\sigma_{\mathrm{HI}}^2 = \frac{\sigma_{\mathrm{V}}^2}{4} \quad \rightarrow \quad \sigma_{\mathrm{HI}} = \frac{\sigma_{\mathrm{V}}}{2} = \frac{0.005}{2}\mathrm{m}\ (\text{or } 2.5\,\mathrm{mm})$$

Similarly, $\sigma_{\mathrm{HRI}} = \dfrac{\sigma_{\mathrm{V}}}{2} = 2.5\,\mathrm{mm}$

$$\sigma_{\mathrm{z}} = \frac{\sigma_{\mathrm{V}}}{2(d_{\mathrm{S}} \sin Z)} = \frac{0.005}{393.9231} = 1.2692833\mathrm{E}-5\ \mathrm{rad}\ (\text{or } 2.62'')$$

$$\sigma_{d_{\mathrm{S}}} = \frac{\sigma_{\mathrm{v}}}{2(\cos Z)} = \frac{0.005}{2(0.17365)}\mathrm{m} = 0.014\,\mathrm{m}$$

The standard deviations of measuring the zenith, slope distance, HI, and HR are 2.6″, 0.014, 0.0025, and 0.0025 m, respectively.

b) The technical specifications for Leica TCRA 702 total station instrument are as follows:
- Standard deviation for horizontal (HZ) and vertical (Z) angles (ISO 17123-3) is 2″.
- Standard deviation for distance measurement (ISO 17123-4) (IR fine mode) is 2 mm ± 3 ppm.

If the TCRA 702 instrument is used in (a) above, determine the expected standard deviations in measuring HI and HR, assuming balanced accuracies and the centering error of 2 mm each for the instrument and target centering procedures.

Solution:

The standard deviations of measuring the zenith and the distance are known from the given specifications; the only unknowns are the errors in HI and HR.

Standard deviation of slope distance (given):

$$\sigma_{d_{\mathrm{S}}} = \sqrt{2^2 + (3 \times 0.2)^2 + 2(2)^2} = 0.0035\,\mathrm{m}\ (\text{or } 3.5\,\mathrm{mm})$$

For zenith angle measurement (in one set), $\sigma_z = 2''$.
From error propagation in Equation (8.12),

$$\sigma^2 = (\cos 100)^2(0.0035)^2 + (200\sin 100)^2\left(\frac{2}{206\,265}\right)^2 + \sigma_{HR}^2 + \sigma_{HI}^2$$

$$= 3.693\,826\,98E-7 + 3.647\,302\,79E-6 + \sigma_{HR}^2 + \sigma_{HI}^2 \tag{8.13}$$

Again using the standard deviation of V from part (a) as $\sigma_V = 0.005$ m in Equation (8.13) and rearranging the terms give

$$(0.005)^2 - 4.016\,685\,483\,35E-6 = \sigma_{HI}^2 + \sigma_{HR}^2$$

$$\sigma_{HI}^2 + \sigma_{HR}^2 = 2.098\,331\,4E-5$$

For equal distribution from the remaining two components (HI and HR), the standard deviation of each component can be given as

$$\sigma_{HI}^2 = \frac{2.098\,331\,452E-5}{2} \quad \rightarrow \quad \sigma_{HR} = \sigma_{HI} = \sqrt{1.049\,165\,726} = \underline{\mathbf{0.003\ 2\,m}}$$

The HI and HR must be measured to an accuracy of 0.003 m in order to achieve an overall standard deviation of 0.005 m in height difference with TCRA 702 instrument used to measure the zenith and slope distance.

c) Continuing from (a) above, if the HI and HR will be measured to an accuracy of ±3 mm, and the Leica TCRA in (b) will be used to measure the slope distance (with centering error of 2 mm each in the instrument and target centering procedures), determine new expected standard deviation of measuring the zenith angle (z).

Solution:

Substitute $\sigma_{d_S} = 0.0035$ m, $\sigma_{HR} = \sigma_{HI} = \pm 3$ mm, and $\sigma_V = 0.005$ m into Equation (8.14) rearranged from Equation (8.12), and solve for σ_z directly:

$$\sigma_V^2 - \sigma_{HR}^2 - \sigma_{HI}^2 - (\cos z)^2\sigma_{d_S}^2 = (d_S \sin z)^2\sigma_z^2 \tag{8.14}$$

$$0.005^2 - 2(0.003)^2 - (0.173\,648)^2(0.003\,5)^2 = (d_S \sin z)^2\sigma_z^2$$

$$6.630\,617E-6 = (d_S \sin z)^2\sigma_z^2$$

$$\sigma_z = \frac{\sqrt{6.630\,617E-6}}{(d_S \sin z)} \quad \rightarrow \quad \sigma_z = \frac{2.575E-3}{(200\sin 100)}$$

$$\sigma_z = \frac{2.575E-3}{196.961\,55} = 1.307\,361E-5\,\text{rad}\,(\text{or}\,2.7'')$$

The new expected standard deviation of measuring the zenith angle (z) is 2.7″.

8.3 Network Design Model

In the design of a complex network to be adjusted by the method of least squares, the procedure for simple preanalysis given in Section 8.3 cannot be applied. A more rigorous design model (variance–covariance matrix of adjusted parameters) from the parametric least squares adjustment must be used as discussed in this section. The variance–covariance matrix of adjusted parameters can be expressed as follows (referring to Section 5.8):

$$C_{\hat{x}} = \sigma_0^2\left(A^T PA\right)^{-1} \tag{8.15}$$

where $C_{\hat{x}}$ is the given variance–covariance matrix desired for the adjusted parameters, A is the first design matrix (or the network configuration discussed in Section 8.2) that can be determined from the approximate coordinates of the network, P is the weight matrix of measurements containing the standard deviations of individual measurements, and σ_0^2 is the a priori variance factor of unit weight (if unknown, the a posteriori variance factor of unit weight s_0^2 should be used). For network design, $\sigma_0^2 = 1$ is to be used; and weight matrix of observations P is the inverse of the covariance matrix (C_ℓ^{-1}) of the observations. Equation (8.15) can then be rewritten as

$$C_{\hat{x}} = \left(A^T C_\ell^{-1} A\right)^{-1} \tag{8.16}$$

Equation (8.16) expresses generally the network design problem. If inverted, Equation (8.16) will provide the covariance matrix of observations (C_ℓ), from which the standard deviations of the corresponding measurements can be extracted from its principal diagonal. Direct inversion of this type of covariance matrix, which is usually done by mathematical programming approach, is beyond the scope of this book. The approach adopted in this book is by using indirect method of inversion with computer software based on trial-and-error procedure known as simulation. If the network involved is simple enough, as in the case of one-dimensional network (e.g. leveling network), a direct inversion to determine the covariance matrix of the observations may be less complicated. Examples of simple network design are provided in the following sections.

8.4 Simple One-dimensional Network Design

With reference to Equations (7.13) and (7.14) in Section 7.3.1, the value ($a_{1-\alpha/2}$) of the *margin of error* at $(1-\alpha)$% confidence level can be derived from the variance–covariance matrix of the adjusted parameters. Assuming s is the unknown standard deviation of a measurement (which is part of the covariance matrix of the adjusted parameters, through the covariance matrix of observations, C_ℓ), one can work forward from Equation (8.16) to obtain the formulas for estimating the margins of errors at $(1 - \alpha)\%$ confidence level as follows. After least squares adjustment of a leveling network, for an example, the value of the margin of error at $(1 - \alpha)\%$ can be given as follows (refer to Section 7.3.1 for further discussion on this):

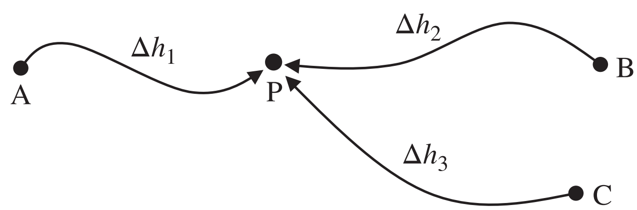
$$a_{1-\alpha} = \pm(\text{SE})z_{1-\alpha/2} \tag{8.17}$$

or

$$a_{1-\alpha} = \pm(\text{SE})t_{1-\alpha/2,\text{df}} \tag{8.18}$$

where the SE in this case is the standard deviation of the adjusted parameter, $z_{1-\alpha/2}$ is the value from the standard normal distribution curve, and $t_{1-\alpha/2,\ \text{df}}$ is from the Student's t-distribution curve. Equation (8.17) should be used when the degrees of freedom of the adjustment is greater than 30; otherwise Equation (8.18) is used. In this case, the square of the calculated SE will be used in the covariance matrix ($C_{\hat{x}}$) of the unknown parameter in Equation (8.16). For example, consider a leveling network in Figure 8.2 where A, B, and C are control points with known heights and Δh_1, Δh_2, and Δh_3 are the three height difference measurements with standard deviations of σ_1, σ_2, and σ_3, respectively. If the relationship among the standard deviations is such that $\sigma_1 = \sigma_2$ and $\sigma_1 = 3\sigma_3$, determine the values of σ_1, σ_2, and σ_3 so that the margin of error at 95% confidence level for the height solution for point P using least adjustment is equal to 8.6 mm.

Figure 8.2 Leveling network.



Since the degrees of freedom in this problem is df < 30, and $s_{\hat{h}_P}$ was computed with degrees of freedom df = 2, the Student's t-value should be used in Equation (8.18):

Given $t_{0.975,\ \text{df}\ =\ 2} = 4.303$, SE $= s_{\hat{h}_P}$, $\alpha = 0.05$, $a_{95\%} = 8.6$ mm, substituting the given quantities into Equation (8.18) gives the following:

$$8.6 = s_{\hat{h}_P} \times 4.303, \text{then}$$

$$s_{\hat{h}_P} = \frac{8.6}{4.303} \quad \rightarrow \quad 2\,\text{mm} \tag{8.19}$$

$$\text{Unknown parameter} = [\text{Elevation of point P}] \quad \text{or} \quad x = [h_P]$$

$$\text{Vector of observations: } \ell = \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \end{bmatrix}$$

Since the problem is based on parametric least squares adjustment, formulate the parametric equations: $\hat{\ell} = f(\hat{x})$

$$\begin{aligned}\Delta h_1 &= h_P - h_A \\ \Delta h_2 &= h_P - h_B \\ \Delta h_3 &= h_P - h_C\end{aligned} \tag{8.20}$$

Form the first design matrix (A) from Equation (8.20):

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial \Delta h_1}{\partial h_P} \\ \frac{\partial \Delta h_2}{\partial h_P} \\ \frac{\partial \Delta h_3}{\partial h_P} \end{bmatrix} \qquad A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{8.21}$$

Form the weight matrix, P.
Actual covariance matrix will be

$$C_\ell = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_2^2 \end{bmatrix} \tag{8.22}$$

Unify the covariance matrix by substituting $\sigma_1 = \sigma_2$ and $\sigma_1 = 3\sigma_3$:

$$C_\ell = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_1^2 & \\ & & \frac{\sigma_1^2}{9} \end{bmatrix} \rightarrow C_\ell^{-1} = \frac{1}{\sigma_1^2} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 9 \end{bmatrix}$$

Form the $A^T C_\ell^{-1} A$ – Matrix:

$$A^T C_\ell^{-1} A = \frac{1}{\sigma_1^2}[11] \tag{8.23}$$

Form the covariance matrix of the adjusted parameters $C_{\hat{x}} = \left(A^T C_\ell^{-1} A\right)^{-1}$:

$$C_{\hat{h}_P} = \left(A^T C_\ell^{-1} A\right)^{-1} \quad \rightarrow \quad C_{\hat{h}_P} = \left(\frac{1}{\sigma_1^2}[11]\right)^{-1}$$

$$C_{\hat{h}_P} = \sigma_1^2[0.091] \tag{8.24}$$

Note that $C_{\hat{h}_P} = s_{\hat{h}_P}^2$. Taking the value of $s_{\hat{h}_P}^2$ from Equation (8.19) and substituting into Equation (8.24) gives

$$4\,\text{mm}^2 = \sigma_1^2[0.091] \quad \rightarrow \quad \sigma_1 = \sqrt{44} = 6.63 \text{ mm}$$

Since $\sigma_1 = \sigma_2$ and $\sigma_1 = 3\sigma_3$,

$$\sigma_2 = 6.63 \text{ mm and } \sigma_3 = \frac{1}{3} \times 6.63 = 2.21 \text{ mm}$$

8.5 Simple Two-dimensional Network Design

Consider a case where the coordinates of control points P and Q are known as P (x_P = 1000.000 m, y_P = 1000.000 m) and Q (x_Q = 1500.000 m, y_Q = 1000.000 m), respectively, and the surveyor is to design a measurement scheme by preanalysis to determine the coordinates of point R. The following four options are to be considered in the preanalysis and the best network chosen based on the one that produces an SE ellipse with the smallest semi-major (assuming the approximate coordinates of point R are x_R = 1250 m, y_R = 1200 m; and an angle can be measured to a precision of 5″, an azimuth to a precision of 3″, and a distance to a precision of 0.003 m):

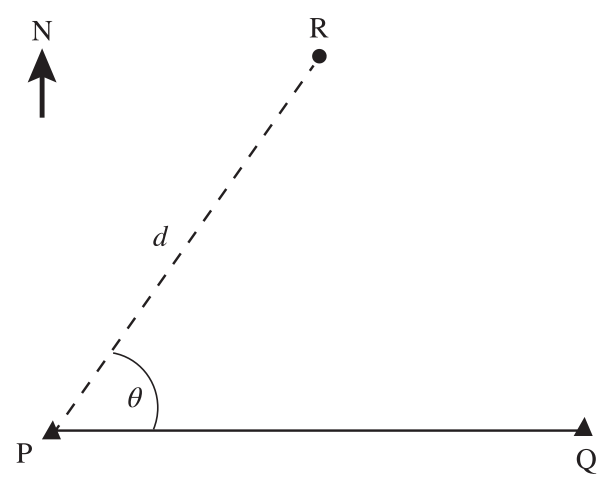
a) Measure angle θ and distance d as shown in Figure 8.3 to establish point R.
b) Measure all the angles θ_1, θ_2, and θ_3, as shown in Figure 8.4 to establish point R.
c) Measure the azimuths Az_1 and Az_2 from points P and Q, respectively, to point R, as shown in Figure 8.5.
d) Measure distances d_1 and d_2 from points P and Q, respectively, to point R, as shown in Figure 8.6.

Option a: Angle θ and distance d are measured according to Figure 8.3.

The parametric model equations can be formulated for the two measurements as follows:

$$d = \sqrt{(x_R - x_P)^2 + (y_R - y_P)^2} \tag{8.25}$$

Figure 8.3 Angle and distance measurements in design option a.



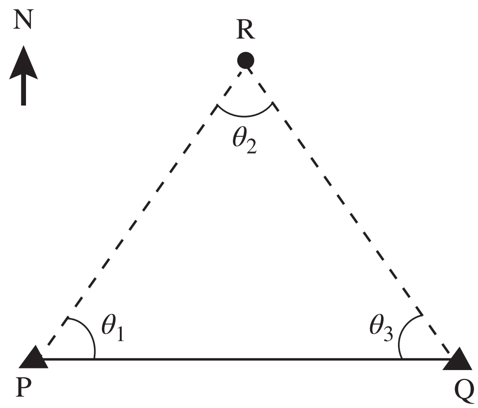


Figure 8.4 Angle measurements only in design option b.

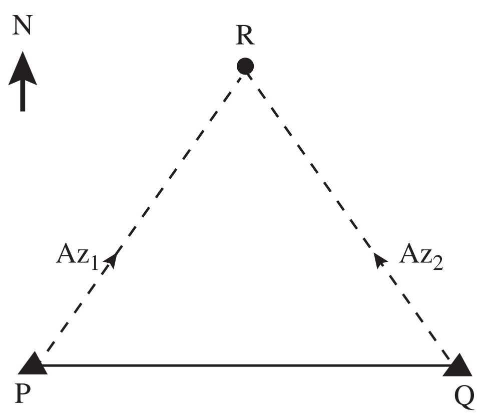


Figure 8.5 Azimuth measurements only in design option c.

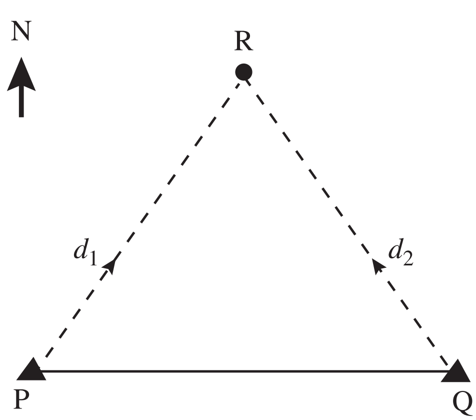


Figure 8.6 Distance measurements only in design option d.

$$\theta = \frac{\pi}{2} - \operatorname{atan}\left(\frac{x_R - x_P}{y_R - y_P}\right) \tag{8.26}$$

The partial derivatives of Equations (8.25) and (8.26) with respect to coordinates (x_R, y_R) can be given as

$$\frac{\partial d}{\partial x_R} = \frac{-(x_P - x_R)}{d_{PR}^0} = 0.7809; \quad \frac{\partial d}{\partial y_R} = \frac{-(y_P - y_R)}{d_{PR}^0} = 0.6247$$

$$\frac{\partial \theta}{\partial x_R} = \frac{(y_P - y_R)}{\left(d_{PR}^0\right)^2} = -0.0020; \quad \frac{\partial \theta}{\partial y_R} = \frac{(x_P - x_R)}{\left(d_{PR}^0\right)^2} = 0.0024$$

where $d_{PR}^0 = 320.1562$ m is the approximate distance from point P to point R based on the given approximate coordinates of point R and the known coordinates of point P.

The first design matrix A can be given from the above partial derivatives as

$$A = \begin{bmatrix} 0.7809 & 0.6247 \\ -0.0020 & 0.0024 \end{bmatrix} \tag{8.27}$$

The weight matrix (P) based on the standard deviations of the distance and angle (in radians) measurements can be given as

$$P = \begin{bmatrix} 1.000\text{E}5 & 0.0 \\ 0.0 & 1.702\text{E}9 \end{bmatrix} \tag{8.28}$$

The covariance matrix (C_x) of the parameters (x_R, y_R), assuming the a priori variance factor is one, can be given as

$$C_x = \left(A^T P A\right)^{-1} = \begin{bmatrix} 0.2899 & -0.2499 \\ -0.2499 & 0.4024 \end{bmatrix} \times \text{E}-4\,\text{m}^2 \tag{8.29}$$

The maximum and minimum eigenvalues of C_x are λ_1 = 6.023E-5 m^2 and λ_2 = 9.00E-6 m^2, respectively; the parameters of the SE ellipse are semi-major axis value, a = 0.0078 m; semi-minor axis value, b = 0.003 m; and the orientation of the semi-major axis, $\beta = 321°20'$.

Option b: Measure all the angles θ_1, θ_2, and θ_3, as shown in Figure 8.4 to establish point R.

The parametric model equations can be formulated for the three measurements as follows:

$$\theta_1 = \frac{\pi}{2} - \operatorname{atan}\left(\frac{x_R - x_P}{y_R - y_P}\right) \tag{8.30}$$

$$\theta_2 = \operatorname{atan}\left(\frac{x_P - x_R}{y_P - y_R}\right) - \operatorname{atan}\left(\frac{x_Q - x_R}{y_Q - y_R}\right) \tag{8.31}$$

$$\theta_3 = \frac{\pi}{2} + \operatorname{atan}\left(\frac{x_R - x_Q}{y_R - y_Q}\right) \tag{8.32}$$

The partial derivatives of Equations (8.30)–(8.32) with respect to coordinates (x_R, y_R) can be given as

$$\frac{\partial \theta_1}{\partial x_R} = \frac{(y_P - y_R)}{\left(d_{PR}^0\right)^2} = -0.0020; \quad \frac{\partial \theta_1}{\partial y_R} = \frac{(x_P - x_R)}{\left(d_{PR}^0\right)^2} = 0.0024$$

$$\frac{\partial \theta_2}{\partial x_R} = \frac{(y_Q - y_R)}{\left(d_{RQ}^0\right)^2} - \frac{(y_P - y_R)}{\left(d_{PR}^0\right)^2} = 0.000;$$

$$\frac{\partial \theta_2}{\partial y_R} = \frac{(x_P - x_R)}{\left(d_{PR}^0\right)^2} - \frac{(x_Q - x_R)}{\left(d_{RQ}^0\right)^2} = -0.0049$$

$$\frac{\partial \theta_3}{\partial x_R} = \frac{-(y_Q - y_R)}{\left(d_{RQ}^0\right)^2} = 0.0020; \quad \frac{\partial \theta_3}{\partial y_R} = \frac{(x_Q - x_R)}{\left(d_{RQ}^0\right)^2} = 0.0024$$

where $d_{RQ}^0 = 320.1562$ m is the approximate distance from point R to point Q based on the given approximate coordinates of point R and the known coordinates of point Q. The first design matrix A can be given from the above partial derivatives as

$$A = \begin{bmatrix} -0.0020 & 0.0024 \\ 0.0 & -0.0049 \\ 0.0020 & 0.0024 \end{bmatrix} \tag{8.33}$$

The weight matrix (P) based on the standard deviations of the angle (in radians) measurements can be given as

$$P = \begin{bmatrix} 1.7018E9 & & \\ & 1.7018E9 & \\ & & 1.7018E9 \end{bmatrix} \tag{8.34}$$

The covariance matrix (C_x) of the parameters (x_R, y_R), assuming the a priori variance factor is one, can be given as

$$C_x = \left(A^T P A\right)^{-1} = \begin{bmatrix} 0.7717 & 0.0 \\ 0.0 & 0.1646 \end{bmatrix} \times \mathrm{E}-4 \text{ m}^2 \tag{8.35}$$

The maximum and minimum eigenvalues of C_x are λ_1 = 7.717E-5 m^2 and λ_2 = 1.646E-5 m^2, respectively; the parameters of the SE ellipse are semi-major axis value, a = 0.0088 m; semi-minor axis value, b = 0.0041 m; and the orientation of the semi-major axis, $\beta = 90°00'$.

Option c: Measure the azimuths Az_1 and Az_2 from points P and Q, respectively, to point R, as shown in Figure 8.5.

The parametric model equations can be formulated for the two measurements as follows:

$$Az_1 = \text{atan}\left(\frac{x_R - x_P}{y_R - y_P}\right) \tag{8.36}$$

$$Az_2 = 2\pi + \text{atan}\left(\frac{x_R - x_Q}{y_R - y_Q}\right) \tag{8.37}$$

The partial derivatives of Equations (8.36)–(8.37) with respect to coordinates (x_R, y_R) can be given as

$$\frac{\partial Az_1}{\partial x_R} = \frac{-(y_P - y_R)}{\left(d^0_{PR}\right)^2} = 0.001\,951\,2; \quad \frac{\partial Az_1}{\partial y_R} = \frac{(x_P - x_R)}{\left(d^0_{PR}\right)^2} = -0.002\,439\,0$$

$$\frac{\partial Az_2}{\partial x_R} = \frac{-(y_Q - y_R)}{\left(d^0_{RQ}\right)^2} = 0.001\,951\,22; \quad \frac{\partial Az_2}{\partial y_R} = \frac{(x_Q - x_R)}{\left(d^0_{RQ}\right)^2} = 0.002\,439\,02$$

The first design matrix A can be given from the above partial derivatives as

$$A = \begin{bmatrix} 0.001\,951\,2 & -0.002\,439\,0 \\ 0.001\,951\,22 & 0.002\,439\,02 \end{bmatrix} \tag{8.38}$$

The weight matrix (P) based on the standard deviations of the angle (in radians) measurements can be given as

$$P = \begin{bmatrix} 4.727\,25\text{E}9 & 0.0 \\ 0.0 & 4.727\,25\text{E}9 \end{bmatrix} \tag{8.39}$$

The covariance matrix (C_x) of the parameters (x_R, y_R), assuming the a priori variance factor is one, can be given as

$$C_x = \left(A^T P A\right)^{-1} = \begin{bmatrix} 0.277\,811 & 0.0 \\ 0.0 & 0.177\,799 \end{bmatrix} \times \text{E}-4\ \text{m}^2 \tag{8.40}$$

The maximum and minimum eigenvalues of C_x are λ_1 = 2.7781E-5 m^2 and λ_2 = 1.777 99E-5 m^2, respectively; the parameters of the SE ellipse are semi-major axis value, a = 0.0053 m; semi-minor axis value, b = 0.0042 m; and the orientation of the semi-major axis, $\beta = 90°00'$.

Option d: Measure distances d_1 and d_2 from points P and Q, respectively, to point R, as shown in Figure 8.6.

The parametric model equations can be formulated for the two measurements as follows:

$$d_1 = \sqrt{(x_R - x_P)^2 + (y_R - y_P)^2} \tag{8.41}$$

$$d_2 = \sqrt{(x_R - x_Q)^2 + (y_R - y_Q)^2} \tag{8.42}$$

The partial derivatives of Equations (8.41)–(8.42) with respect to coordinates (x_R, y_R) can be given as

$$\frac{\partial d_1}{\partial x_R} = \frac{-(x_P - x_R)}{d^0_{PR}} = 0.7809; \quad \frac{\partial d_1}{\partial y_R} = \frac{-(y_P - y_R)}{d^0_{PR}} = 0.6247$$

$$\frac{\partial d_2}{\partial x_R} = \frac{-(x_Q - x_R)}{d^0_{RQ}} = -0.7809; \quad \frac{\partial d_2}{\partial y_R} = \frac{-(y_Q - y_R)}{d^0_{RQ}} = 0.6247$$

The first design matrix A can be given from the above partial derivatives as

$$A = \begin{bmatrix} 0.7809 & 0.6247 \\ -0.7809 & 0.6247 \end{bmatrix} \tag{8.43}$$

The weight matrix (P) based on the standard deviations of the distance measurements can be given as

$$P = \begin{bmatrix} 1.111\text{E}5 & 0.0 \\ 0.0 & 1.111\text{E}5 \end{bmatrix} \tag{8.44}$$

The covariance matrix (C_x) of the parameters (x_R, y_R), assuming the a priori variance factor is one, can be given as

$$C_x = (A^T PA)^{-1} = \begin{bmatrix} 0.073\,80 & 0.0 \\ 0.0 & 0.1153\,13 \end{bmatrix} \times \text{E}-4\,\text{m}^2 \tag{8.45}$$

The maximum and minimum eigenvalues of C_x are λ_1 = 1.153 13E-5 m^2 and λ_2 = 7.38E-6 m^2, respectively; the parameters of the SE ellipse are semi-major axis value, a = 0.0034 m; semi-minor axis value, b = 0.0027 m; and the orientation of the semi-major axis, $\beta = 00°00'$. The summary of all of the options is given in Table 8.1.

As it can be seen in Table 8.1, option d (measurement of two distances) seems to be the best design; it produces an error circle and has the least standard semi-major axis value of 0.003 m, and only two measurements are required. The worst design is associated with measuring the three angles in the triangular network (option b).

Table 8.1 Summary of results of preanalysis of simple two-dimensional network.

Option	Semi-major axis (a) (m)	Semi-minor axis (b) (m)	Orientation of semi-major axis (β)
a	0.008	0.003	321°20′
b	0.009	0.004	90°00′
c	0.005	0.004	90°00′
d	0.003	0.003	00°00′

Example 8.4 Continuing from Section 8.6, assume that after determining the coordinates of point R based on option c, distance P–R and angle θ (in option a) were measured to improve the precisions of the coordinates of point R. Recalculate the covariance matrix of the coordinates of point R using the concept of weighted station approach, and determine from the calculated SE ellipses if there is any improvement on either of the two combined options.

Solution:

Using the cofactor matrix of the weight constraint adjusted parameters from Equation (6.58),

$$C_{\hat{x}} = \left(P_x + A^T PA\right)^{-1} \tag{8.46}$$

where P_x is the a priori weight matrix (from option "c") for the coordinates of point R, which can be given from Equation (8.40) as

$$P_x = \left(A^T PA\right) = \begin{bmatrix} 3.5996 & 0.0 \\ 0.0 & 5.6243 \end{bmatrix} \times \text{E} + 4\ \text{m}^2 \tag{8.47}$$

From option "a" (Equation (8.29), the following is obtained:

$$\left(A^T PA\right) = \begin{bmatrix} 7.4236 & 4.6102 \\ 4.6102 & 5.3481 \end{bmatrix} \times \text{E} + 4\,\text{m}^2 \tag{8.48}$$

Substituting Equations (8.47) and (8.48) into Equation (8.46) gives

$$C_x = \begin{bmatrix} 0.1101 & -0.0462 \\ -0.0462 & 0.1106 \end{bmatrix} \times \text{E} - 4\,\text{m}^2 \tag{8.49}$$

The maximum and minimum eigenvalues of C_x are λ_1 = 1.566E-5 m^2 and λ_2 = 6.41E-6 m^2, respectively; the parameters of the SE ellipse are semi-major axis value, a = 0.0040 m; semi-minor axis value, b = 0.0025 m; and the orientation of the semi-major axis, $\beta = 315°09'$. As it can be seen in Table 8.1, the combined adjustment done in this example has $a = 0.004$ m, which is an improvement on either of the two options combined in the example.

Example 8.5 Consider two-dimensional survey network in Figure 8.7, in which the coordinates of point P are to be determined from three fixed points 1, 2, and 3. The planned measurements are distances s_1, s_2, and s_3. The observations will be uncorrelated. The approximate coordinates of the fixed and new points taken from a large-scale map are given in Table 8.2.

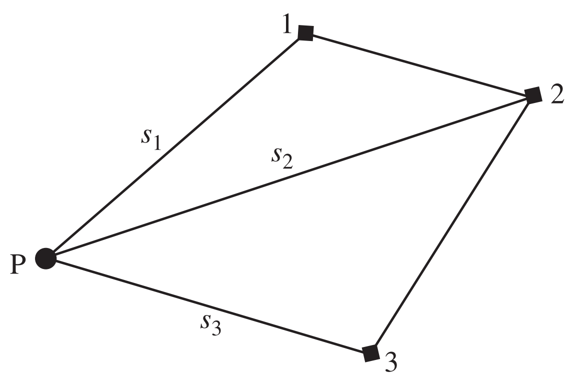


Figure 8.7 Two-dimensional network.

Table 8.2 Approximate coordinates.

Point	*X* (m)	*Y* (m)
1	600	800
2	900	700
3	600	100
P	200	400

What should be the accuracy of the three distances in order to obtain the semi-major axis of the absolute error ellipse at 95% confidence ($a_{95\%}$) of less than 10 mm? In solving this problem, the parametric equations, $\ell = f(\hat{x})$, with the parameters as $x = \begin{bmatrix} x \\ y \end{bmatrix}$ is formulated as follows:

$$s_1 = \left[(x-600)^2 + (y-800)^2\right]^{\frac{1}{2}} \tag{8.50}$$

$$s_2 = \left[(x-900)^2 + (y-700)^2\right]^{\frac{1}{2}} \tag{8.51}$$

$$s_3 = \left[(x-600)^2 + (y-100)^2\right]^{\frac{1}{2}} \tag{8.52}$$

Solution:

Form the first design matrix, A, from Equations (8.50)–(8.52):

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{x-600}{s_1} & \frac{y-800}{s_1} \\ \frac{x-900}{s_2} & \frac{y-700}{s_2} \\ \frac{x-600}{s_3} & \frac{y-100}{s_3} \end{bmatrix} \qquad A = \begin{bmatrix} -0.707 & -0.707 \\ -0.919 & -0.394 \\ -0.800 & 0.600 \end{bmatrix}$$

Form the weight matrix (P) of the observations, assuming the same standard deviation (s) for all the measurements:

$$P = \begin{bmatrix} \left(\frac{1}{s}\right)^2 & 0 & 0 \\ 0 & \left(\frac{1}{s}\right)^2 & 0 \\ 0 & 0 & \left(\frac{1}{s}\right)^2 \end{bmatrix} \quad \text{or} \quad P = s^{-2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Form the matrix of coefficients of normal equations (N):

$$N = s^{-2}A^TA = s^{-2}\begin{bmatrix} 1.984\,83 & 0.382\,07 \\ 0.382\,07 & 1.015\,17 \end{bmatrix}$$

$$Q = N^{-1} = s^2\begin{bmatrix} 0.543\,17 & -2.044\,28 \\ -2.044\,28 & 1.061\,993 \end{bmatrix}$$

Use the cofactor Q to determine the eigenvalues as follows:

$$\lambda_1 = \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2 + z\right) \quad z = \left[\left(\sigma_x^2 - \sigma_y^2\right)^2 + 4\sigma_{xy}^2\right]^{1/2}$$

$$z = 0.660\,558s^2 \qquad \lambda_1 = 1.132\,862s^2$$

Determine the semi-major axis value of the 95% error ellipse using Equation (7.52):

$$\chi^2_{0.95,df=2} = 5.99$$

$$a_{95\%} = \sqrt{\lambda_1 \times 5.99} \;\rightarrow\; \underline{2.604\,97s} \tag{8.53}$$

Equate the calculated semi-major axis value in Equation (8.53) with the given value $a_{95\%} \le 10$ mm, and solve for the unknown standard deviation (s): 2.604 97s = 10 mm

$$s = \frac{10}{2.604\,97}\,\text{mm} \quad \rightarrow \quad \underline{3.8\,\text{mm}}$$

Each distance in the survey network must be measured to an accuracy of 3.8 mm in order to achieve $a_{95\%} \le 10$ mm absolute error ellipse at 95% confidence level for the unknown point P.

Example 8.6 Consider the survey network in Figure 8.8. The coordinates of point P are to be determined from three fixed points 1, 2, 3. The planned measurements are angles β_1, β_2, and β_3 with standard deviation σ_β. The observations will be uncorrelated. The approximate coordinates of the fixed and new points taken from a large-scale map are as given in Table 8.2. Answer the following.

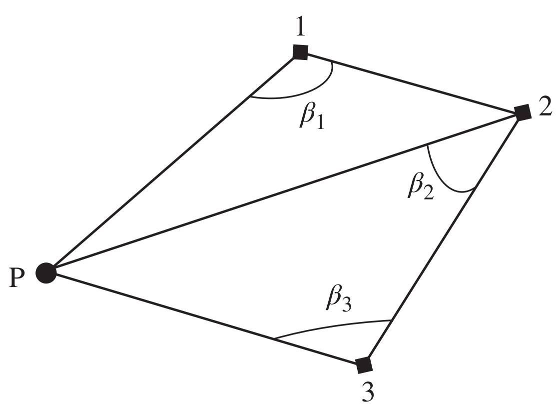


Figure 8.8 Two-dimensional network including angle measurements.

a) If the planned measurements are to have standard deviations of $\sigma_\beta = 2''$, calculate the expected positional error of point P at 95% confidence level.

Solution:

Formulate the parametric equations, $\ell = f(x)$, with the parameters as $x = \begin{bmatrix} x \\ y \end{bmatrix}$:

$$\beta_1 = \arctan\left(\frac{x-600}{y-800}\right) - \arctan\left(\frac{900-600}{700-800}\right) \tag{8.54}$$

$$\beta_2 = \arctan\left(\frac{x-900}{y-700}\right) - \arctan\left(\frac{600-900}{100-700}\right) \tag{8.55}$$

$$\beta_3 = \arctan\left(\frac{900-600}{700-100}\right) - \arctan\left(\frac{x-600}{y-100}\right) \tag{8.56}$$

The approximate distances calculated using the approximate coordinates of point P from Table 8.2 are $s_1 = 565.685$, $s_2 = 761.577$, and $s_3 = 500.000$.

Form the first design matrix, A, with respect to Equations (8.54)–(8.56):

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{y-800}{s_1^2} & -\left(\frac{x-800}{s_1^2}\right) \\ \frac{y-700}{s_2^2} & -\left(\frac{x-900}{s_2^2}\right) \\ -\left(\frac{y-100}{s_3^2}\right) & \frac{x-600}{s_3^2} \end{bmatrix} \quad A = \begin{bmatrix} -0.0125 & 0.001\,25 \\ -0.000\,517\,2 & 0.001\,206\,9 \\ -0.001\,20 & -0.001\,60 \end{bmatrix}$$

Form the weight matrix of the observations, P:

$$P = \begin{bmatrix} \left(\frac{206\,265}{2}\right)^2 & 0 & 0 \\ 0 & \left(\frac{206\,265}{2}\right)^2 & 0 \\ 0 & 0 & \left(\frac{206\,265}{2}\right)^2 \end{bmatrix}$$

or

$$P = \begin{bmatrix} 1.063\,631\,256\text{E}10 & 0 & 0 \\ 0 & 1.063\,631\,256\text{E}10 & 0 \\ 0 & 0 & 1.063\,631\,256\text{E}10 \end{bmatrix}$$

Form the matrix of the coefficients of normal equations, N, and the cofactor matrix, Q, of the adjusted coordinates of point P:

$$N = A^T P A = \begin{bmatrix} 347\,81.153\,1 & -283\,7.309\,11 \\ -283\,7.309\,1 & 593\,41.643\,82 \end{bmatrix}$$

$$Q = N^{-1} = \begin{bmatrix} 2.886\,4\text{E}-5 & 1.380\,082\text{E}-6 \\ 1.380\,082\text{E}-6 & 1.691\,773\text{E}-5 \end{bmatrix}$$

Standard deviations of the adjusted coordinates of point P:

$$\sigma_x = \sqrt{2.8864\text{E}-5} \rightarrow \underline{0.0054\text{ m}}$$

$$\sigma_y = \sqrt{1.691\,773\text{E}-5} \rightarrow \underline{0.0041\text{ m}}$$

Determine the eigenvalues and the 95% confidence error ellipse using chi-square approach from Equations (7.52) and (7.53):

$$\lambda_1 = \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2 + z\right) \quad \lambda_2 = \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2 - z\right) \quad z = \left[\left(\sigma_x^2 - \sigma_y^2\right)^2 + 4\sigma_{xy}^2\right]^{1/2}$$

$$z = 1.226\,078\,1\text{E}-5 \quad \lambda_1 = 2.902\,115\text{E}-5 \quad \lambda_2 = 1.676\,036\,7\text{E}-5$$

$$\chi^2_{0.95,df=2} = 5.99 \quad k = \sqrt{5.99} = 2.4474$$

$$a_{\text{st}} = \sqrt{\lambda_1} = 5.387\text{E}-3 \quad b_{\text{st}} = \sqrt{\lambda_2} = 4.094\text{E}-3$$

$$a_{95\%} = a_{\text{st}} \times k \quad \rightarrow \quad \underline{0.0132\ \text{m}}$$

$$b_{95\%} = b_{\text{st}} \times k \rightarrow \underline{0.0100\,\text{m}}$$

$$\theta = \text{atan}\left(\frac{\lambda_1 - \sigma_y^2}{\sigma_{xy}}\right) \qquad \theta = 83.495°$$

b) If the planned measurements are to be made with the same accuracy, $\sigma_\beta = s$, what should be the numerical value of the accuracy of the three angles in order to obtain the semi-major axis ($a_{95\%}$) of the 95% confidence absolute error ellipse of less than or equal to 10 mm at point P?

Solution:

The parametric equations and the A-matrix in question (a) are still applicable in this problem. Assume all the three distances would be measured with the same standard deviation (s) (with s in seconds); the following weight matrix is formed:

$$P = \begin{bmatrix} \left(\frac{206\,265}{s}\right)^2 & 0 & 0 \\ 0 & \left(\frac{206\,265}{s}\right)^2 & 0 \\ 0 & 0 & \left(\frac{206\,265}{s}\right)^2 \end{bmatrix}$$

or

$$P = s^{-2}\begin{bmatrix} 4.254\,525\text{E}10 & 0 & 0 \\ 0 & 4.254\,525\text{E}10 & 0 \\ 0 & 0 & 4.254\,525\text{E}10 \end{bmatrix}$$

Form the matrix of coefficients of normal equations, N, and the cofactor matrix of the adjusted coordinates, Q:

$$N = s^{-2}A^TA = s^{-2}\begin{bmatrix} 1.391\,246\,123\text{E}10 & -11\,349.236\,38 \\ -11\,349.236\,37 & 2.373\,641\,75\text{E}-5 \end{bmatrix}$$

$$Q = N^{-1} = s^2\begin{bmatrix} 7.215\,95\text{E}-6 & 3.450\,204\text{E}-7 \\ 3.450\,204\text{E}-7 & 4.229\,43\text{E}-6 \end{bmatrix}$$

Compute the semi-major axis value of the 95% confidence error ellipse from the cofactor matrix, Q:

$$\lambda_1 = \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2 + z\right) \qquad z = \left[\left(\sigma_x^2 - \sigma_y^2\right)^2 + 4\sigma_{xy}^2\right]^{1/2}$$

$$z = 3.065\,195\text{E}-6s^2 \qquad \lambda_1 = 7.255\,29\text{E}-6s^2$$

Using 95% error ellipse formula in Equation (7.52),

$$\chi^2_{0.95,\text{df}=2} = 5.99$$

$$a_{95\%} = \sqrt{\lambda_1 \times 5.99} \quad \rightarrow \quad 0.006\,592\,36s \tag{8.57}$$

Equate the given $a_{95\%} \leq 10$ mm to the computed value in Equation (8.57), and solve for the value of s:

$$0.006\,592\,36s = 0.010$$

$$s = \frac{0.010}{0.006\,592\,36}'' \quad \rightarrow \quad 1.5''$$

The standard deviation for each angle measurement should be 1.5″.

Example 8.7 A detailed survey point P shown in Figure 8.9 was to be laid out by independent relocation traverse surveys from different control points within a simultaneously adjusted network. The point P with known coordinates

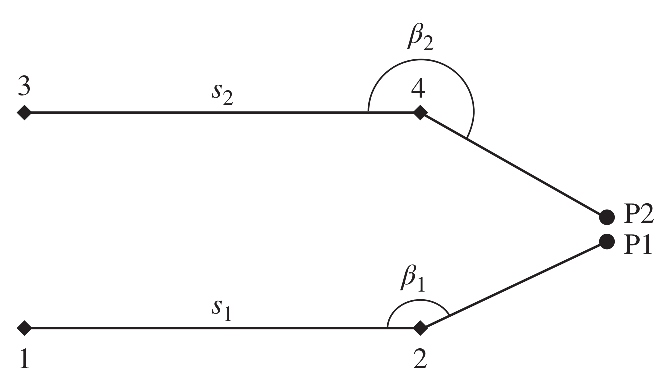


Figure 8.9 Relocation traverse surveys.

(Y_{P1}, X_{P1}) had been marked earlier on the ground as point P1 by the survey from control points 1 and 2. The second survey to the same point was carried from control points 3 and 4, and different coordinates P2 (Y_{P2}, X_{P2}) were obtained. The following variance–covariance matrix for points P1 and P2 are obtained and given in the order $x = [Y_{P1} \;\; X_{P1} \;\; Y_{P2} \;\; X_{P2}]^T$ as follows:

$$C_{x_P} = 10^{-5}\begin{bmatrix} 5.134 & -0.784 & 0.928 & 0.166 \\ & 12.179 & 0.121 & 0.959 \\ & & 11.096 & -1.603 \\ & & & 8.220 \end{bmatrix}$$

Answer the following.

a) What are the maximum differences ΔX and ΔY between points P1 and P2 that could be allowed at the 95% confidence level for the given accuracy of the control and of the measurements of β_1, β_2, s_1, s_2?

Solution:

Equation for the coordinate differences in form of $p = f(x)$ is formulated as follows, where $x = [Y_{P1} \;\; X_{P1} \;\; Y_{P2} \;\; X_{P2}]^T$ is a vector of the adjusted coordinates of points P1 and P2 and $p = [\Delta x \;\; \Delta y]^T$ is a vector of corresponding coordinate differences:

$$\Delta x = X_{P2} - X_{P1} \tag{8.58}$$

$$\Delta y = Y_{P2} - Y_{P1} \tag{8.59}$$

The Jacobian matrix of Equations (8.58) and (8.59) with respect to the adjusted coordinates is given as

$$B = \frac{\partial p}{\partial x} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

By variance–covariance propagation laws,

$$C_p = BC_{x_p}B^T$$

$$C_p = \begin{bmatrix} 1.8481E-4 & -2.674E-5 \\ -2.674E-5 & 1.4374E-4 \end{bmatrix}$$

$$\sigma_{\Delta X} = \sqrt{1.8481E-4} \quad \rightarrow \quad 0.014\,\text{m}$$

$$\sigma_{\Delta Y} = \sqrt{1.4374E-4} \quad \rightarrow \quad 0.012\,\text{m}$$

At 95% confidence, the allowable values of ΔX and ΔY are determined as their error margins at 95% confidence level, using Equation (7.13): $\Delta = (SE)z_{1-\alpha/2}$ (for $z_{1-0.05/2} = 1.96$)

$$\Delta X_{\max} = 1.96 \times 0.014\,\text{m} \quad \rightarrow \quad 0.0274\ \text{m}$$

$$\Delta Y_{\max} = 1.96 \times 0.012\ \text{m} \quad \rightarrow \quad 0.0235\ \text{m}$$

b) What is the expected maximum distance between the points P1 and P2 to be marked on the ground from the two layouts?

Solution:

The expected distance is the semi-major axis of the relative error ellipse connecting P1 and P2 at 95% confidence level. By error propagation done in (a), the variance–covariance matrix for the differences in coordinates of the two points P1 and P2 is

$$C_\text{p} = \begin{bmatrix} 1.8481\text{E}-4 & -2.674\text{E}-5 \\ -2.674\text{E}-5 & 1.4374\text{E}-4 \end{bmatrix}$$

Determine the eigenvalues and the 95% confidence relative error ellipse from C_p:

$$\lambda_1 = \frac{1}{2}\left(\sigma_x^2 + \sigma_y^2 + z\right)$$

$$z = \left[\left(\sigma_x^2 - \sigma_y^2\right) + 4\sigma_{xy}^2\right]^{\frac{1}{2}}$$

$$z = 6.743\,04\text{E}-5;\quad \lambda_1 = 1.979\,9\text{E}-4;\quad \lambda_2 = 1.305\,60\text{E}-4$$

Using the 95% error ellipse formulas in Equations (7.52) and (7.53) with the value of $\chi^2_{1-\alpha,\text{df}=2} = 5.991$,

$$a = \sqrt{\lambda_1 \times 5.991} \quad \rightarrow \quad \underline{0.0344\ \text{m}}$$

$$b = \sqrt{\lambda_2 \times 5.991} \quad \rightarrow \quad \underline{0.0280\,\text{m}}$$

$$\theta = \text{atan}\left(\frac{\lambda_1 - \sigma_y^2}{\sigma_{xy}}\right) \quad \theta = -63.76^\circ$$

The maximum expected distance at 95% confidence is 0.034 m along the bearing of −63.76°.

8.6 Simulation of Three-dimensional Survey Scheme

Designing a survey scheme for the purpose of deciding the best choice of equipment and procedures for three-dimensional positioning usually involves the process of preanalysis or what is called computer simulation. The computer simulation steps are well documented (cf. Nickerson 1979; Cross 1985):

1) The simulation process usually starts with the input into the computer software, the standard deviations of the potential observables (such as horizontal and zenith angles and distances). The standard deviations of the observables are usually derived from the available equipment to be used in the survey scheme.
2) Input the potential geometry or preliminary location of points (as first design matrix A) using the given as approximate coordinates, taken from large-scale maps, aerial photographs, or other sources and supported with field reconnaissance survey to ensure intervisibility of points.
3) Simulate the quality of the network (using appropriate computer software) based on the initial design given in steps 1–2, and check the simulated quality (standard deviations and absolute and relative error ellipses of unknown parameters to determined later) against the expected positioning tolerance, which are sometimes the limit on relative error ellipses or absolute station error ellipses at 95% confidence level.
4) If step 3 is not satisfied, consider modifying and repeating steps 1–2 until step 3 is satisfied. Otherwise, consider the design complete.

Steps 1–2 constitute the initial network design, which is preanalyzed in steps 3–4. The following example illustrates the simulation process based on the use of MicroSurvey STAR∗NET v8 as the preanalysis or simulation software.

8.6.1 Typical Three-dimensional Micro-network

Eleven wall targets on four walls (A, B, C, and D) shown in Figure 8.10 are to be coordinated three dimensionally as a part of a micro-network establishment in an industrial environment. The targets are to be positioned to a relative positioning tolerance of 10.0 mm relative to the fixed point 1 at 95% confidence level, assuming the azimuth of line 1–2 will be considered known and fixed for the network datum. The approximate coordinates of the wall targets, possible locations of the instrument, and two well-calibrated scale bars (to provide scale) are shown in Table 8.3 and Figure 8.11. The scale bars whose positions will remain fixed throughout the project are 2 m in length with the calibrated accuracy of 0.02 mm; they will be used to improve precision of distance measurements instead of measuring distances to the wall targets since the lengths involved are less than 20 m. For example, using a total station with standard

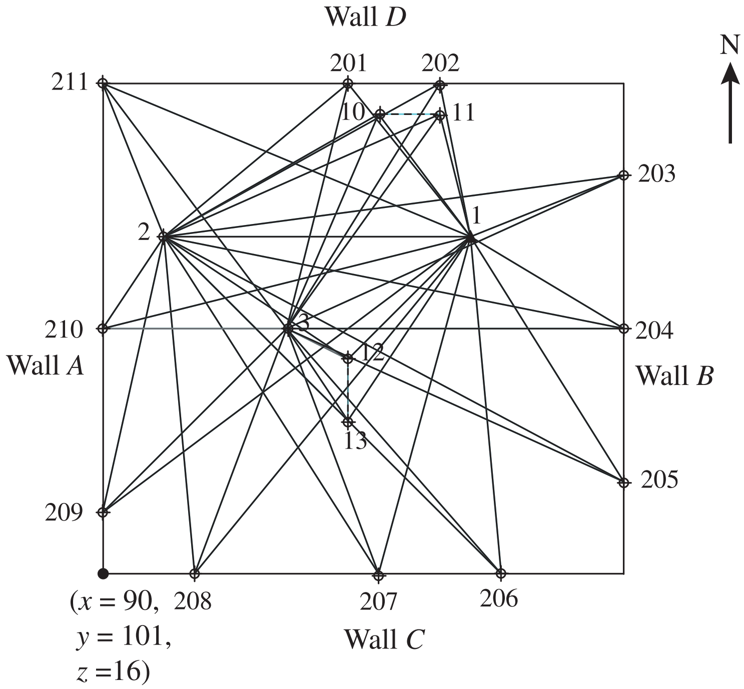


Figure 8.10 The plan locations of the wall targets, instrument, and scale bar locations in a micro-network survey.

Table 8.3 Approximate coordinates of wall targets.

Point no.	Northing (m)	Easting (m)	Elevation (m)	Description of point
201	117	98	20	Wall target
202	117	101	20	Wall target
203	114	107	20	Wall target
204	109	107	20	Wall target
205	104	107	20	Wall target
206	101	103	20	Wall target
207	101	99	20	Wall target
208	101	93	20	Wall target
209	103	90	20	Wall target
210	109	90	20	Wall target
211	117	90	20	Wall target
1	112	102	16	Instrument location (fixed)
2	112	92	16	Possible setup point (fixed)
3	109	96	16	Possible setup point (free)
10	116	99	18	First marker on scale bar 1
11	116	101	18.5	Second marker on scale bar 1
12	108	98	16	First marker on scale bar 2
13	106	98	16	Second marker on scale bar 2

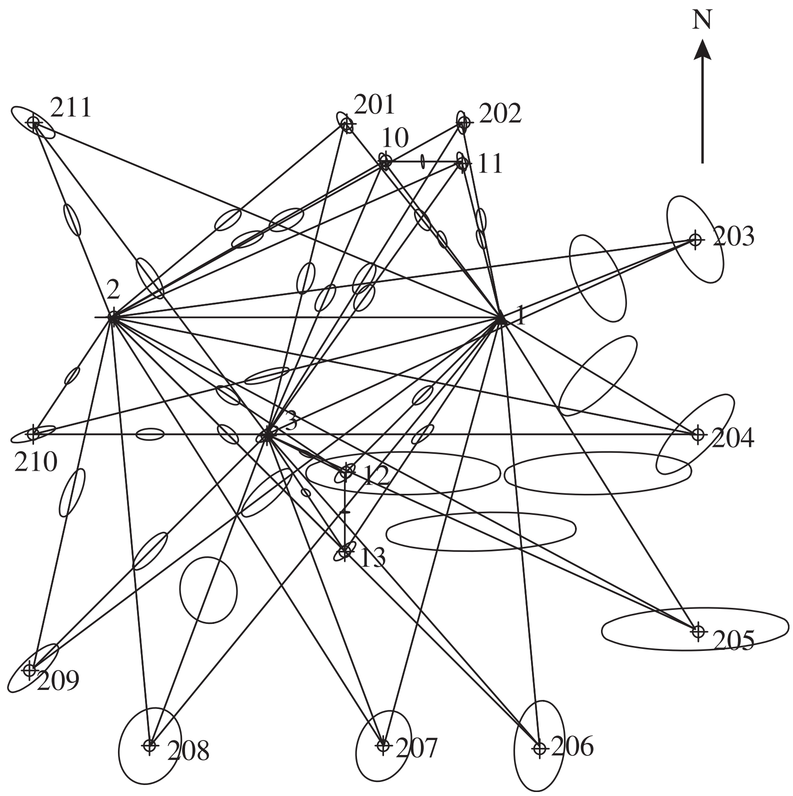


Figure 8.11 The two-dimensional view of the simulated survey scheme for a micro-network survey project.

deviation of distance measurement as 3 mm ± 2 ppm will produce constant precision of 3 mm for any distance measurement in this project, thereby reducing the precision of the whole project. The approximate coordinates provided in Table 8.3 for the wall targets, scale bar markers (10, 11, 12, 13), and the instrument locations (1, 2, 3) were extracted from the building drawings. Point 3 is given approximate coordinates to start with; these values are free to be changed when manipulating the location of the instrument in order to achieve the best geometry in relation to other target points to be measured to. The locations of the scale bars can also be changed during the preanalysis if that change will provide better results.

8.6.2 Simulation Results

The MicroSurvey STAR∗NET v8 software was used in the simulation process. An instrument considered has a standard deviation for one set of direction measurements as 5″; point 1 and the azimuth of line 1–2 are fixed for datum definition, with the scale bars providing the scale. The total number of points whose coordinates are to be determined is 17 (with four of them representing the markers on the two scale bars); the total of 45 zenith angles and 45 horizontal directions are to be measured to the wall targets, markers on the scale bars, and targets on the other unoccupied setup stations, from three setup stations (1, 2, and 3) with the two scale bars providing additional two distance measurements and the azimuth of line 1–2 considered also as measurement. The total number of degrees of freedom for the adjustment is 76. The result of the simulation is given in Figure 8.11, showing the 95% confidence station and relative error ellipses and the survey scheme.

In Figure 8.11, it can be seen that the maximum station coordinate error ellipse at 95% confidence level is at station 205 with a semi-major axis value of 8 mm and its orientation along the azimuth 88°33′; the remaining 95% confidence station coordinate error ellipses have their semi-major axes values less than 5 mm. The maximum relative error ellipse at 95% confidence (the main consideration for this project) is on line 2–205 with a semi-major axis value of 8.5 mm (oriented along the azimuth 90°) with the remaining 95% relative error ellipses having their semi-major axes values less than 5 mm. This design is acceptable (but not necessarily the best) since the achieved 8.5 mm for the 95% confidence relative error ellipse is less than the required relative positioning tolerance of 10.0 mm. The MicroSurvey STAR∗NET code for the three-dimensional design is given in Table 8.4.

Table 8.4 The MicroSurvey STAR∗NET 8 code for the three-dimensional design.

```
# Three-dimensional design
.3D
.ORDER NE AtFromTo
.UNITS Meters DMS
C 10 116.0 99 18
C 11 116 101 18.5
C 201 117 98 20
C 202 117 101 20
C 203 114 107 20
C 204 109 107 20
C 205 104 107 20
C 206 101 103 20
C 207 101 99 20
C 208 101 93 20
C 209 103 90 20
C 210 109 90 20
C 211 117 90 20
```

(*Continued*)

Table 8.4 (Continued)

```
C 1 112 102 16 ! ! !
C 2 112 92 16
C 3 109 96 16
C 12 108 98 16
C 13 106 98 16
# Measurements- Fixed bearing
B 1-2 ? !
#Scale bar distance
D 10-11 ? 0.00002
D 12-13 ? 0.00002
# Zenith angle measurements
V 1-209 ? 5
V 1-210 ? 5
V 1-211 ? 5
V 1-201 ? 5
V 1-202 ? 5
V 1-10 ? 5
V 1-11 ? 5
V 1-12 ? 5
V 1-13 ? 5
V 2-209 ? 5
V 2-210 ? 5
V 2-211 ? 5
V 2-201 ? 5
V 2-202 ? 5
V 2-10 ? 5
V 2-11 ? 5
V 2-12 ? 5
V 2-13 ? 5
V 3-209 ? 5
V 3-210 ? 5
V 3-211 ? 5
V 3-201 ? 5
V 3-202 ? 5
V 3-10 ? 5
V 3-11 ? 5
V 3-12 ? 5
V 3-13 ? 5
#Wall A & D
# Horizontal direction measurements
DB 1
DN 209 ? 5
```

Table 8.4 (Continued)

```
DN 210 ? 5
DN 211 ? 5
DN 201 ? 5
DN 202 ? 5
DN 10 ? 5
DN 11 ? 5
DN 12 ? 5
DN 13 ? 5
DE
DB 2
DN 209 ? 5
DN 210 ? 5
DN 211 ? 5
DN 201 ? 5
DN 202 ? 5
DN 10 ? 5
DN 11 ? 5
DN 12 ? 5
DN 13 ? 5
DE
DB 3
DN 209 ? 5
DN 210 ? 5
DN 211 ? 5
DN 201 ? 5
DN 202 ? 5
DN 10 ? 5
DN 11 ? 5
DN 12 ? 5
DN 13 ? 5
DE
# To Wall B
# Zenith angles
V 1-205 ? 5
V 1-204 ? 5
V 1-203 ? 5
V 2-205 ? 5
V 2-204 ? 5
V 2-203 ? 5
V 3-205 ? 5
V 3-204 ? 5
V 3-203 ? 5
```

(*Continued*)

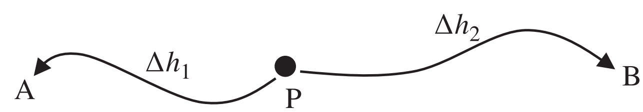
Table 8.4 (Continued)

```
# Horizontal directions
DB 1
DN 205 ? 5
DN 204 ? 5
DN 203 ? 5
DE
DB 2
DN 205 ? 5
DN 204 ? 5
DN 203 ? 5
DE
DB 3
DN 205 ? 5
DN 204 ? 5
DN 203 ? 5
DE
# To Wall C
# Zenith angles
V 1-208 ? 5
V 1-207 ? 5
V 1-206 ? 5
V 2-208 ? 5
V 2-207 ? 5
V 2-206 ? 5
V 3-208 ? 5
V 3-207 ? 5
V 3-206 ? 5
# Horizontal directions
DB 1
DN 208 ? 5
DN 207 ? 5
DN 206 ? 5
DE
DB 2
DN 208 ? 5
DN 207 ? 5
DN 206 ? 5
DE
DB 3
DN 208 ? 5
DN 207 ? 5
DN 206 ? 5
DE
```

Problems

8.1 Given the leveling network in Figure P8.1 where A and B are control points with known heights; Δh_1 and Δh_2 are two height difference measurements with standard deviations of σ_1 and σ_2, respectively; and $\sigma_1 = 0.25\sigma_2$. Determine the values of σ_1 and σ_2 so that the 90% confidence interval of the height solution for point P using least adjustment is equal to 10 mm.

Figure P8.1



8.2 You are to design two-dimensional FOD network for the monitoring of an object point OP (on a deformable body) to a relative positioning tolerance of 4 mm (at 95% confidence level) with the coordinates of point RBR1402 and y-coordinate of point RBR1408 fixed in a minimal constraint adjustment. The approximate coordinates (x^0) of the reference points for possible location of the instrument and the object point OP are as shown in the following table; they were extracted from a large scale topographic map of the region. In the design, you are to assume that the object point OP and the reference points RBR1402 and RBR1408 will not change; you are therefore left with the manipulation of points RBR1410 and BC1001 for the FOD design.

Approximate coordinates of network points

Point no.	Northing (m)	Easting (m)
RBR1402	4629.6	1306.8
RBR1408	4742.4	1191.0
OP	4695.0	1273.5
RBR1410	4768.8	1279.7
BC1001	4655.6	1314.3

Using appropriate measurement scheme with distances introduced in one direction only, perform the design work by completing the following. Based on the nature of the topography and the environment of the network location, you can assume positions of the moveable points (RBR1402 and RBR1408) can be shifted by ±25 m in the FOD design. Assume that you have only access to Leica TC703 total station. The specifications for TC703 are 3″ for direction measurements and 2 mm ± 2 ppm for distance measurements; and the centering error for the instrument and target is 0.2 mm (based on the use of forced centering devices). Perform the following tasks:

1) Input the accuracies of measuring all possible directions and distances with the Leica TC 703 total station into the simulation software.
2) Starting from the approximate coordinates given in the table, perform a SOD by considering using Leica TC 703. Give the summary of your best result (giving the worst relative 95% error ellipse and a plot of your design).
3) Perform a combined FOD (moving only points RBR1402 and RBR1408) and SOD of the network. Give the summary of your best result (giving the worst relative 95% error ellipse and a plot of your design).
4) Assuming the tolerance is achieved in step 4, delete distance or angular measurements that will have minimal effect (still barely satisfying 4.0 mm relative accuracy) on the design in step 3.
5) Conclude whether it is possible to achieve the relative positioning tolerance specified. In addition to the foregoing information, include the following:
   a) Maximum semiaxes at 95% level for your "best" network design.
   b) Provide for the best design (step 4), the final precisions of measurements, number, and types of observations.
   c) Coordinates of best location of reference points RBR1402 and RBR1408 in step 4.
   d) A plot showing your design with the appropriate 95% error ellipses from step 4.

9

Concepts of Three-dimensional Geodetic Network Adjustment

CHAPTER MENU

OBJECTIVES

After studying this chapter, you should be able to:

1) Formulate parametric model equations relating spatial observables, such as distances, zenith (vertical) angles, and azimuths (directions or angles), with the three-dimensional *X*, *Y*, *Z* Cartesian coordinates in conventional terrestrial (CT) system.
2) Formulate parametric model equations relating spatial observables, such as distances, zenith (vertical) angles, and azimuths (directions or angles), with the three-dimensional curvilinear geodetic coordinates (latitude, longitude, and ellipsoidal height) in geodetic system.
3) Formulate parametric model equations relating spatial observables, such as distances, zenith (vertical) angles, and azimuths (directions or angles), with the three-dimensional local Cartesian coordinates (*n*, *e*, *u*) in local astronomic (LA) system.

Understanding Least Squares Estimation and Geomatics Data Analysis, First Edition.
John Olusegun Ogundare.

Companion website: www.wiley.com/go/ogundare/Understanding-lse-and-gda

9.1 Introduction

The adjustments of three-dimensional geodetic networks have been discussed in detail in a number of technical reports and books, such as Wolf (1963, 1975), Heiskanen and Moritz (1967), Vincenty and Bowring (1978), Vincenty (1979), Dragomir et al. (1982), Vanicek and Krakiwsky (1986), and Leick (2004). The discussion in this section is based on them. Further details can be found in those reports and books. There are three main classes of three-dimensional networks: those based on terrestrial measurements, such as spatial distance, horizontal and vertical angles, height differences, etc.; those based on photogrammetric and remote sensing measurements; and those based on the measurements made from tracking stations to orbiting satellites. This section is mainly interested in the networks based on terrestrial measurements.

9.2 Three-dimensional Coordinate Systems and Transformations

In the adjustment of three-dimensional geodetic networks, the measurements are not reduced to the ellipsoid as in two-dimensional cases, and computations are not done on the ellipsoid or on the conformal mapping planes; the computations are generally done in a three-dimensional Cartesian coordinate system. The differential shifts of coordinates are in linear units in a rectangular horizon system (local coordinate system) centered on the point where the measurements are made; this means that there will be as many local coordinate systems as there are measurement points. One of the important local coordinate systems commonly used is local geodetic coordinate system. Typical observables in modern surveying are horizontal angles (or directions), slope distances, zenith angles, GPS vectors, astronomic latitudes, longitudes, azimuths, and height differences. One of the limitations in classical three-dimensional adjustment is the uncertainty in the vertical refraction when measuring zenith angles.

The conventional terrestrial (CT) Cartesian coordinate system is a global system with the origin at the Earth's center of mass; the X, Y, Z Cartesian coordinates are in the equatorial system with X-axis passing though the Greenwich meridian, Z-axis is parallel to the mean rotation axis of the Earth, and Y-axis is perpendicular to the Z–X plane in a right-handed system. This is illustrated in Figure 9.1. An ellipsoid associating with latitude (ϕ), longitude (λ), and ellipsoidal height (h) can be positioned so as to be coaxial with the X, Y, Z of the CT system such that Z-axis coincides with the ellipsoid's rotation axis and the ellipsoid's center coinciding with the origin of the X, Y, Z system. The ellipsoid so-positioned and oriented with the CT system is a *reference ellipsoid* or global

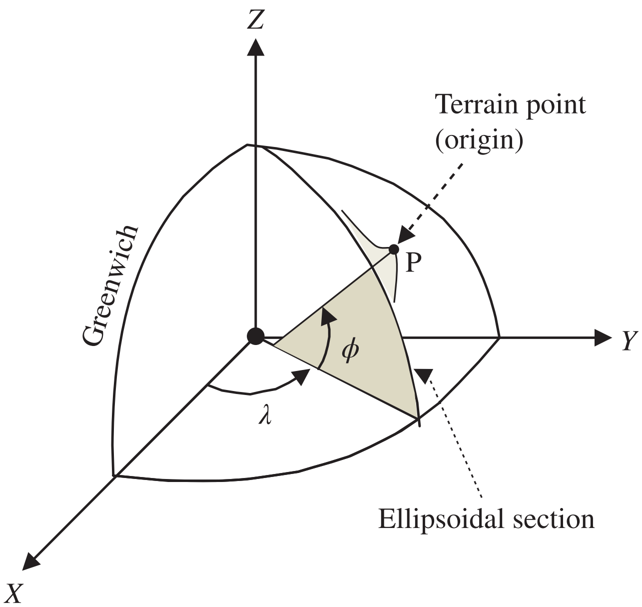


Figure 9.1 Relationship between the conventional terrestrial (CT) system and the geodetic system.

geodetic (G) system. The relationship between the Cartesian coordinates (X, Y, Z) and the G system coordinates (ϕ, λ, h) can be given as follows:

$$X = X_0 + (N + h)\cos\phi\cos\lambda \quad (9.1)$$

$$Y = Y_0 + (N + h)\cos\phi\sin\lambda \quad (9.2)$$

$$Z = Z_0 + \left(N\left(1 - e^2\right) + h\right)\sin\phi \quad (9.3)$$

where X_0, Y_0, and Z_0 are the coordinates of the center of the reference ellipsoid with respect to the CT system, N is the *radius of curvature in the prime vertical direction*, and M is the *radius of curvature in the meridian plane* given as follows:

$$N = \frac{a}{\left(1 - e^2\sin^2\phi\right)^{1/2}} \quad (9.4)$$

$$M = \frac{a(1 - e^2)}{\left(1 - e^2\sin^2\phi\right)^{3/2}} \quad (9.5)$$

with a and b as the semi-major axis and semi-minor axis values of the reference ellipsoid, respectively, and e as the first eccentricity of the ellipsoid. As an example, the parameters of the Geodetic Reference System of 1980 (GRS80 ellipsoid) are as follows:

$$a = 6\ 378\ 137.0 \text{ m}$$

$$b = 6\ 356\ 752.314\ 1 \text{ m}$$

$$e^2 = 0.006\ 694\ 380\ 023$$

By taking the partial derivatives of Equations (9.1)–(9.3) with respect to ϕ, λ, and h, the relationship between the coordinate differences (dX, dY, dZ) in Cartesian geodetic coordinate system and the coordinate differences in the curvilinear geodetic coordinate system (dϕ, dλ, dh) can be given as follows:

$$\begin{bmatrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{bmatrix} = \begin{bmatrix} -(M+h)\sin\phi\cos\lambda & -(N+h)\cos\phi\sin\lambda & \cos\phi\cos\lambda \\ -(M+h)\sin\phi\sin\lambda & (N+h)\cos\phi\cos\lambda & \cos\phi\sin\lambda \\ (M+h)\cos\phi & 0 & \sin\phi \end{bmatrix} \begin{bmatrix} \mathrm{d}\phi \\ \mathrm{d}\lambda \\ \mathrm{d}h \end{bmatrix} \tag{9.6}$$

where dϕ and dλ are in radians.

9.2.1 Local Astronomic Coordinate Systems and Transformations

Measurements are usually made in the local astronomic (LA) system. The natural astronomic (physically meaningful) quantities usually measured between any two given points i and j are the spatial distance (s_{ij}), astronomic latitude (Φ_i), longitude (Λ_i), azimuth (Az_{ij}), vertical angle (ν_{ij}) (or zenith angles z_{ij}), and orthometric height (H). The spatial orientation of this coordinate system is completely specified by the astronomic latitude and longitude with the Z-axis coinciding with the direction of the Conventional Terrestrial Pole (CTP). The LA coordinate system is a *topocentric coordinate system* or a local coordinate system as illustrated in Figure 9.2 and defined as follows:

- *Origin*: At the instrument setup station.
- *Z (or u)-axis*: Along the vertical (the direction of gravity) at the setup point.
- *X (or n)-axis*: A line tangent at the origin and aligned along the astronomical meridian, pointing toward the true north.

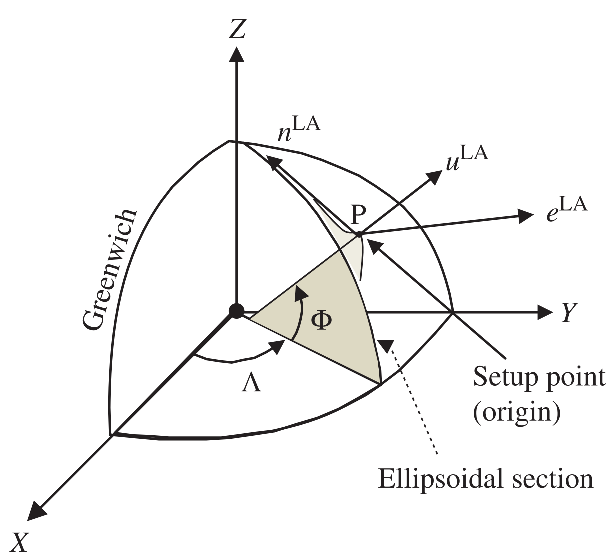
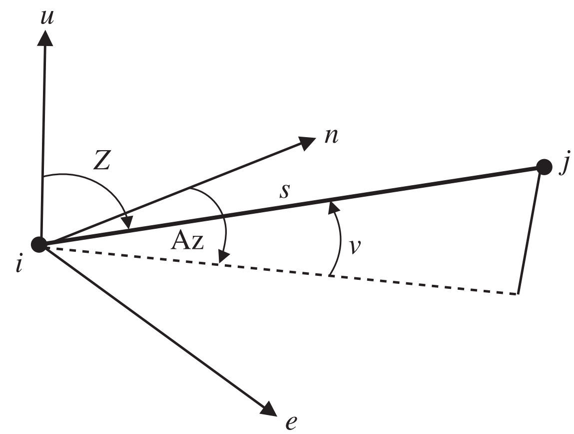


Figure 9.2 Relationship between the conventional terrestrial (CT) system and the local astronomic (LA) system.

Figure 9.3 Relationship between the azimuth (Az), vertical angle (v), zenith Angle (Z), and slope distance (s) in the local astronomic (LA) coordinate system.



- *X–Y (n–e) plane*: Tangent to the geoid at the instrument setup point.
- *Y – (e) axis*: Defined by certain azimuth such that the coordinate system forms a right-handed system.
- Geocentric X_0, Y_0, Z_0 coordinates and the orthometric height H_0 are assigned to the origin.

In LA system, the north–east (n–e) plane shown in Figure 9.3 coincides with the physical horizontal plane. By using CT coordinate system, one can conveniently describe astro-geodetic networks extending over a large area such as states, provinces, continent, and the entire terrestrial globe.

The geoid in the region of measurement is defined as being tangent to the reference ellipsoid at the origin and the deflection of the vertical and the geoid undulations relative to the reference ellipsoid. With the instrument station i as the origin of the local coordinate system in Figure 9.3 and the target at point j, the coordinate differences between points i and j can be given as follows:

$$\mathrm{d}n_{ij} = s_{ij} \cos v_{ij} \cos \mathrm{Az}_{ij} \tag{9.7}$$

$$\mathrm{d}e_{ij} = s_{ij} \cos v_{ij} \sin \mathrm{Az}_{ij} \tag{9.8}$$

$$\mathrm{d}u_{ij} = s_{ij} \sin v_{ij} \tag{9.9}$$

where s_{ij} is the slope distance, v_{ij} is the vertical angle, and Az_{ij} is the azimuth of line i to j. The inverses of Equations (9.7)–(9.9) can be given as

$$s_{ij} = \sqrt{\mathrm{d}n_{ij}^2 + \mathrm{d}e_{ij}^2 + \mathrm{d}u_{ij}^2} \tag{9.10}$$

$$\mathrm{Az}_{ij} = a\tan\left(\frac{\mathrm{d}e_{ij}}{\mathrm{d}n_{ij}}\right) \tag{9.11}$$

$$v_{ij} = a\sin\left(\frac{\mathrm{d}u_{ij}}{s_{ij}}\right) \tag{9.12}$$

Total station instruments collect survey data in three dimensions at any given setup station, which is usually considered as the origin of that LA coordinate system; this system provides a natural system in which to perform the adjustment of the data. The relationship between the coordinate differences (dn, de, du) in the LA coordinate system and the coordinate differences (dX, dY, dZ) in the CT coordinate system can be given as

$$\begin{bmatrix} \mathrm{d}n_{ij} \\ \mathrm{d}e_{ij} \\ \mathrm{d}u_{ij} \end{bmatrix}^{\mathrm{LA}} = \begin{bmatrix} -\sin\Phi_i\cos\Lambda_i & -\sin\Phi_i\sin\Lambda_i & \cos\Phi_i \\ -\sin\Lambda_i & \cos\Lambda_i & 0 \\ \cos\Phi_i\cos\Lambda_i & \cos\Phi_i\sin\Lambda_i & \sin\Phi_i \end{bmatrix} \begin{bmatrix} \mathrm{d}X_{ij} \\ \mathrm{d}Y_{ij} \\ \mathrm{d}Z_{ij} \end{bmatrix}^{\mathrm{CT}} \tag{9.13}$$

where Φ_i and Λ_i are the astronomic latitude and astronomic longitude at point i corrected for the effect of polar motion so that they refer to the Conventional International Origin (CIO) of the CT system.

9.3 Parametric Model Equations in Conventional Terrestrial System

Equation (9.13) is exact, forming the basis of relating a measured quantity (e.g. a distance, an angle, a GPS vector, leveled height difference, etc.) to either the LG or LA coordinate differences between the stations involved in the measurement. By combining Equations (9.10)–(9.13), the following can be obtained (Vincenty and Bowring 1978):

$$s_{ij} = \sqrt{\mathrm{d}X_{ij}^2 + \mathrm{d}Y_{ij}^2 + \mathrm{d}Z_{ij}^2} \tag{9.14}$$

$$\mathrm{Az}_{ij} = a\tan\left(\frac{-\mathrm{d}X_{ij}\sin\Lambda_i + \mathrm{d}Y_{ij}\cos\Lambda_i}{-\mathrm{d}X_{ij}\sin\Phi_i\cos\Lambda_i - \mathrm{d}Y_{ij}\sin\Phi_i\sin\Lambda_i + \mathrm{d}Z_{ij}\cos\Phi_i}\right) \tag{9.15}$$

$$v_{ij} = a\sin\left(\frac{\mathrm{d}X_{ij}\cos\Phi_i\cos\Lambda_i + \mathrm{d}Y_{ij}\cos\Phi_i\sin\Lambda_i + \mathrm{d}Z_{ij}\sin\Phi_i}{\sqrt{\mathrm{d}\,X_{ij}^2 + \mathrm{d}\,Y_{ij}^2 + \mathrm{d}\,Z_{ij}^2}}\right) \tag{9.16}$$

$$z_{ij} = a\cos\left(\frac{\mathrm{d}X_{ij}\cos\Phi_i\cos\Lambda_i + \mathrm{d}Y_{ij}\cos\Phi_i\sin\Lambda_i + \mathrm{d}Z_{ij}\sin\Phi_i}{\sqrt{\mathrm{d}\,X_{ij}^2 + \mathrm{d}\,Y_{ij}^2 + \mathrm{d}\,Z_{ij}^2}}\right) \tag{9.17}$$

where z_{ij} is the zenith angle from point i to point j and Φ_i and Λ_i define the direction of gravity at the given point i and serve as reference direction in space to which Az_{ij} and v_{ij} (or z_{ij}) are referred. The Φ_i and Λ_i relate LA system to the CT system and are treated as additional unknown parameters in the adjustment. It should be mentioned that Φ_i and Λ_i may be replaced by the corresponding geodetic latitude (ϕ_i) and geodetic longitude (λ_i) in the coefficients of partial derivatives without losing accuracy. If observed latitudes and longitudes are available, they may be introduced to the parametric equations as observed parameters in the adjustment. Since horizontal angle is the difference between two azimuths, Equation (9.15) can be used to formulate horizontal angle equation. Equations (9.14)–(9.17) constitute the adjustment model of the usual parametric equations ($\hat{\ell} = f(\hat{x})$, for $\hat{\ell}$ as a vector of observations and $\hat{x}$ as vector of unknown parameters). The parameters to be estimated in the equations using the method of least squares adjustment are $\mathrm{d}X_{ij}$, $\mathrm{d}Y_{ij}$, $\mathrm{d}Z_{ij}$, where

$$\begin{bmatrix} \mathrm{d}X_{ij} \\ \mathrm{d}Y_{ij} \\ \mathrm{d}Z_{ij} \end{bmatrix} = \begin{bmatrix} X_j - X_i \\ Y_j - Y_i \\ Z_j - Z_i \end{bmatrix} \tag{9.18}$$

Equation (9.14) can be rewritten in symbolic forms as follows:

$$\mathrm{Az}_{ij} = f\left(X_i, Y_i, Z_i, X_j, Y_j, Z_j\right) \tag{9.19}$$

$$v_{ij} = f\left(X_i, Y_i, Z_i, X_j, Y_j, Z_j\right) \tag{9.20}$$

$$s_{ij} = f\left(X_i, Y_i, Z_i, X_j, Y_j, Z_j\right) \tag{9.21}$$

The spatial distance s_{ij} given in Equation (9.14) relates to the CT system and can be rewritten as

$$\left[\left(X_j - X_i\right)^2 + \left(Y_j - Y_i\right)^2 + \left(Z_j - Z_i\right)^2\right]^{1/2} - s_{ij} = 0 \tag{9.22}$$

The linearized distance equation (required for the least squares adjustment) can be obtained by finding the partial derivatives of Equation (9.22) with respect to the unknown coordinates of points i and j given by

$$r_{\mathrm{s}} = \mathrm{d}X_{ij}\frac{\left(\delta X_j - \delta X_i\right)}{s_{ij}^0} + \mathrm{d}Y_{ij}\frac{\left(\delta Y_j - \delta Y_i\right)}{s_{ij}^0} + \mathrm{d}Z_{ij}\frac{\left(\delta Z_j - \delta Z_i\right)}{s_{ij}^0} + s_{ij}^0 - s_{ij} \tag{9.23}$$

where $\mathrm{d}X_{ij}$, $\mathrm{d}Y_{ij}$, $\mathrm{d}Z_{ij}$, s_{ij}^0 are calculated values using approximate coordinates, s_{ij} is the measured distance and r_{s} is the residual, and δX_i, δY_i, δZ_i, δX_j, δY_j, and δZ_j are the unknown corrections to be determined and applied to the approximate Cartesian coordinates of points i and j in CT system. Equation (9.23) can also be given in matrix form as follows:

$$r_{\mathrm{s}} = \begin{bmatrix} -\frac{\mathrm{d}X_{ij}}{s_{ij}^0} & -\frac{\mathrm{d}Y_{ij}}{s_{ij}^0} & -\frac{\mathrm{d}Z_{ij}}{s_{ij}^0} & \frac{\mathrm{d}X_{ij}}{s_{ij}^0} & \frac{\mathrm{d}Y_{ij}}{s_{ij}^0} & \frac{\mathrm{d}Z_{ij}}{s_{ij}^0} \end{bmatrix} \begin{bmatrix} \delta X_i \\ \delta Y_i \\ \delta Z_i \\ \delta X_j \\ \delta Y_j \\ \delta Z_j \end{bmatrix} + s_{ij}^0 - s_{ij} \tag{9.24}$$

or

$$r_{\text{s}} = a_{11}\delta X_i + a_{12}\delta Y_i + a_{13}\delta Z_i + a_{14}\delta X_j + a_{15}\delta Y_j + a_{16}\delta Z_j + s_{ij}^0 - s_{ij} \tag{9.25}$$

Similarly, the linearized azimuth equation (9.15) and linearized vertical angle equation (9.16) can be given, respectively, as follows:

$$r_{\text{A}} = a_{21}\delta X_i + a_{22}\delta Y_i + a_{23}\delta Z_i + a_{24}\delta X_j + a_{25}\delta Y_j + a_{26}\delta Z_j + \text{Az}_{ij}^0 - \text{Az}_{ij} \tag{9.26}$$

$$r_{\text{v}} = a_{31}\delta X_i + a_{32}\delta Y_i + a_{33}\delta Z_i + a_{34}\delta X_j + a_{35}\delta Y_j + a_{36}\delta Z_j + v_{ij}^0 - v_{ij} \tag{9.27}$$

where

$$a_{11} = \frac{\partial s_{ij}}{\partial X_i} = \frac{-\,\text{d}X_{ij}}{s_{ij}^0} = -a_{14} \tag{9.28}$$

$$a_{12} = \frac{\partial s_{ij}}{\partial Y_i} = \frac{-\,\text{d}Y_{ij}}{s_{ij}^0} = -a_{15} \tag{9.29}$$

$$a_{13} = \frac{\partial s_{ij}}{\partial Z_i} = \frac{-\,\text{d}Z_{ij}}{s_{ij}^0} = -a_{16} \tag{9.30}$$

$$a_{21} = \frac{\partial \text{Az}_{ij}}{\partial X_i} = \frac{-\sin\Phi_i \cos\Lambda_i \sin\text{Az}_{ij} + \sin\Lambda_i \cos\text{Az}_{ij}}{s_{ij}\cos v_{ij}} = -a_{24} \tag{9.31}$$

$$a_{22} = \frac{\partial \text{Az}_{ij}}{\partial Y_i} = \frac{-\sin\Phi_i \sin\Lambda_i \sin\text{Az}_{ij} - \cos\Lambda_i \cos\text{Az}_{ij}}{s_{ij}\cos v_{ij}} = -a_{25} \tag{9.32}$$

$$a_{23} = \frac{\partial \text{Az}_{ij}}{\partial Z_i} = \frac{\cos\Phi_i \sin\text{Az}_{ij}}{s_{ij}\cos v_{ij}} = -a_{26} \tag{9.33}$$

$$a_{31} = \frac{\partial v_{ij}}{\partial X_i} = \frac{-s_{ij}\cos\Phi_i \cos\Lambda_i + \sin v_{ij}\text{d}X_{ij}}{s_{ij}^2 \cos v_{ij}} = -a_{34} \tag{9.34}$$

$$a_{32} = \frac{\partial v_{ij}}{\partial Y_i} = \frac{-s_{ij}\cos\Phi_i \sin\Lambda_i + \sin v_{ij}\text{d}Y_{ij}}{s_{ij}^2 \cos v_{ij}} = -a_{35} \tag{9.35}$$

$$a_{33} = \frac{\partial v_{ij}}{\partial Z_i} = \frac{-s_{ij}\sin\Phi_i + \sin v_{ij}\text{d}Z_{ij}}{s_{ij}^2 \cos v_{ij}} = -a_{36} \tag{9.36}$$

For a number of distance measurements, Equations (9.28)–(9.30) must be repeated for each measurement, making sure that the matrix elements a_{11}, a_{12}, etc. relate to appropriate parameters and columns in the overall design matrix A; similarly, for a number of azimuth (bearing) measurements, Equations (9.31)–(9.33) must be repeated for each measurement; and the same thing applies to Equations (9.34)–(9.36) for vertical angle measurements. Representing Equations (9.25)–(9.27) in matrix form will give the following:

$$\begin{bmatrix} r_{\mathrm{s}} \\ r_{\mathrm{A}} \\ r_{\mathrm{v}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix} \begin{bmatrix} \delta X_i \\ \delta Y_i \\ \delta Z_i \\ \delta X_j \\ \delta Y_j \\ \delta Z_j \end{bmatrix} + \begin{bmatrix} s_{ij}^0 - s_{ij} \\ Az_{ij}^0 - Az_{ij} \\ v_{ij}^0 - v_{ij} \end{bmatrix} \tag{9.37}$$

or

$$r = A\delta + w \tag{9.38}$$

where A is the first design matrix, r is a vector of residuals, δ is a vector of unknown corrections to the approximate Cartesian coordinates in CT system, and w is a vector of misclosures.

9.4 Parametric Model Equations in Geodetic System

Sometimes it is preferred to work with the differences in geodetic latitude, longitude, and height ($\mathrm{d}\phi_{ij}$, $\mathrm{d}\lambda_{ij}$, $\mathrm{d}h_{ij}$) as parameters instead of the Cartesian coordinate differences ($\mathrm{d}X_{ij}$, $\mathrm{d}Y_{ij}$, $\mathrm{d}Z_{ij}$); in this case, Equation (9.6) should be related to Equation (9.38). Equation (9.6) can be formulated for points i and then point j; for example, for point i the following will be obtained:

$$\begin{bmatrix} \delta X_i \\ \delta Y_i \\ \delta Z_i \end{bmatrix} = \begin{bmatrix} -(M_i + h_i)\sin\phi_i\cos\lambda_i & -(N_i + h_i)\cos\phi_i\sin\lambda_i & \cos\phi_i\cos\lambda_i \\ -(M_i + h_i)\sin\phi_i\sin\lambda_i & (N_i + h_i)\cos\phi_i\cos\lambda_i & \cos\phi_i\sin\lambda_i \\ (M_i + h_i)\cos\phi_i & 0 & \sin\phi_i \end{bmatrix} \begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \end{bmatrix} \tag{9.39}$$

or

$$\delta_i = J_i \begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \end{bmatrix} \tag{9.40}$$

Similarly, for point j, the following can be obtained:

$$\delta_j = J_j \begin{bmatrix} \delta\phi_j \\ \delta\lambda_j \\ \delta h_j \end{bmatrix} \tag{9.41}$$

Equations (9.40) and (9.41) can be combined to give

$$\begin{bmatrix} \delta X_i \\ \delta Y_i \\ \delta Z_i \\ \delta X_j \\ \delta Y_j \\ \delta Z_j \end{bmatrix} = \begin{bmatrix} J_i & 0 \\ 0 & J_j \end{bmatrix} \begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \\ \delta\phi_j \\ \delta\lambda_j \\ \delta h_j \end{bmatrix} \tag{9.42}$$

If Equation (9.37) is partitioned according to points i and j, the following can be obtained:

$$r = \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} \begin{bmatrix} \delta X_i \\ \delta Y_i \\ \delta Z_i \\ \delta X_j \\ \delta Y_j \\ \delta Z_j \end{bmatrix} + w \tag{9.43}$$

By substituting Equation (9.42) into Equation (9.43), the following are obtained:

$$r = \begin{bmatrix} A_i J_i & 0 \\ 0 & A_j J_j \end{bmatrix} \begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \\ \delta\phi_j \\ \delta\lambda_j \\ \delta h_j \end{bmatrix} + w \tag{9.44}$$

or

$$\begin{bmatrix} r_{\mathrm{s}} \\ r_{\mathrm{A}} \\ r_{\mathrm{v}} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{bmatrix} \begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \\ \delta\phi_j \\ \delta\lambda_j \\ \delta h_j \end{bmatrix} + \begin{bmatrix} s_{ij}^0 - s_{ij} \\ Az_{ij}^0 - Az_{ij} \\ v_{ij}^0 - v_{ij} \end{bmatrix} \tag{9.45}$$

Equation (9.45) can be given in matrix form in Equation (9.38) or in long form of linearized parametric equations based on the partial derivatives with respect to the unknown geodetic latitude, longitude, and height coordinates of points i and j as follows.

For *spatial distance measurement*, the parametric equation can be given as

$$r_s = b_{11}\delta\phi_i + b_{12}\delta\lambda_i + b_{13}\delta h_i + b_{14}\delta\phi_j + b_{15}\delta\lambda_j + b_{16}\delta h_j + s_{ij}^0 - s_{ij} \quad (9.46)$$

For astronomical azimuth measurement, the parametric equation can be given as

$$r_A = b_{21}\delta\phi_i + b_{22}\delta\lambda_i + b_{23}\delta h_i + b_{24}\delta\phi_j + b_{25}\delta\lambda_j + b_{26}\delta h_j + b_{27}\delta\Phi_i + b_{28}\delta\Lambda_i + \mathrm{Az}_{ij}^0 - \mathrm{Az}_{ij} \quad (9.47)$$

where the astronomical coordinates (Φ_i, Λ_i) of the setup station are treated as unknown with the corrections as $\delta\Phi_i$ and $\delta\Lambda_i$ to be determined. For the vertical angle measurement, the parametric equation can be given as

$$r_v = b_{31}\delta\phi_i + b_{32}\delta\lambda_i + b_{33}\delta h_i + b_{34}\delta\phi_j + b_{35}\delta\lambda_j + b_{36}\delta h_j + b_{37}\delta\Phi_i + b_{38}\delta\Lambda_i - \delta v + v_{ij}^0 - v_{ij} \quad (9.48)$$

where δv is the unknown residual vertical angle refraction correction and the astronomical coordinates are considered unknown. The coefficients of Equations (9.46)–(9.48) are given in Vincenty and Bowring (1978), Vincenty (1979), and Vanicek and Krakiwsky (1986) as follows:

$$b_{11} = -(M_i + h_i)\cos \mathrm{Az}_{ij} \cos v_{ij} \quad (9.49)$$

$$b_{12} = -(N_i + h_i)\cos\phi_i \sin \mathrm{Az}_{ij} \cos v_{ij} \quad (9.50)$$

$$b_{13} = -\sin v_{ij} \quad (9.51)$$

$$b_{14} = -(M_j + h_j)\cos \mathrm{Az}_{ji} \cos v_{ji} \quad (9.52)$$

$$b_{15} = -(N_j + h_j)\cos\phi_j \sin \mathrm{Az}_{ji} \cos v_{ji} \quad (9.53)$$

$$b_{16} = -\sin v_{ji} \quad (9.54)$$

$$b_{21} = \frac{(M_i + h_i)\sin \mathrm{Az}_{ij}}{s_{ij}\cos v_{ij}} \quad (9.55)$$

$$b_{22} = -\frac{(N_i + h_i)\cos\phi_i \cos \mathrm{Az}_{ij}}{s_{ij}\cos v_{ij}} \quad (9.56)$$

$$b_{23} = 0 \quad (9.57)$$

$$b_{24} = -\frac{\left[(M_j + h_j)\left(\sin\phi_i \sin\phi_j \cos\Delta\lambda \sin \mathrm{Az}_{ij} + \sin\phi_j \sin\Delta\lambda \cos \mathrm{Az}_{ij} + \cos\phi_i \cos\phi_j \sin \mathrm{Az}_{ij}\right)\right]}{s_{ij}\cos v_{ji}} \quad (9.58)$$

$$b_{25} = \frac{\left[\left(N_j + h_j\right)\cos\phi_j\left(\cos\Delta\lambda\cos\mathrm{Az}_{ij} - \sin\phi_i\sin\Delta\lambda\sin\mathrm{Az}_{ij}\right)\right]}{s_{ij}\cos\nu_{ij}} \tag{9.59}$$

$$b_{26} = 0 \tag{9.60}$$

$$b_{27} = \sin\mathrm{Az}_{ij}\tan\nu_{ij} \tag{9.61}$$

$$b_{28} = \sin\phi_i - \cos\phi_i\cos\mathrm{Az}_{ij}\tan\nu_{ij} \tag{9.62}$$

$$b_{31} = \frac{(M_i + h_i)\cos\mathrm{Az}_{ij}\sin\nu_{ij}}{s_{ij}} \tag{9.63}$$

$$b_{32} = \frac{(N_i + h_i)\cos\phi_i\sin\mathrm{Az}_{ij}\sin\nu_{ij}}{s_{ij}} \tag{9.64}$$

$$b_{33} = \frac{-\cos\nu_{ij}}{s_{ij}} \tag{9.65}$$

$$b_{34} = \frac{-\left[\left(M_j + h_j\right)\left(\cos\phi_i\sin\phi_j\cos\Delta\lambda - \sin\phi_i\cos\phi_j - \cos\mathrm{Az}_{ji}\sin\nu_{ij}\cos\nu_{ji}\right)\sec\nu_{ij}\right]}{s_{ij}} \tag{9.66}$$

$$b_{35} = \frac{-\left[\left(N_j + h_j\right)\cos\phi_j\left(\cos\phi_i\sin\Delta\lambda - \sin\mathrm{Az}_{ji}\sin\nu_{ij}\cos\nu_{ji}\right)\sec\nu_{ij}\right]}{s_{ij}} \tag{9.67}$$

$$b_{36} = \frac{\left(\cos\phi_i\cos\phi_j\cos\Delta\lambda + \sin\phi_i\sin\phi_j + \sin\nu_{ij}\sin\nu_{ji}\right)\sec\nu_{ij}}{s_{ij}} \tag{9.68}$$

$$b_{37} = \cos\mathrm{Az}_{ij} \tag{9.69}$$

$$b_{38} = \cos\phi_i\sin\mathrm{Az}_{ij} \tag{9.70}$$

According to Vincenty and Bowring (1978), the following is acceptable:

$$\cos\nu_{ji} = \cos\nu_{ij}\left(\frac{a + h_i}{a + h_j}\right) \tag{9.71}$$

where a is the semi-major axis value of the reference ellipsoid and h_i and h_j are the ellipsoidal heights of points i and j, respectively.

Parametric equation for a *total station direction measurement* can be formulated from an azimuth equation by subtracting orientation parameter (γ) from the azimuth equation; in this case there will be an approximate value (γ^0) of the orientation parameter and an unknown correction ($\delta\gamma$) subtracted from Equation (9.47). Parametric equation for *horizontal angle measurement* will be obtained by subtracting parametric equations for two corresponding azimuth measurements. If astronomical latitude (Φ_i) and longitude (Λ_i) have also been measured, two more parametric equations can be added to the linearized model as

$$r_\Phi = \delta\Phi_i + \Phi_i^0 - \Phi_i \tag{9.72}$$

$$r_\Lambda = \delta\Lambda_i + \Lambda_i^0 - \Lambda_i \tag{9.73}$$

If height difference is observed, the parametric equation for the height difference ($\mathrm{d}h_{ij}$) can be added to the linearized model as

$$r_{\mathrm{d}h} = -\delta h_i + \delta h_j + \mathrm{d}h_{ij}^0 - \mathrm{d}h_{ij} \tag{9.74}$$

In adjusting horizontal networks in three dimensions, only approximate geodetic heights of the network points are needed in the model; note also that accurate vertical angles (or zenith angles) and astronomic latitudes and longitudes are not usually measured in horizontal networks.

9.5 Parametric Model Equations in Local Astronomic System

In order to allow easier interpretation of parameters, the geodetic coordinate differences ($\mathrm{d}\phi$, $\mathrm{d}\lambda$, $\mathrm{d}h$) in Equation (9.6) can be transformed into local Cartesian coordinate differences ($\mathrm{d}n$, $\mathrm{d}e$, $\mathrm{d}u$) (in local geodetic coordinate system) by combining Equations (9.6) and (9.13) as follows. In this case, the coordinate differences in Equation (9.13) relate to a particular point and not to two points. Note that Φ_i and Λ_i may be replaced by the corresponding geodetic latitude (ϕ_i) and geodetic longitude (λ_i) in Equation (9.13) without losing accuracy. For a particular point i, the following is obtained:

$$\begin{bmatrix} \delta n_i \\ \delta e_i \\ \delta u_i \end{bmatrix} = \begin{bmatrix} M_i + h_i & 0 & 0 \\ 0 & (N_i + h_i)\cos\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \end{bmatrix} \tag{9.75}$$

or

$$\begin{bmatrix} \delta n_i \\ \delta e_i \\ \delta u_i \end{bmatrix} = H_i \begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \end{bmatrix} \tag{9.76}$$

where (δn_i, δe_i, δu_i) are linear coordinate differences at point i in the directions of north, east, and plumb line, respectively, and the matrix H is evaluated for the station i with latitude ϕ_i. From Equation (9.76), the following is obtained:

$$\begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \end{bmatrix} = H_i^{-1} \begin{bmatrix} \delta n_i \\ \delta e_i \\ \delta u_i \end{bmatrix} \tag{9.77}$$

Equation (9.77) converts local Cartesian coordinate differences to geodetic curvilinear coordinates at point i. Forming similar equation for point j, the following equation is obtained:

$$\begin{bmatrix} \delta\phi_i \\ \delta\lambda_i \\ \delta h_i \\ \delta\phi_j \\ \delta\lambda_j \\ \delta h_j \end{bmatrix} = \begin{bmatrix} H_i^{-1} & 0 \\ 0 & H_j^{-1} \end{bmatrix} \begin{bmatrix} \delta n_i \\ \delta e_i \\ \delta u_i \\ \delta n_j \\ \delta e_j \\ \delta u_j \end{bmatrix} \tag{9.78}$$

By combining Equations (9.44) and (9.78) the following is obtained:

$$r = \begin{bmatrix} A_i J_i H_i^{-1} & 0 \\ 0 & A_j J_j H_j^{-1} \end{bmatrix} \begin{bmatrix} \delta n_i \\ \delta e_i \\ \delta u_i \\ \delta n_j \\ \delta e_j \\ \delta u_j \end{bmatrix} \tag{9.79}$$

or

$$\begin{bmatrix} r_{\mathrm{s}} \\ r_{\mathrm{A}} \\ r_{\mathrm{v}} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \end{bmatrix} \begin{bmatrix} \delta n_i \\ \delta e_i \\ \delta u_i \\ \delta n_j \\ \delta e_j \\ \delta u_j \end{bmatrix} + \begin{bmatrix} s_{ij}^0 - s_{ij} \\ Az_{ij}^0 - Az_{ij} \\ v_{ij}^0 - v_{ij} \end{bmatrix} \tag{9.80}$$

Equation (9.80) can be expressed in matrix form in Equation (9.38) or in long form (with additional terms added as needed) as follows.

For *spatial distance measurement*, the parametric equation can be given as

$$r_{\mathrm{s}} = c_{11}\delta n_i + c_{12}\delta e_i + c_{13}\delta u_i + c_{14}\delta n_j + c_{15}\delta e_j + c_{16}\delta n_j + s_{ij}^0 - s_{ij} \tag{9.81}$$

For astronomical azimuth measurement, the parametric equation can be given as

$$r_{\mathrm{A}} = c_{21}\delta n_i + c_{22}\delta e_i + c_{23}\delta u_i + c_{24}\delta n_j + c_{25}\delta e_j + c_{26}\delta u_j + c_{27}\delta\Phi_i + c_{28}\delta\Lambda_i + \mathrm{Az}_{ij}^0 - \mathrm{Az}_{ij} \tag{9.82}$$

where the astronomical coordinates (Φ_i, Λ_i) of the setup station are treated as unknown with the corrections as $\delta\Phi_i$ and $\delta\Lambda_i$ to be determined. For the vertical angle measurement, the parametric equation can be given as

$$r_{\mathrm{v}} = c_{31}\delta n_i + c_{32}\delta e_i + c_{33}\delta u_i + c_{34}\delta n_j + c_{35}\delta e_j + c_{36}\delta u_j + c_{37}\delta\Phi_i + c_{38}\delta\Lambda_i - \delta v + v_{ij}^0 - v_{ij} \tag{9.83}$$

where δv is the unknown residual vertical angle refraction correction and the astronomical coordinates are considered unknown. The coefficients of Equations (9.81)–(9.83) are given in Vincenty and Bowring (1978), Vincenty (1979), and Vanicek and Krakiwsky (1986) as follows:

For *spatial distance measurement*, for example, the coefficients in Equation (9.81) can be determined by taking the partial derivatives of Equation (9.10) with respect to the unknowns, for example, with respect to n_i:

$$c_{11} = \frac{\partial s_{ij}}{\partial n_i} = \frac{-\,\mathrm{d}n_{ij}}{s_{ij}} \tag{9.84}$$

Substituting Equation (9.7) into Equation (9.84) gives

$$c_{11} = -\cos v_{ij} \cos \mathrm{Az}_{ij} \tag{9.85}$$

Similarly,

$$c_{12} = \frac{\partial s_{ij}}{\partial e_i} = -\cos v_{ij} \sin \mathrm{Az}_{ij} \tag{9.86}$$

$$c_{13} = \frac{\partial s_{ij}}{\partial u_i} = -\sin v_{ij} \tag{9.87}$$

$$c_{14} = \frac{\partial s_{ij}}{\partial n_j} = -\cos v_{ji} \cos \mathrm{Az}_{ji} \tag{9.88}$$

$$c_{15} = \frac{\partial s_{ij}}{\partial e_j} = -\cos v_{ji} \sin \mathrm{Az}_{ji} \tag{9.89}$$

$$c_{16} = \frac{\partial s_{ij}}{\partial u_j} = -\sin v_{ji} \tag{9.90}$$

For *astronomic azimuth measurement*, the coefficients in Equation (9.82) can be determined by taking the partial derivatives of Equation (9.11) with respect to the unknowns, for example, with respect to n_i:

$$c_{21} = \frac{\partial \mathrm{Az}_{ij}}{\partial n_i} = \frac{\mathrm{d}e_{ij}}{\mathrm{d}\,e_{ij}^2 + \mathrm{d}\,n_{ij}^2} \tag{9.91}$$

Substituting Equations (9.7) and (9.8) into Equation (9.91) gives

$$c_{21} = \frac{s_{ij} \cos v_{ij} \sin \mathrm{Az}_{ij}}{s_{ij}^2 \cos^2 v_{ij} \left(\sin^2 \mathrm{Az}_{ij} + \cos^2 \mathrm{Az}_{ij}\right)} \tag{9.92}$$

which can be simplified to

$$c_{21} = \frac{\sin \mathrm{Az}_{ij}}{s_{ij} \cos v_{ij}} \tag{9.93}$$

Similarly, the following are obtained:

$$c_{22} = \frac{-\cos \mathrm{Az}_{ij}}{s_{ij} \cos v_{ij}} \tag{9.94}$$

$$c_{23} = 0 \tag{9.95}$$

$$c_{24} = \frac{-\sin \mathrm{Az}_{ij}}{s_{ij} \cos v_{ij}} \left[\cos \phi_i \cos \phi_j + \sin \phi_i \sin \phi_j \cos \left(\lambda_j - \lambda_i\right) + \sin \phi_j \sin \left(\lambda_j - \lambda_i\right) \cot \mathrm{Az}_{ij} \right] \tag{9.96}$$

$$c_{25} = \frac{\cos \mathrm{Az}_{ij}}{s_{ij} \cos v_{ij}} \left[\cos \left(\lambda_j - \lambda_i\right) - \sin \phi_i \sin \left(\lambda_j - \lambda_i\right) \tan \mathrm{Az}_{ij} \right] \tag{9.97}$$

$$c_{26} = \frac{-\sin \mathrm{Az}_{ij}}{s_{ij} \cos v_{ij}} \left[\sin \left(\phi_j - \phi_i\right) + \cos \phi_j \sin \left(\lambda_j - \lambda_i\right) \cot \mathrm{Az}_{ij} \right] \tag{9.98}$$

$$c_{27} = \sin \mathrm{Az}_{ij} \tan v_{ij} \tag{9.99}$$

$$c_{28} = \sin \phi_i - \cos \phi_i \cos \mathrm{Az}_{ij} \tan v_{ij} \tag{9.100}$$

For *vertical angle measurement*, the coefficients in Equation (9.83) can be determined by taking the partial derivatives of Equation (9.12) with respect to the unknowns, for example, with respect to n_i:

$$c_{31} = \frac{\partial v_{ij}}{\partial n_i} = \frac{\mathrm{d}u_{ij} \times \mathrm{d}n_{ij}}{\left(s_{ij}^2 - \mathrm{d}\,u_{ij}^2\right)^{1/2} s_{ij}^2} \tag{9.101}$$

Substituting Equations (9.7) and (9.9) into Equation (9.101) (taking note that $\left(s_{ij}^2 - \mathrm{d}u_{ij}^2\right)^{1/2}$ is the same as $s_{ij} \cos v_{ij}$) gives

$$c_{31} = \frac{s_{ij}^2 \cos v_{ij} \cos \mathrm{Az}_{ij} \sin v_{ij}}{s_{ij} \cos v_{ij} \times s_{ij}^2} \tag{9.102}$$

which can be simplified to

$$c_{31} = \frac{\cos \mathrm{Az}_{ij} \sin v_{ij}}{s_{ij}} \tag{9.103}$$

Similarly, the following are obtained:

$$c_{32} = \frac{\sin \mathrm{Az}_{ij} \sin v_{ij}}{s_{ij}} \tag{9.104}$$

$$c_{33} = \frac{-\cos v_{ij}}{s_{ij}} \tag{9.105}$$

$$c_{34} = \frac{-\cos\phi_i \sin\phi_j \cos\left(\lambda_j - \lambda_i\right) + \sin\phi_i \cos\phi_j + \sin v_{ij} \cos v_{ji} \cos \mathrm{Az}_{ji}}{s_{ij} \cos v_{ij}} \tag{9.106}$$

$$c_{35} = \frac{-\cos\phi_i \sin\left(\lambda_j - \lambda_i\right) + \sin v_{ij} \cos v_{ji} \cos \mathrm{Az}_{ji}}{s_{ij} \cos v_{ij}} \tag{9.107}$$

$$c_{36} = \frac{\cos\phi_i \cos\phi_j \cos\left(\lambda_j - \lambda_i\right) + \sin\phi_i \sin\phi_j + \sin v_{ij} \sin v_{ji}}{s_{ij} \cos v_{ij}} \tag{9.108}$$

$$c_{37} = \cos \mathrm{Az}_{ij} \tag{9.109}$$

$$c_{38} = \cos\phi_i \sin \mathrm{Az}_{ij} \tag{9.110}$$

9.6 General Comments on Three-dimensional Adjustment

If the weight matrix (P) of measurements is expressed as usual, the parametric least squares adjustment solution for the corrections to the unknown parameters can be given as

$$\delta = -\left(A^T P A\right)^{-1} A^T P w \tag{9.111}$$

where the first design matrix A and the vector of misclosures w can be formulated from Equations (9.37), (9.45), or (9.80) with the associated parameters. The adjusted parameters ($\hat{x}$) and the adjusted observations ($\hat{\ell}$) will be given as follows:

$$\hat{x} = x^0 + \delta \tag{9.112}$$

$$\hat{\ell} = \ell + r \tag{9.113}$$

where

$$r = A\delta + w \tag{9.114}$$

x^0 is a vector of the approximate values of the parameters, ℓ is a vector of measurements, and r is the vector of residuals expressed by Equation (9.38).

If the azimuth (Equation (9.82)) is used for the total station direction measurement, an orientation parameter with coefficient of −1 can be added to the equation. In the least squares adjustment process, the adjusted positions of the previous iteration must be used at the current point of expansion. This is required irrespective of whether the partial derivatives are expressed in terms of Cartesian coordinates, geodetic latitudes, longitudes, and ellipsoidal heights or using azimuths and vertical angle measurements.

Note also that observations are not reduced to the marks on the ground, but to the line in space between the instrument and the target at the time of measurement. After the adjustment the reduction to the marks is determined indirectly "by applying the residuals, refraction corrections, and scale corrections (with the signs reversed) to the values computed by the inverse formula from adjusted coordinates of the marked point" (Vincenty 1979). The final values obtained will be identical to what would have been observed if the heights of the instruments and the target were zero.

In three-dimensional parametric adjustment, the provisional positions (usually, the geodetic coordinates) are required as input. Typically, the astronomic coordinates Φ_i and Λ_i are used to define the direction of plumb line, and their geodetic counterparts ϕ_i and λ_i with ellipsoidal height (h_i) are the true point coordinates that are essentially equivalent to the Cartesian X, Y, Z coordinates. If the astronomic coordinates are unknown, the geodetic values may be used instead (Vincenty 1979). It is also to be known that the same results will be obtained without using the geodetic latitudes, longitudes, and ellipsoidal heights anywhere in the computations since the ellipsoid is not considered at all in three-dimensional computations.

Note that since traditional observations are taken to some elevated target by an instrument at a height (HI) above the setup station, the geodetic heights of each station must be increased by the instrument heights (HIs) when computing geodetic coordinates. The approximate geodetic coordinates may be determined by first adjusting measurements in the map projection plane and converting the coordinates to geodetic equivalents later. The average geoidal heights for the region may also be used in order to determine the approximate ellipsoidal heights of points.

It should also be mentioned that vertical angles are subject to large systematic errors due to deflection of the vertical refraction and should not be used in an adjustment on a regular basis. If they must be used, the systematic errors must be corrected for or another terms to take care of the errors introduced in the model as unknown. This must be done with an understanding of the risk of over-parameterization by not adding too many unknowns to the model. Reciprocal distance measurements may be difficult to make because of heights of instrument and target changes at both ends; in this case, the forward and backward distance measurements should be treated as different. One choice for minimal constraint adjustment is to fix the coordinates (ϕ, λ, h) or (X, Y, Z) of one station, the azimuth or the longitude of another station, and the heights of two additional stations.

Differential Leveling Observations: Orthometric height differences obtained from differential leveling procedure can be included in three-dimensional geodetic network model. This will require, however, that the height differences be corrected for geoid undulation differences ($\mathrm{d}N_{ij}$) between points i and j that are being considered. The adjustment parametric model can be given as

$$r_{\mathrm{d}H} = \delta u_j - \delta u_i + \mathrm{d}H_{ij} + \mathrm{d}N_{ij} - \mathrm{d}h_{ij} \tag{9.115}$$

where $\mathrm{d}H_{ij}$ is the elevation difference between the stations and $\mathrm{d}h_{ij}$ is the change in the ellipsoidal heights between the stations. It can be seen from Equation (9.115) that orthometric height difference equation cannot be formulated without a reference ellipsoid.

9.7 Adjustment Examples

The simple examples given in this section are mainly for the purpose of illustrating how to implement the equations discussed in this chapter. The examples can be solved using Microsoft Excel and MATLAB software applications or other programming environments. The author, however, used the MATLAB software application in processing the given measurements.

9.7.1 Adjustment in Cartesian Geodetic System

Three-dimensional data with regard to the network of points P1, P2, and P3 in Figure 9.4 were collected as shown in Table 9.1. Point P1 is a control point that is to be kept fixed during adjustment; the geodetic coordinates of the point and the approximate geodetic coordinates of P2 and P3 are provided in Table 9.2.

The adjustment in Cartesian system process starts with the use of Equations (9.1)–(9.5) with the results summarized in Tables 9.3 and 9.4 (note that the negative values of longitudes are used since the points are in the "West" region).

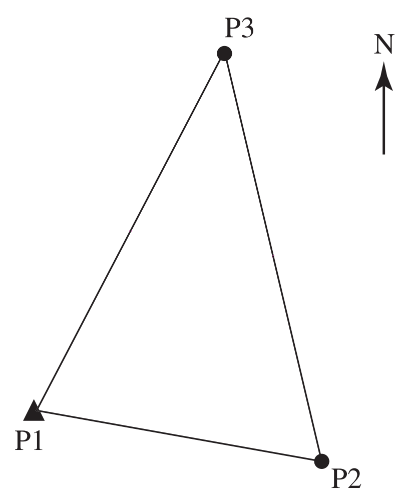


Figure 9.4 Sample 3D geodetic network.

Table 9.1 Field measurements.

Leg	Distance (m)	Zenith angle	Bearing
P1–P2	330.305 ± 0.003	89°53′10″ ± 5″	103°30′10″ ± 5″
P1–P3	584.140 ± 0.004	90°18′15″ ± 5″	21°14′30″ ± 5″
P2–P3	631.160 ± 0.003	90°20′40″ ± 5″	350°00′20″ ± 5″

Table 9.2 Initial geodetic coordinates of network points.

Point	Latitude (ϕ)	Longitude (λ)	Ellipsoidal height (m)
P1	49°05′24.73726″N	127°23′56.95384″W	291.895
P2	49°05′22.24485″N	127°23′41.12441″W	292.448
P3	49°05′42.33153″N	127°23′46.52178″W	289.560

Table 9.3 Radii of curvature calculations based on GRS80 ellipsoid.

Point	*N* (m)	*M* (m)
P1	6 390 365.3638	6 371 948.7421
P2	6 390 365.1070	6 371 947.9739
P3	6 390 367.1767	6 371 954.1651

Table 9.4 Initial Cartesian geodetic coordinates.

Point	*X* (m)	*Y* (m)	*Z* (m)
P1	−2 541 849.080 2	−3 324 702.344 2	4 797 354.789 3
P2	−2 541 629.586 0	−3 324 944.056 7	4 797 304.782 3
P3	−2 541 429.789 9	−3 324 502.379 0	4 797 708.960 7

The network in Figure 9.4 with the measurements in Table 9.1 is to be adjusted three-dimensionally using the method of least squares by fixing the three-dimensional Cartesian coordinates (X, Y, Z) of point P1 (by assigning standard deviation of 0.000 01 m to each coordinate) and the

Z coordinate of point P2 (by assigning standard deviation of 0.05 m to the coordinate).

9.7.1.1 Solution Approach

In adjusting the network, Equation (9.37) or Equation (9.38) will be used in order to be able to constrain the X, Y, Z of point P1 and Z of point P3. The first design matrix (A) is derived from Equations (9.37) and (9.38) with additional constraint equations due to the fixing of the X_1, Y_1, Z_1 coordinates of point P1 and of the Z_2 coordinate of point P2. These additional constraint equations, which constitute the 10th to 13th observations, can be expressed as follows:

$$r_{10} = \delta X_1 \tag{9.116}$$

$$r_{11} = \delta Y_1 \tag{9.117}$$

$$r_{12} = \delta Z_1 \tag{9.118}$$

$$r_{13} = \delta Z_2 \tag{9.119}$$

where the misclosures are zero (since the corresponding coordinates are the measurements); r_{10}, r_{11}, r_{12}, and r_{13} are the residuals of the observations; and δX_1, δY_1, δZ_1, and δZ_2 are the coordinate changes. The observations in Equations (9.116)–(9.118) are given very small standard deviations (0.001 mm) to ensure they are fixed after adjustment, and the observation in Equation (9.119) is given a standard deviation of 0.05 m since it is not well known. The first nine equations formulated from Equations (9.25)–(9.27) are based on three distances, three bearings, and three zenith angles. In formulating the A-matrix, the elements of the matrix due to the three distance measurements are derived from Equations (9.28)–(9.30), those due to the bearing measurements are derived from Equations (9.31)–(9.33), and those due to the zenith angles are derived from Equations (9.34)–(9.36), remembering that 90° minus zenith angle will give the vertical angle. The size of A-matrix is 13 observations by 9 unknown parameters (with the fixed parameters constrained by Equations (9.116)–(9.119) and to be highly weighted using some specified variances). The elements of A-matrix must also correspond with the appropriate parameters. For example, distance P1–P2 will have a_{11}, a_{12}, a_{13} values in the columns corresponding to δX_1, δY_1, δZ_1 and a_{14}, a_{15}, a_{16} in columns corresponding to δX_2, δY_2, and δZ_2, respectively; distance P1–P3 will have a_{11}, a_{12}, a_{13} values in the columns corresponding to X_1, Y_1, Z_1 and a_{14}, a_{15}, a_{16} in columns corresponding to δX_3, δY_3, and δZ_3, respectively; constraint Equation (9.116) will have 1.0 in column corresponding to δX_1 in row

10 corresponding to the order of the equation; etc. The computed A-matrix for this problem is given as follows:

$$A = \begin{bmatrix} -0.665 & 0.732 & 0.151 & 0.665 & -0.732 & -0.151 & 0.0 & 0.0 & 0.0 \\ -0.718 & -0.342 & -0.606 & 0.0 & 0.0 & 0.0 & 0.718 & 0.342 & 0.606 \\ 0.0 & 0.0 & 0.0 & -0.317 & -0.699\,8 & -0.640\,4 & 0.316\,6 & 0.699\,8 & 0.640\,4 \\ -0.002 & -0.001 & -0.002 & 0.002 & 0.001\,3 & 0.001\,9 & 0.0 & 0.0 & 0.0 \\ -0.001 & 0.001 & 0.000\,4 & 0.0 & 0.0 & 0.0 & 0.001\,0 & -0.001\,3 & -0.000\,4 \\ 0.0 & 0.0 & 0.00 & -0.001\,4 & 0.000\,8 & -0.000\,18 & 0.001\,4 & -0.000\,8 & 0.000\,2 \\ 0.001 & 0.002 & -0.002\,2 & -0.001\,2 & -0.001\,6 & 0.002\,3 & 0.0 & 0.0 & 0.0 \\ 0.001 & 0.001 & -0.001\,3 & 0.0 & 0.0 & 0.0 & -0.000\,7 & -0.000\,9 & 0.001\,3 \\ 0.0 & 0.0 & 0.0 & 0.000\,6 & 0.000\,8 & -0.001\,2 & -0.000\,6 & -0.000\,8 & 0.001\,2 \\ 1.000 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.000 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.000 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.00 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The least squares corrections to the approximate coordinates determined using the given geodetic coordinates are as follows:

$$\delta = -\left(A^T P A\right)^{-1} A^T P w$$
$$= [0.0 \;\; 0.0 \;\; 0.0 \;\; -0.0007 \;\; 0.0009 \;\; 0.0067 \;\; -0.0303 \;\; 0.0028 \;\; 0.0236]^T$$

The adjusted geodetic Cartesian coordinates are

$$x^0 + \delta = \begin{bmatrix} \hat{X}_1 \\ \hat{Y}_1 \\ \hat{Z}_1 \\ \hat{X}_2 \\ \hat{Y}_2 \\ \hat{Z}_2 \\ \hat{X}_3 \\ \hat{Y}_3 \\ \hat{Z}_3 \end{bmatrix} = \begin{bmatrix} -2\,541\,849.080\,2 \\ -3\,324\,702.344\,2 \\ 4\,797\,354.789\,3 \\ -2\,541\,629.586\,8 \\ -3\,324\,944.055\,8 \\ 4\,797\,304.789\,0 \\ -2\,541\,429.820\,2 \\ -3\,324\,502.376\,1 \\ 4\,797\,708.984\,3 \end{bmatrix}$$

where the approximate coordinates (x^0) were determined from the given geodetic coordinates and given in Table 9.4. The adjusted observations are given in Table 9.5 (note that the parameters adjusted as observations gave the same results as the adjusted parameters).

Table 9.5 Adjusted observations.

Leg	Distance (m)	Zenith angle	Bearing
P1–P2	330.3057 ± 0.0028	89°53′08.2″ ± 4.4″	103°30′08.0″ ± 3.3″
P1–P3	584.1406 ± 0.0036	90°18′18.4″ ± 3.7″	21°14′23.9″ ± 3.0″
P2–P3	631.1588 ± 0.0036	90°20′36.4″ ± 3.5″	350°00′24.1″ ± 2.9″

Table 9.6 Computed corrections to the geodetic coordinates of network points.

Point	Latitude ($\delta\phi$)	Longitude ($\delta\lambda$)	Ellipsoidal height (δh) (m)
P1	0.000 00″	0.000 00″	0.000 0
P2	+0.000 15″	−0.000 06″	+0.004 9
P3	+0.000 11″	−0.001 27″	+0.028 4

The covariance matrix ($C_{\hat{\ell}}$) of the adjusted observations is obtained using the following formula:

$$C_{\hat{\ell}} = s_0^2\left[A\left(A^{\mathrm{T}}PA\right)^{-1}A\right] \tag{9.120}$$

where the a posteriori variance factor of unit weight (s_0^2) was determined after the adjustment as 0.9101 with the number of degrees of freedom as 4. By using the transformation formula in Equation (9.42), three block diagonal submatrices of the coefficient matrix J can be formed for this problem; each block is a 3×3 submatrix formed using Equation (9.39), giving the overall J matrix of size 9×9. By inversion of the formula in Equation (9.42), the corrections to the given geodetic coordinates (ϕ, λ, h) in Table 9.2 are obtained, and the results are given in Table 9.6.

9.7.2 Adjustment in Curvilinear Geodetic System

Continuing from the example given in Section 9.7.1, assume that the geodetic network is to be adjusted in curvilinear (ϕ, λ, h) coordinate system (i.e. adjustment in curvilinear system) as discussed in Section 9.4. In this case, Equations (9.45)–(9.70) will be used. Assume the latitude, longitude, and the ellipsoidal height of point P1 and the ellipsoidal height of point P2 will be fixed to constraint the three-dimensional least squares adjustment of the measurements in Table 9.1. The constraint equations to be used with the measurement equations in Equation (9.45) can be expressed as follows:

$$r_{10} = \delta\phi_1 \tag{9.121}$$

$$r_{11} = \delta\lambda_1 \tag{9.122}$$

$$r_{12} = \delta h_1 \tag{9.123}$$

$$r_{13} = \delta h_2 \tag{9.124}$$

where the misclosures are zero (since the coordinates are directly measured) and $\delta\phi_1$, $\delta\lambda_1$, δh_1, and δh_2 are the geodetic coordinate changes. The observations in Equations (9.121)–(9.122) are given very small standard deviations (0.001″), the observation in Equation (9.123) is given 0.01 mm standard deviation, and the observation in Equation (9.124) is given a standard deviation of 0.05 m since it is not well known. The first design matrix A has a size of 13 equations by 9 unknown parameters; the unknown parameters include the fixed coordinates of point P1. The resulting A-matrix is given as follows; the adjustment results are given in Tables 9.7–9.9.

$$A = \begin{bmatrix}
-1.49E6 & 4.07E6 & -1.98E-3 & 1.49E6 & -4.07E6 & -1.98E-3 & 0.0 & 0.0 & 0.0 \\
-5.94E6 & -1.52E6 & 5.37E-3 & 0.0 & 0.0 & 0.0 & 5.94E6 & 1.52E6 & 5.37E-3 \\
0.0 & 0.0 & 0.0 & -6.28E6 & 7.26E5 & 6.03E-3 & 6.28E6 & -7.26E5 & 6.03E-3 \\
-1.88E4 & -2.96E3 & 0.0 & 1.88E4 & 2.96E3 & 0.0 & 0.0 & 0.0 & 0.0 \\
3.95E3 & -6.68E3 & 0.0 & 0.0 & 0.0 & 0.0 & -3.95E3 & 6.68E3 & 0.0 \\
0.0 & 0.0 & 0.0 & -1.75E3 & -6.53E3 & 0.0 & 1.75E3 & 6.53E3 & 0.0 \\
8.9 & -24.4 & -3.03E-3 & -8.69E3 & 23.8 & 3.03E-3 & 0.0 & 0.0 & 0.0 \\
-55.0 & -13.9 & -1.71E-3 & 0.0 & 0.0 & 0.0 & 53.71 & 13.71 & 1.71E-3 \\
0.0 & 0.0 & 0.0 & -59.97 & 6.94 & -1.58E-3 & 58.99 & -6.82 & 1.58E-3 \\
1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0
\end{bmatrix}$$

The a posteriori variance factor of unit weight for the adjustment is 0.6936 with the number of degrees of freedom as 4.

Table 9.7 Computed corrections to the geodetic coordinates of network points.

Point	Latitude ($\delta\phi$)	Longitude ($\delta\lambda$)	Ellipsoidal height (δh) (m)
P1	0.000 00″	0.000 00″	0.000 0
P2	+0.000 12″	+0.000 22″	+0.004 9
P3	+0.000 06″	−0.001 13″	+0.028 5

Table 9.8 Adjusted geodetic coordinates of network points.

Point	Latitude (ϕ)	Longitude (λ)	Ellipsoidal height (m)
P1	49°05′24.73726″N	127°23′56.95384″W	291.8950
P2	49°05′22.24497″N	127°23′41.12463″W	292.4529
P3	49°05′42.33159″N	127°23′46.52065″W	289.5885

Table 9.9 Adjusted observations.

Leg	Distance (m)	Zenith angle	Bearing
P1–P2	330.3044 ± 0.0025	89°53′08.2″ ± 3.8″	103°30′08.3″ ± 2.9″
P1–P3	584.1405 ± 0.0031	90°18′18.4″ ± 3.2″	21°14′25.1″ ± 2.6″
P2–P3	631.1590 ± 0.0031	90°20′36.3″ ± 3.0″	350°00′23.2″ ± 2.6″

9.7.3 Adjustment in Local System

Continuing from the example given in Section 9.7.1, assume that the geodetic network is to be adjusted in the LA (n, e, u) coordinate system (i.e. adjustment in local system) as discussed in Section 9.5. In this case, Equations (9.80)–(9.110) will be used. Assume the local coordinates (n, e, u) of point P1 and the u-coordinate of points P2 will be fixed to constraint the three-dimensional least squares adjustment of the measurements in Table 9.1. The constraint equations to be used with the measurement equations in Equation (9.80) can be expressed as follows:

$$r_{10} = \delta n_1 \tag{9.125}$$

$$r_{11} = \delta e_1 \tag{9.126}$$

$$r_{12} = \delta u_1 \tag{9.127}$$

$$r_{13} = \delta u_2 \tag{9.128}$$

where the misclosures are zero (since the coordinates are measured directly) and δn_1, δe_1, δu_1, and δu_2 are the coordinate changes. The constraint measurements in Equations (9.125)–(9.127) are assigned very small standard deviations of 0.01 mm, and the u-coordinate value for point P2 in Equation (9.128) is assigned a standard deviation of 0.05 m since it is not well known. The first design matrix A formed from Equation (9.80) with the constraint Equations (9.125)–(9.128) is given as a matrix of 13 rows (for the number of equations) by 9 columns (for the number of unknown coordinate corrections) as follows:

$$A=\begin{bmatrix} -0.233 & 0.972 & 1.98E-3 & 0.233 & -0.972 & -2.03E-3 & 0.0 & 0.0 & 0.0 \\ -0.932 & -0.362 & 5.37E-3 & 0.0 & 0.0 & 0.0 & 0.932 & 0.362 & -5.31E-3 \\ 0.0 & 0.0 & 0.0 & -0.985 & 0.173 & 6.0E-3 & 0.985 & -0.173 & -6.0E-3 \\ -2.9E-3 & -7.1E-4 & 0.0 & 2.9E-3 & 7.1E-4 & -7.1E-8 & 0.0 & 0.0 & 0.0 \\ 6.2E-4 & -1.6E-3 & 0.0 & 0.0 & 0.0 & 0.0 & -6.2E-4 & 1.6E-3 & -1.1E-7 \\ 0.0 & 0.0 & 0.0 & -2.8E-4 & -1.6E-3 & 0.0 & 2.75E-4 & 1.6E-3 & 5.4E-8 \\ 1.4E-6 & -5.8E-6 & -3.0E-3 & 1.4E-6 & -1.6E-6 & 1.0 & 0.0 & 0.0 & 0.0 \\ -8.6E-6 & -3.3E-6 & -1.7E-3 & 0.0 & 0.0 & 0.0 & -8.7E-6 & 8.5E-6 & 1.0 \\ 0.0 & 0.0 & 0.0 & -9.4E-6 & 1.66E-6 & -1.6E-3 & -9.6E-6 & 9.4E-6 & 1.0 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The least squares computed corrections of δn, δe, and δu are given as

$$\begin{aligned}\delta &= -\left(A^T PA\right)^{-1} A^T Pw \\ &= [0.0\ \ 0.0\ \ 0.0\ \ 0.0037\ \ 0.0044\ \ 0.0\ \ 0.0019\ \ -0.023\ \ 0.0]^T\end{aligned}$$

The computed corrections ($\delta n, \delta e, \delta u$) are for each instrument setup points. In order to interpret the vector of corrections determined in this section, one should take note that LA system is not a unified coordinate system since each instrument setup point is a separate coordinate system with its separate origin. This is the reason why the initial coordinates of the instrument points must be given in unified coordinate system such as the Cartesian geodetic (X, Y, Z) coordinate system or the curvilinear geodetic (ϕ, λ, h) coordinate system. Since the computed corrections actually do not relate to the same coordinate system, the geodetic coordinates must be computed to represent the coordinates of the instrument points (in a unified coordinate system). In this case, the corrections must be transformed into changes in the geodetic coordinates using the inverse formula of Equation (9.78).

The H-matrix based on Equations (9.75)–(9.78) is based on three diagonal submatrix blocks formed for each geodetic point using Equation (9.75); this results in a matrix of size 9×9. The determined H-matrix is given as follows:

$$H = \text{diag}([6.372E6\ \ 4.185E6\ \ 1.0\ \ 6.372E6\ \ 4.185E6\ \ 1.0\ \ 6.372E6\ \ 4.185E6\ \ 1.0])$$

Table 9.10 Adjusted observations.

Leg	Distance (m)	Zenith angle	Bearing
P1–P2	330.3044 ± 0.0027	89°53′10.0″ ± 4.6″	103°30′08.3″ ± 3.2″
P1–P3	584.1406 ± 0.0035	90°18′19.7″ ± 3.3″	21°14′25.0″ ± 2.9″
P2–P3	631.1589 ± 0.0035	90°20′35.3″ ± 3.3″	350°00′23.2″ ± 2.8″

The computed changes in the curvilinear geodetic coordinates ($\delta\phi$, $\delta\lambda$, δh) are given as follows:

$$\begin{bmatrix} \delta\phi_1 \\ \delta\lambda_1 \\ \delta h_1 \\ \delta\phi_2 \\ \delta\lambda_2 \\ \delta h_2 \\ \delta\phi_3 \\ \delta\lambda_3 \\ \delta h_3 \end{bmatrix} = H^{-1} \begin{bmatrix} \delta n_1 \\ \delta e_1 \\ \delta u_1 \\ \delta n_2 \\ \delta e_2 \\ \delta u_2 \\ \delta n_3 \\ \delta e_3 \\ \delta u_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \delta\phi_1 \\ \delta\lambda_1 \\ \delta h_1 \\ \delta\phi_2 \\ \delta\lambda_2 \\ \delta h_2 \\ \delta\phi_3 \\ \delta\lambda_3 \\ \delta h_3 \end{bmatrix} = \begin{bmatrix} 0.000\ 00'' \\ 0.000\ 00'' \\ 0.000\ 0\ \text{m} \\ +0.000\ 12'' \\ +0.000\ 22'' \\ +0.000\ 1\ \text{m} \\ +0.000\ 06' \\ -0.001\ 13'' \\ 0.000\ 0\ \text{m} \end{bmatrix}$$

The adjusted observations and their corresponding standard deviations are given in Table 9.10.

The computed a posteriori variance factor of unit weight is 0.8509 with the number of degrees of freedom as 4.