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A TESTING PROCEDURE FOR USE IN GEODETIC NETWORKS

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PREFACE

This paper was presented as the report of I.A.G. Special Study Group No. 1.14 (Specifications for Triangulation and Trilateration Nets) at the XIVth General Assembly of the International Geodetic Association, Lucerne 1967.

For this publication the text has been amended or extended in some parts; this concerns mainly the sections 2, 10, 11 and 12. The diagram shown in figure 3-3 was recomputed and has been replaced by three new diagrams computed with the values $\beta_0 = 0.70$, 0.80 and 0.90 respectively. The assistance of the members of my staff in revising the original paper is greatfully acknowledged. The author is greatly indebted to J. E. ALBERDA for translating the manuscript. In this case it was certainly more than a translation since new words had to be found for new concepts. Moreover during many helpful discussions Mr. ALBERDA has suggested several modifications to the text to improve the clarity for the reader. Sincere thanks are also due to Miss CARLA SMIT for her unusual skill at mathematical typing and to John Wiley and Sons, New York, for their permission to print a few quotations from A. L. ACKOFF, Scientific Method. Optimizing Applied Research Decisions.

Delft, January 1968 W. BAARDA

CONTENTS

SUMMARY

In this paper an outline for the continuation of the work in I.A.G. Special Study Group No. 1.14 is given, based on the paper "Statistical Concepts in Geodesy". As a first step, a testing procedure for geodetic networks is discussed, using one-sided F-tests; a definition of the concept "reliability of geodetic networks" is proposed.

1. The testing procedure is a part of the so-called unlinking of the computing model and consequently a part of the prediction. This new theory was developed to obtain a clearer line of thought in testing and to overcome existing confusion by pointing out the interconnection of consequences. The theory is as much as possible connected to current methods; the purpose was to find a theory applicable in practice.

2. For geodetic networks it is important to have the possibility to make checks soon after completion of the measurements in order to make partial remeasurements possible while towers and signals are still standing. Remeasurement shortly afterwards, even remeasuring slightly too much, will in the long run prove to be cheaper than hidden {gross} errors. The latter will often lead to strong local distortion of the network and render difficult the effective control of densification measurements or possibly even prevent that the objectives of such measurements are attained.

3. Because of the random character of observations it is impossible to signalize {gross} errors with certainty. At the best only statements having a certain probability of success, can be made. The order of magnitude of this probability β_0, has to be agreed upon; it is one of the parameters in the theory. β_0 leads to a lower bound for the order of magnitude of a function λ_0 of gross errors which can just be signalized by a test; λ_0 is determined by one specified alternative hypothesis, provided that a second parameter is fixed. Multidimensional tests are reduced to a one-dimensional test via this particular alternative hypothesis. The second parameter then is the significance level α_0 of this one-dimensional test. The significance level α of a b-dimensional test is then dependent on b, if in combination with the same β_0 the same bound for the signalization of errors λ_0 is required. This leads, on the one hand, to the designed testing procedure, and on the other hand to the definition of the "reliability of geodetic networks".

4. Planning the precision as well as the reliability of geodetic networks requires a quantification of the demands following from the purpose of the network. This quantification is the most difficult problem and it is certainly not solved yet. In view of the difficulty of foreseeing future applications of the network, it is questionable whether more than the formulation of partial or relative purposes and their quantifications can ever be attained. It will be necessary to come to a conclusion in order to make a justified choice of values $\{\alpha_0, \beta_0\}$, possibly by applying the decision theory approach.

5. Methods for "data-snooping" follow from the reasoning developed. Connections with publications by BJERHAMMAR are given, whereas a comparison is made with related problems from mathematical statistics.

6. The reasoning is generally applicable and need not be restricted to geodetic networks. It can also be applied when arbitrary distribution functions are used, possibly along with F-statistics, provided the power function of the tests can be computed. In this paper only the method using F-distributions has been worked out.

1 THE PROBLEMS

The report of I.A.G. Special Study Group No. 1.14 presented at the Berkeley Assembly of 1963 [1] can be considered as the conclusion of the first phase of the work of this group. A limited number of recommendations for the structure of networks were carefully worded. The recommendations have a practical character, they are more founded on experience than on theoretical considerations. A coherent theory is still lacking, so that for the time being it was not possible to proceed on the road chosen. This is also connected with the fact that it was difficult to broaden the field of work of the Study Group.

In continuation of the views expressed in the introduction of [1], the President of SSG No. 1.14, Professor ASPLUND, proposed in a circular letter to the members, dated August, 1965, to initiate a second phase in the work of the group, in which the emphasis should be laid on preliminary theoretical considerations. Lack of time compelled Prof. ASPLUND to hand over the Presidency of the group to the author of this paper.

Preliminary discussions took place in November, 1964, in Stockholm, and it was agreed to start from an earlier study [2], in which a sort of programme had been developed for further and more coherent specifications. At that time the theory seemed to be developed far enough to serve as a basis for discussion. The most important problem was considered to be:

(1.1) the construction of an artificial covariance matrix, to serve as a mathematical translation of the lower limit of precision required by the purposes of geodetic networks {cf. [3] pages 71 and 37}.

The formation of such a matrix is only possible if one disengages oneself from coordinates and similar dimensioned quantities, so that the quantification of variances is freed from the effects of systematic errors in scale and azimuth. The relevant theory, already sketched in [2], has since been developed to the so-called "Polygon Theory in the Complex Plane". A practical test on a proposed solution of (1.1) was therefore possible, but the solution proved not to satisfy essential theoretical requirements.

Further preliminary study showed that a second problem had to be solved:

(1.2) the use of statistical tests in connection with the adjustment of networks – in particular the choice of critical regions and the computation of the values of model errors which can just be detected with a certain probability β {see [3] pages 73 and 16 to 32} – in order to assure the reliability of geodetic networks in relation to their purpose in society.

This second problem is a notorious one in mathematical statistics. A generally accepted satisfactory solution is not known. In geodesy one is confronted with this problem by the extensive mechanization and rationalization of adjustment computations. In these computations consciously {or perhaps unconsciously} a choice is made from different simplified methods. Therefore it seems to be acceptable to try the same for solving problem (1.2).

The paper [3] was written to give a survey of the connection between statistical and functional problems in geodesy. In its final form, [3] is meant to be a basis for further considera-

tions and discussions in SSG 1.14. In particular it tries to indicate the problems that should be solved in the first instance. This may explain a certain one-sidedness, which is present in spite of the fact that of course much more literature was consulted than the list of references on page 54 in [3] shows.

Since [3] was written, considerable progress has been made, in particular with respect to problem (1.2). This makes it possible to present now some more concrete ideas. Because of the computations that had to be executed, it was not possible to distribute this paper among the members of the Study Group before the Lucerne Assembly in 1967. Reactions can therefore only be assimilated in a subsequent paper.

The study of the problem (1.1), in particular, has led to a careful investigation on the quantities that must be deemed essential for the linking of a mathematical computing model to observations. This gave rise to considerations about model theory, as sketched in [3] pages 44 ff. Although this approach does not exclude more general probability distributions, the probability distribution of Laplace-Gauss is accepted for simplicity. But one should be aware of more complicated situations, as indicated by ROMANOWSKI {[4] and references indicated} and BJERHAMMAR [5]. Working in phases or steps with testing may cause truncation of distribution functions. But it will be better to reserve problems of this type for SSG 1.24, although some consequences must be discussed in this paper.

The study on computing models, in [2] and [3] is mainly restricted to the plane. This corresponds with most investigations into the precision of geodetic networks, also with those developed in previous papers of SSG 1.14. However, this restriction causes difficulties when astronomical observations are analysed, not to mention satellite observations. The classical line of thought to overcome all or part of these difficulties was the extrapolation of results obtained in the plane to the curved surface of the reference ellipsoid used as a quasi-spatial computing model. On this way one meets mathematical difficulties, as were found in an attempt to generalize the "Polygon Theory in the Complex Plane" to an ellipsoidal theory; cf. also the remarks in [3] pages 43 and 52.

For the purposes of this Study Group, a generalization to the so-called "Spatial Polygon Theory" is more important. This is a truly spatial computing model of which an indication is given in [3] page 52.

In the theory that was developed for the plane, the adoption of a coordinate system is not essential, and consequently the direction of north is not essential either. It is interesting to search for the analogue of this matter in the spatial theory. That the idea is not new is shown by a remark made by JOHN A. O'KEEFE in 1958, cf [6] page 28:

"What I contend is that north is obsolete, that the concept of north should not be a part of modern geodesy".

Restricting oneself to what is called "geometric geodesy" and leaving gravimetric geodesy out of consideration, this seems indeed possible. However, this has consequences for the use of astronomical measurements as a contribution to the construction of a geodetic network. The absolute character of the astronomical coordinate system aimed at can then perhaps be relativated, by which possibly the importance of sources of systematic errors can be diminished. It is clear that this has a repercussion on the concept "time", whereas "simultaneousness" gains in importance, as with satellite observations. Thus one obtains a possibility to broaden the field of study, which is nevertheless closely connected to the

considerations in [3] pages 45 ff. In course of time one will certainly have to consider also the third problem:

(1.3) the consequences for geodetic networks of a relativation of the concepts north and time.

It will be clear that in this way the restriction on the field of work, which had to be accepted in [1], can be lifted. Also, one need not recoil from the search for closer ties with gravimetric geodesy. For some indications about this, see [3] page 53. As things are now, it does not seem possible to eliminate the concept "north" in this field, although at first sight its importance is not great.

In the following sections of this paper, only the problem (1.2) will be treated, as a continuation and sharpening of [3]. In this way, the discussion on the *reliability* of networks can be started.

2 ADJUSTMENT IN ONE STEP EFFECTS OF ALTERNATIVE HYPOTHESES

It is assumed that a vector of observations (x^i) has been obtained, to which by way of the registration R a computing model H_0 has been linked up. See (2.2.1) and (2.2.2) in [3]. We use "condition model" for "laws of nature":

$$\left.\begin{array}{l}\{\text{covariance matrix of } (\underset{\sim}{x}^i)\} = \sigma^2(g^{ij}) = \sigma^2(\bar{g}_{ji})^{-1} \\ \text{condition model: } (u_i^\varrho)(\tilde{x}^i - a_0^i) = (0)\end{array}\right\} \quad \ldots\ldots\ldots \quad (2.1)$$

With $(a_\alpha{}^i)$ from, see (2.2.3) in [3]:

$$(u_i^\varrho)(a_\alpha^i) = (0)$$

a first orthogonalization with respect to the probability distribution of $(\underset{\sim}{x}^i)$ is reached by the transformation, entirely aimed at the condition model:

$$\left.\begin{array}{l}\begin{pmatrix}\underset{\sim}{y}^\varrho \\ \underset{\sim}{y}_\beta\end{pmatrix} = \begin{pmatrix}(u_i^\varrho) \\ (a_\beta^j)^*(\bar{g}_{ji})\end{pmatrix}(\underset{\sim}{x}^i - a_0^i) \\ \{i,j = 1, \ldots, m;\ \varrho,\tau = 1, \ldots, b;\ \alpha,\beta = b+1, \ldots, m\}\end{array}\right\} \quad \ldots\ldots\ldots \quad (2.2)$$

from which follows, see (2.2.11) in [3]:

$$\left.\begin{array}{l}\{\text{covariance matrix of } \begin{pmatrix}\underset{\sim}{y}^\varrho \\ \underset{\sim}{y}_\beta\end{pmatrix}\} = \sigma^2\begin{pmatrix}g^{\varrho\tau} & 0 \\ 0 & g_{\beta\alpha}\end{pmatrix} = \sigma^2\begin{pmatrix}\bar{g}_{\tau\varrho} & 0 \\ 0 & \bar{g}^{\alpha\beta}\end{pmatrix}^{-1} \\ \text{condition model: } (\tilde{y}^\varrho) = (0)\end{array}\right\} \quad (2.3)$$

An illustration of adjustment and testing can be obtained by mapping on to a standardized*) sample space with:

$$\{\text{metric of coordinate system}\} = \{\text{covariance matrix}\}^{-1} \quad \ldots\ldots\ldots \quad (2.4)$$

The loci of the representations of vectors (x^i) with equal probability density and consequently also the standard {hyper} ellipsoid of $(\underset{\sim}{x}^i)$ then are hyperspheres.

---

*) Terminology used by TIENSTRA in [7]. The name sample space comes from mathematical statistics, see also [8], [9] and [10].

A two-dimensional sketch is given in figure 2-1, in which is also pictured an alternative hypothesis H_a, as defined in (3.1.5) in [3], e.i. *only* a possible shifting of the probability distribution from H_0 is considered. In this figure we have the following notation:

"measured point" P = representation of vector (x^i)
"adjusted point" $P_{(\beta)}$ = representation of vector (X^i)
= projection of P on y_β-subspace
$P^{(\varrho)}$ = projection of P on y^ϱ-subspace
M_0 = representation of vector $(\tilde{x}^i)$
{situation in y^β-subspace unknown}
$M_0^{(\varrho)}$ = representation of vector (a_o^i)
= projection of M_0 on y^ϱ-subspace
M_a = representation of vector $(\tilde{x}^i + \widetilde{\nabla x}^i)$
$M_a^{(\varrho)}$ = projection of M_a on y^ϱ-subspace

$$\text{distance}^2\{\overline{M_0P}\}^2 = \left(\frac{x^j-\tilde{x}^j}{\sigma}\right)^*(\bar{g}_{ji})\left(\frac{x^i-\tilde{x}^i}{\sigma}\right) = \{\overline{M_0P_{(\beta)}}\}^2 + \{\overline{PP_{(\beta)}}\}^2$$

$$\{\overline{M_0P_{(\beta)}}\}^2 = \left(\frac{y_\alpha-\tilde{y}_\alpha}{\sigma}\right)^*(\bar{g}^{\varrho\beta})\left(\frac{y_\beta-\tilde{y}_\beta}{\sigma}\right)$$

$$\{\overline{PP_{(\beta)}}\}^2 = \{\overline{M_0^{(\varrho)}P^{(\varrho)}}\}^2 = \left(\frac{y^\tau}{\sigma}\right)^*(\bar{g}_{\tau\varrho})\left(\frac{y^\varrho}{\sigma}\right)$$

$$\{\overline{M_0^{(\varrho)}P_{(\beta)}}\}^2 = \left(\frac{y_\alpha}{\sigma}\right)^*(\bar{g}^{\alpha\beta})\left(\frac{y_\beta}{\sigma}\right)$$

$$\{\overline{M_0^{(\varrho)}P}\}^2 = \left(\frac{x^j-a_0^j}{\sigma}\right)^*(\bar{g}_{ji})\left(\frac{x^i-a_0^i}{\sigma}\right) = \{\overline{M_0^{(\varrho)}P_{(\beta)}}\}^2 + \{\overline{M_0^{(\beta)}P^{(\varrho)}}\}^2$$

with analogous formulas for ∇-quantities, such as:

$$\{\overline{M_0M_a}\}^2 = \left(\frac{\widetilde{\nabla x}^j}{\sigma}\right)^*(\bar{g}_{ji})\left(\frac{\widetilde{\nabla x}^i}{\sigma}\right) = \{\overline{M_0M_{a(\beta)}}\}^2 + \{\overline{M_aM_{a(\beta)}}\}^2$$

$$\{\overline{M_0M_{a(\beta)}}\}^2 = \left(\frac{\widetilde{\nabla y}_\alpha}{\sigma}\right)^*(\bar{g}^{\alpha\beta})\left(\frac{\widetilde{\nabla y}_\beta}{\sigma}\right)$$

$$\{\overline{M_aM_{a(\beta)}}\}^2 = \{\overline{M_0^{(\varrho)}M_a^{(\varrho)}}\}^2 = \left(\frac{\widetilde{\nabla y}^\tau}{\sigma}\right)^*(\bar{g}_{\tau\varrho})\left(\frac{\widetilde{\nabla y}^\varrho}{\sigma}\right)$$

$$\left.\begin{array}{l}\text{(2.2.14) in [3]: } \overline{M_0^{(\varrho)}P^{(\varrho)}} = \left\{\dfrac{1}{\sigma^2}E\right\}^{\frac{1}{2}} \\ \text{(3.1.6) in [3]: } \overline{M_0^{(\varrho)}M_a^{(\varrho)}} = \lambda^{\frac{1}{2}}\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.5)$$

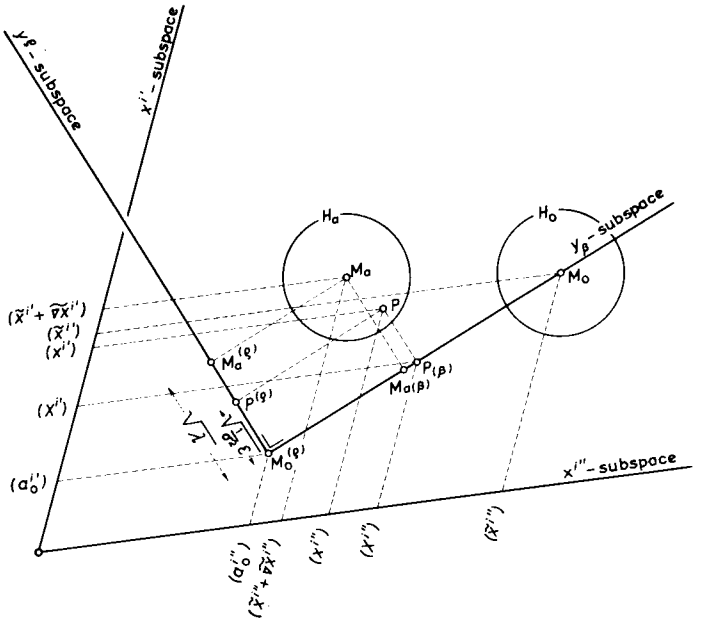

Fig. 2-1

Because in H_0 only limited assumptions concerning $(\tilde{x}^i)$ are made, see (2.1), the transformation (2.2) is aimed at a maximum use of these assumptions. Of all the quadratic forms mentioned above, we can only use a single one for the theory of testing, because its probability distribution can be derived.

With the central and non-central χ^2-and F-distributions we have:

$$\left.\begin{aligned} \left\{\frac{1}{\sigma^2}\underline{E}|H_0\right\} &\equiv \underline{\chi}^2_b \equiv b\cdot\underline{F}_{b,\infty} \\ \left\{\frac{1}{\sigma^2}\underline{E}|H_a\right\} &\equiv \underline{\chi}'^2_{b,\lambda} \equiv b\cdot\underline{F}'_{b,\infty,\lambda} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.6)$$

The importance of λ as a quadratic ∇-quantity follows from, see the formulas from [3] quoted:

(see (2.7) on page 12)

$$
\begin{array}{ll}
(3.2.1): & \left(E\left\{\frac{\underline{x}^i}{\sigma} \mid H_a\right\}\right) = \left(E\left\{\frac{\underline{x}^i}{\sigma} \mid H_0\right\}\right) + \left(\frac{\widetilde{\nabla x}^i}{\sigma}\right) = \left(\frac{\tilde{x}^i}{\sigma}\right) + \left(\frac{\widetilde{\nabla x}^i}{\sigma}\right) \\
 & \left(\frac{\underline{y}^\varrho}{\sigma}\right) = (u_i^\varrho)\left(\frac{\underline{x}^i - a_0^i}{\sigma}\right) \\
 & \left(\frac{\widetilde{\nabla y}^\varrho}{\sigma}\right) = (u_i^\varrho)\left(\frac{\widetilde{\nabla x}^i}{\sigma}\right) \\
(3.2.3): & \left(E\left\{\frac{\underline{y}^\varrho}{\sigma} \mid H_a\right\}\right) = \left(E\left\{\frac{\underline{y}^\varrho}{\sigma} \mid H_0\right\}\right) + \left(\frac{\widetilde{\nabla y}^\varrho}{\sigma}\right) = (0) + \left(\frac{\widetilde{\nabla y}^\varrho}{\sigma}\right) \\
 & \frac{\underline{E}}{\sigma^2} = \left(\frac{\underline{y}^\tau}{\sigma}\right)^* (\bar{g}_{\tau\varrho}) \left(\frac{\underline{y}^\varrho}{\sigma}\right) \\
 & \lambda = \left(\frac{\widetilde{\nabla y}^\tau}{\sigma}\right)^* (\bar{g}_{\tau\varrho}) \left(\frac{\widetilde{\nabla y}^\varrho}{\sigma}\right) \\
(3.1.13): & E\left\{\frac{\underline{E}}{\sigma^2} \mid H_a\right\} = E\left\{\frac{\underline{E}}{\sigma^2} \mid H_0\right\} + \lambda = b + \lambda
\end{array}
\quad (2.7)
$$

Here it is essential that, because of the restriction imposed by the formulation of the null hypothesis H_0, we must always work in a subspace, the y^ϱ-subspace. As follows from (2.5) we have the following inequality:

$$\lambda^{\frac{1}{2}} \leqq \overline{M_0 M_a} \quad \text{(2.8')}$$

with the extreme case:

$$M_a \text{ in } y_\beta\text{-subspace: } \lambda = 0 \quad \text{(2.8'')}$$

From this it follows that a model error in means can be present *without* manifesting itself in adjustment and test. It is clear that this is very undesirable in view of the later use of geodetic networks, and every effort should be made to build up a network in such a way that possible errors of this type can be signalized {which does not mean the same as: traced!}.

These "possible" model errors are now described by a number of alternative hypotheses, *which in principle do not have to occur silmultaneously* {this is essential for the line of thought leading from (3.1.9) to (3.1.10) in [3]}. The hypotheses are, see (3.2.21) in [3], with parameters ∇_p:

H_{a_p}	$\left(\frac{\widetilde{\nabla_p x}^i}{\sigma}\right) = (c_p^i) \cdot \nabla_p$ $-\infty < \nabla_p < +\infty$	$p = \ldots$

(2.9)

From (2.9) with (2.7) follows for each p-value separately, the direction of $\overrightarrow{M_0 M_{a_p}}$ and consequently, in the projection on the y^{ϱ}-subspace, the direction of $\overrightarrow{M_0^{(\varrho)} M_{a_p}^{(\varrho)}}$. ∇_p acts here as a kind of scale parameter.

On account of the following equality:

$$(u_j^{\tau})^*(\bar{g}_{\tau\varrho})(u_i^{\varrho}) = (\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})$$

it also follows that, see (3.2.5) and (3.2.7) in [3]:

$$\left.\begin{aligned} \lambda_p &= N_p \cdot \{\nabla_p\}^2 \\ N_p &= (c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})(c_p^i) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2.10)$$

By decomposing $(g^{\varrho\tau})$ into the product of two triangular matrices according to CHOLESKY, with a subsequent orthogonal transformation, it proves to be always possible to change over to variates $\underline{w}_p^{\bar{\varrho}}$:

$$\left.\begin{aligned} \{\text{covariance matrix } (\underline{w}_p^{\bar{\varrho}})\} &= (\delta^{\bar{\varrho}\bar{\tau}}) = \text{unit matrix} \\ (\tilde{w}_p^{\bar{\varrho}}) &= (0) \\ (|\widetilde{\nabla} w_p^1|, \widetilde{\nabla} w_p^2, \ldots, \widetilde{\nabla} w_p^b) &= (\sqrt{\lambda_p}, 0, \ldots, 0) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2.11)$$

(2.11) is illustrated by figure 2-2 as a sketch of the y^{ϱ}-subspace of figure 2-1.

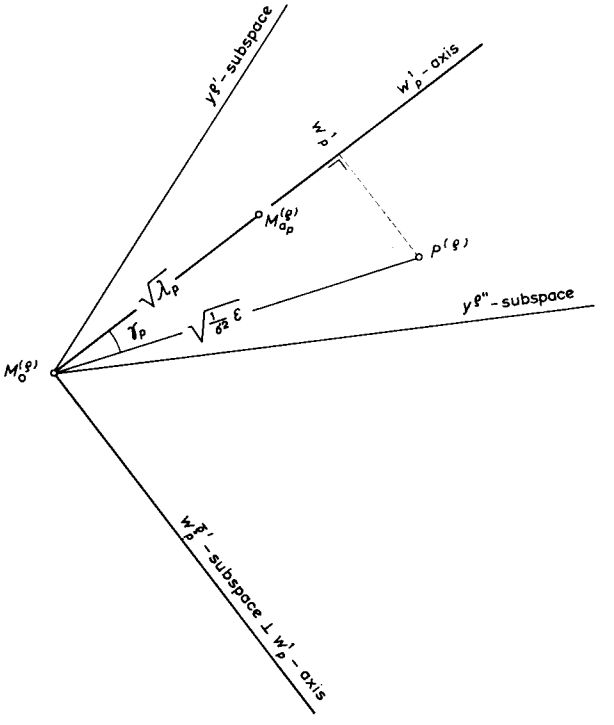

Fig. 2-2

The transformation to be executed, of which only a part is relevant, can be written as:

$$\begin{pmatrix} \underline{w}_p^1 \\ \underline{w}_p^2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{w}_p^b \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{N_p}}(c_p^j)^*(u_j^\tau)^*(\bar{g}_{\tau\varrho}) \\ (C_\varrho^2) \\ \cdot \\ \cdot \\ \cdot \\ (C_\varrho^b) \end{pmatrix} \left(\frac{\underline{y}^\varrho}{\sigma}\right) \qquad \ldots\ldots\ldots\ldots\ldots (2.12)$$

or, with (2.7) and (2.9), compare (2.10):

$$|\tilde{\nabla} w_p^1| = \frac{1}{\sqrt{N_p}}(c_p^j)^*(u_j^\tau)^*(\bar{g}_{\tau\varrho})(u_i^\varrho)(c_p^i)\cdot|\nabla_p| = \sqrt{N_p}\cdot|\nabla_p| = \sqrt{\lambda_p}$$

For completeness, (2.7) is supplemented by:

$$\boxed{\begin{aligned} &\frac{\underline{E}}{\sigma^2} = \sum_{\bar{\varrho}=1}^{b} \underline{w}_p^{\bar{\varrho}}\underline{w}_p^{\bar{\varrho}} \\ &\lambda_p = \tilde{\nabla} w_p^1 \cdot \tilde{\nabla} w_p^1 \\ &E\left\{\frac{\underline{E}}{\sigma^2} \mid H_a\right\} = b + \lambda_p \end{aligned}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.13)$$

From (2.12), several forms can be deduced which may contribute to a better insight in the results of computations. With γ_p and other notations defined in figure 2-2, we have:

$$\boxed{\begin{aligned} \underline{w}_p^1 &= \frac{1}{\sqrt{N_p}}(c_p^j)^*(\bar{g}_{ji})\left(\frac{-\underline{\varepsilon}^i}{\sigma}\right) \\ &= \frac{1}{\sqrt{N_p}}(c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})\left(\frac{\underline{x}^i - a_0^i}{\sigma}\right) \\ &= \cos\gamma_p \cdot \sqrt{\frac{1}{\sigma^2}\underline{E}} \end{aligned}} \qquad \ldots\ldots\ldots (2.14)$$

As was already remarked in [3] pages 30, 31, it is in most cases practically impossible to form vectors (c_p^i). But it is possible to calculate the effect of a model error in one mean value $\tilde{x}^{i''}$. Of course this is only a choice from several possibilities, but it is a choice that makes it possible to arrive at a *convention* concerning statements about *reliability of geodetic networks.*

All vectors (c_p^i), with $p = i''$ running from 1 to m, can then be taken together to form a matrix, viz. a unit matrix $(\delta^i_{i''})$.

Instead of (2.9) one obtains:

CONVENTIONAL group of m alternative hypotheses		
$H_{a_{i''}}$	$\left(\frac{\widetilde{\nabla}_{i''}x^i}{\sigma}\right) = (\delta^i_{i''})\cdot\nabla_{i''}$	$i, i'' = 1, \ldots, m$

. . . . . (2.15)

From (2.10), (2.11), (2.14) follows then:

$$N_{i''} = (\bar{g}_{i''i'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i''})$$

$$\begin{aligned} \underline{w}^1_{i''} &= \frac{1}{\sqrt{N_{i''}}}(\bar{g}_{i''i})\left(\frac{-\underline{\varepsilon}^i}{\sigma}\right) \\ &= \frac{1}{\sqrt{N_{i''}}}(\bar{g}_{i''i'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})\left(\frac{\underline{x}^i - a^i_0}{\sigma}\right) \\ &= \cos\gamma_{i''}\cdot\sqrt{\frac{1}{\sigma^2}\underline{E}} \end{aligned} \qquad \ldots\ldots\ldots \text{(2.16)}$$

$$|\widetilde{\nabla}w^1_{i''}| = \sqrt{N_{i''}}\cdot|\nabla_{i''}| = \sqrt{\lambda_{i''}}$$

A simplification of the notation is possible on account of (2.15) where only the case $i'' = i$ does not result in zero values. Then:

$\widetilde{\nabla}w^1_i$ is the projection on the y^e-subspace of $\frac{\widetilde{\nabla}x^i}{\sigma}\sqrt{\underline{g}_{ii}}$, with $\widetilde{\nabla}x^i \equiv \widetilde{\nabla}_i x^i$

. . . . . (2.17)

Remark concerning the meaning of $\underline{w}_p{}^1$ and N_p *)

Several formulas of this paper can be developed further. An example will be given concerning formulas relating to variates $\underline{w}^1$ and quantities N.

Introduce the variates $(\underline{x}_j)$, sometimes called reciprocal variates of $(\underline{x}^i)$:

$$(\underline{x}_{j''}) = (\bar{g}_{j''i})(\underline{x}^i) \qquad \{j, i = 1, \ldots, m\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2.18)}$$

---

*) The author is indebted to J. C. P. DE KRUIF for his remark that an elucidation as given here might be important with a view to inclusion of the computations in the algorithm of the adjustment problem.

variates; matrix of weight coefficients
$(\underline{x}_{j''} - a^0_{j''}) = (\bar{g}_{j''i})(\underline{x}^i - a^i_0)$
$\overline{(x_{j''}),(x_{i''})^*} = (\bar{g}_{j''i''})$
least-squares estimators; matrix of weight coefficients
$(\underline{X}_{j''} - a^0_{j''}) = (\bar{g}_{j''i})(\underline{X}^i - a^i_0)$
$\overline{(X_{j''}),(X_{i''})^*} \equiv (G_{j''i''}) = (\bar{g}_{j''i'})(G^{i'j'})(\bar{g}_{j'i''})$
correction variates; matrix of weight coefficients
$(\underline{\varepsilon}_{j''}) \equiv (\underline{X}_{j''} - \underline{x}_{j''}) = (\bar{g}_{j''i})(\underline{\varepsilon}^i)$
$\overline{(\varepsilon_{j''}),(\varepsilon_{i''})^*} = (\bar{g}_{j''i''} - G_{j''i''}) = (\bar{g}_{j''i'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i''})$

. . . . . (2.19)

Next, introduce the derived variates:

$$(\underline{x}_q) = (c^j_q)^*(\bar{g}_{ji})(\underline{x}^i) \qquad \{q, p = \ldots\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.20)$$

Then one obtains, for the right-hand column see (2.19):

variates; matrix of weight coefficients		
$(\underline{x}_q - a^0_q)$	$(c^j_q)^*(\bar{g}_{ji})(\underline{x}^i - a^i_0)$	$(c^j_q)^*(\underline{x}_j - a^0_j)$
$\overline{(x_q),(x_p)^*}$	(g_{qp})	
(g_{qp})	$(c^j_q)^*(\bar{g}_{ji})(c^i_p)$	$(c^j_q)^*(\bar{g}_{ji})(c^i_p)$
least-squares estimators; matrix of weight coefficients		
$(\underline{X}_q - a^0_q)$	$(c^j_q)^*(\bar{g}_{ji})(\underline{X}^i - a^i_0)$	$(c^j_q)^*(\underline{X}_j - a^0_j)$
$\overline{(X_q),(X_p)^*}$	(G_{qp})	
(G_{qp})	$(c^j_q)^*(\bar{g}_{ji'})(G^{i'j'})(\bar{g}_{j'i})(c^i_p)$	$(c^j_q)^*(G_{ji})(c^i_p)$
correction variates; matrix of weight coefficients		
$(\underline{X}_q - \underline{x}_q)$	$(\underline{\varepsilon}_q)$	
$(\underline{\varepsilon}_q)$	$(c^j_q)^*(\bar{g}_{ji})(\underline{\varepsilon}^i)$	$(c^j_q)^*(\underline{\varepsilon}_j)$
$\overline{(\varepsilon_q),(\varepsilon_p)^*}$	$(g_{qp} - G_{qp})$	
$(g_{qp} - G_{qp})$	$(c^j_q)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})(c^i_p)$	$(c^j_q)^*(\bar{g}_{ji} - G_{ji})(c^i_p)$

(2.21)

A comparison of (2.16) with (2.19) gives:

$N_i \equiv \overline{\varepsilon_i, \varepsilon_i} = \bar{g}_{ii} - G_{ii}$ as the i-th element of the main diagonal of $\overline{(\varepsilon_j),(\varepsilon_i)}^*$	
$\underline{w}_i^1 \equiv -\dfrac{\underline{\varepsilon}_i}{\sigma_{\varepsilon_i}}$	$\{\sigma_{\varepsilon_i}\}^2 = \sigma^2 \cdot N_i$

. . . . . (2.22)

(2.10) and (2.14), compared with (2.21), give:

$N_p \equiv \overline{\varepsilon_p, \varepsilon_p} = g_{pp} - G_{pp} = (c_p^j)^*(\bar{g}_{ji} - G_{ji})(c_p^i)$ as the p-th element of the main diagonal of $\overline{(\varepsilon_q),(\varepsilon_p)}^*$	
$\underline{w}_p^1 \equiv -\dfrac{\underline{\varepsilon}_p}{\sigma_{\varepsilon_p}}$	$\{\sigma_{\varepsilon_p}\}^2 = \sigma^2 \cdot N_p$

. . . . . (2.23)

Further, $\underline{w}_p^1$ could be written as a function of the $(\underline{w}_j^1)$ whenever this would be a real advantage. This might be meaningful in the sense that (2.15) is seen as an intermediate step towards the use of (2.9) {possibly in a later stage}. In this case, the complete computation of the matrix $\overline{(\varepsilon_j),(\varepsilon_i)}^*$ may be considered.

Testing $\underline{w}_p^1$ amounts to the same as testing $-\varepsilon_p$, provided the standard deviation σ_{ε_p} is taken into account.

Instead of (3.2), one obtains:

$$B_p^{(a)}: \quad -F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}} < \frac{-\underline{\varepsilon}_p}{\sigma_{\varepsilon_p}} < +F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}} \qquad . . . . . (2.24)$$

3 ADJUSTMENT IN ONE STEP CRITICAL REGIONS FOR F-TESTS

In [3] the problem of choosing a critical region was fully discussed, in particular the choice of α. In this section it will be tried to develop a line of thought offering possibilities for practical use.

One alternative hypothesis

For this we return to the general case (2.9) of an alternative hypothesis. Its influence makes itself felt in the direction of the w_p^1-axis, see the figures 2-1 and 2-2. From (2.11) it appears that only the mean of $\underline{w}_p^1$ is affected by a model error, the variates $\underline{w}_p^2, \ldots, \underline{w}_p^b$ are undisturbed.

Essentially, the test on this one H_{a_p} can be considered as a one-dimensional case, viz. the testing of $\underline{w}_p^1$, whereas the disturbing stochastic influences of $\underline{w}_p^2, \ldots, \underline{w}_p^b$ are eliminated.

In view of (2.11) we have:

$$\left.\begin{aligned} \{\underline{w}_p^1 \underline{w}_p^1 | H_0\} &\equiv \underline{F}_{1,\infty} \\ \{\underline{w}_p^1 \underline{w}_p^1 | H_{a_p}\} &\equiv \underline{F}'_{1,\infty,\lambda_p} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.1)$$

After the choice of an α-value α_0 one obtains, compare (2.2.4) in [3]:

$K_p^{(\varrho)}$: rejection region

$B_p^{(\varrho)}$: acceptance region

$K_p^{(\varrho)}$: $\underline{w}_p^1 \underline{w}_p^1 > F_{1-\alpha_0;\,1,\infty}$

$B_p^{(\varrho)}$	$\underline{w}_p^1 \underline{w}_p^1 < F_{1-\alpha_0;\,1,\infty}$	or:
	$-F^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty} < \underline{w}_p^1 < +F^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty}$	

$$\ldots\ldots\ldots\ldots \quad (3.2)$$

But one should realize that (3.2) can also be taken as b-dimensional regions without a change of α_0, because no restrictions are imposed on $\underline{w}_p^2, \ldots, \underline{w}_p^b$. This is illustrated by figure 3-1, with a partial representation of the details of figure 2-2.

The power function of the test is in this case, compare (3.1.7) in [3]:

$$\begin{aligned} \beta_p &= P\{(\underline{w}_p^{\bar{\varrho}}) \in K_p^{(\varrho)} | H_a\} = \\ &= 1 - P\{-F^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty} < \underline{w}_p^1 < +\underline{F}^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty} | H_a\} \\ &= \text{function } \beta\{\alpha_0, \lambda_p, 1, \infty\} \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.3)$$

We follow now the line of thought in [3] after (3.1.7). Wanted is a {lower} bound for λ_p which can just be detected with probability $\beta_{p,0}$:

$$\lambda_{p,0} = \lambda\{\alpha_0, \beta_{p,0}, 1, \infty\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.4)$$

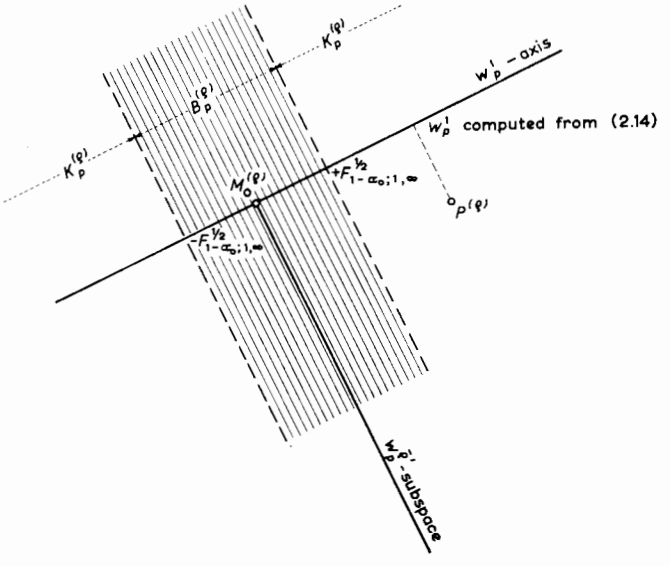

Fig. 3-1

Now the essence in the following exposition is, that this β-value $\beta_{p,0}$, as well as α_0, is always left unchanged, irrespective of the number p of the alternative hypothesis.

Then it follows from (3.4) that:

$\beta_{p,0} = \beta_0$	for every value of p
$\lambda_{p,0} = \lambda_0$	$\lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (3.5)

But then a bound for the parameter ∇_p can be computed from (2.10):

$$|\nabla_{p,0}| = \sqrt{\frac{\lambda_0}{N_p}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.6)$$

This can further be used to compute bounds for this influence on functions of observation variates, see (3.2.6) and (3.2.22) in [3].

Several alternative hypotheses

For *each* alternative hypothesis formulated we can now follow the procedure (3.1)–(3.6). If the respective figures 3-1 are combined into one, then we get following picture, see fig. 3-2.

One could indeed work like this, but the computation of the often large number of w_p^1's is an enormous work, whereas rejection of H_0 will not just happen once but perhaps many times {for example, in figure 3-2 in two out of three tests}.

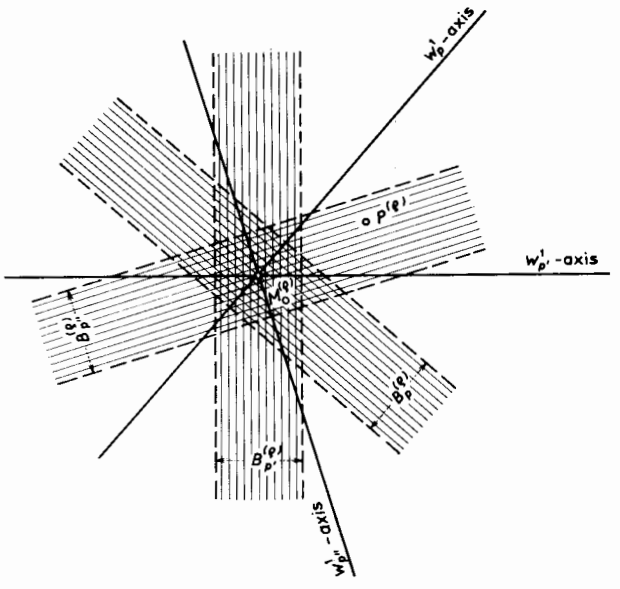

Fig. 3-2

The overlap of the regions $B_p^{(\varrho)}$ offers the possibility to revert to more intuitive considerations, such as given in [3] page 17, in which the vector (x^i) becomes more unacceptable as the distance to the mid-point of the probability distribution increases. For the y^ϱ-subspace this means: as the distance $\overline{P^{(\varrho)}M_0^{(\varrho)}}$ increases.

As the most symmetric solution, this implies for the standardized sample space the choice of a hypersphere centred in $M_0^{(\varrho)}$ as the boundary of $B^{(\varrho)}$.

For a certain single H_{a_p} this means that also the values $w_p^2, \ldots, w_p^b$ outside $B^{(\varrho)}$ are cut off, so no doubt it will have to be accepted that:

$$\left.\begin{aligned} &P\{(\underline{w}_p^{\bar{\varrho}}) \in B^{(\varrho)} | H_0\} = 1-\alpha \\ &\alpha > \alpha_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.7)$$

The test statistic is then, see (2.13) and (2.7) with (2.6), and also (2.2.17) in [3]:

$$\left\{ \sum_{\bar{\varrho}=1}^{b} \underline{w}_p^{\bar{\varrho}} \underline{w}_p^{\bar{\varrho}} = \frac{1}{\sigma^2} \underline{\underline{E}} = b \frac{\hat{\underline{\sigma}}^2}{\sigma^2} | H_0 \right\} \equiv b \cdot \underline{F}_{b,\infty} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3.8)$$

so that for the acceptance region we have:

$B^{(\varrho)}$	$\dfrac{E}{\sigma^2} < b \cdot F_{1-\alpha;\, b, \infty}$	(3.9)
	independent of the value of p, hence usable for all H_{a_p}	

For the power function we have on the analogy of (3.3):

$$\beta_p = \beta\{\alpha, \lambda_p, b, \infty\} \qquad (3.10)$$

In [3] the question had been raised what should be the values α and β to be introduced for the computation of the bound for λ_p.

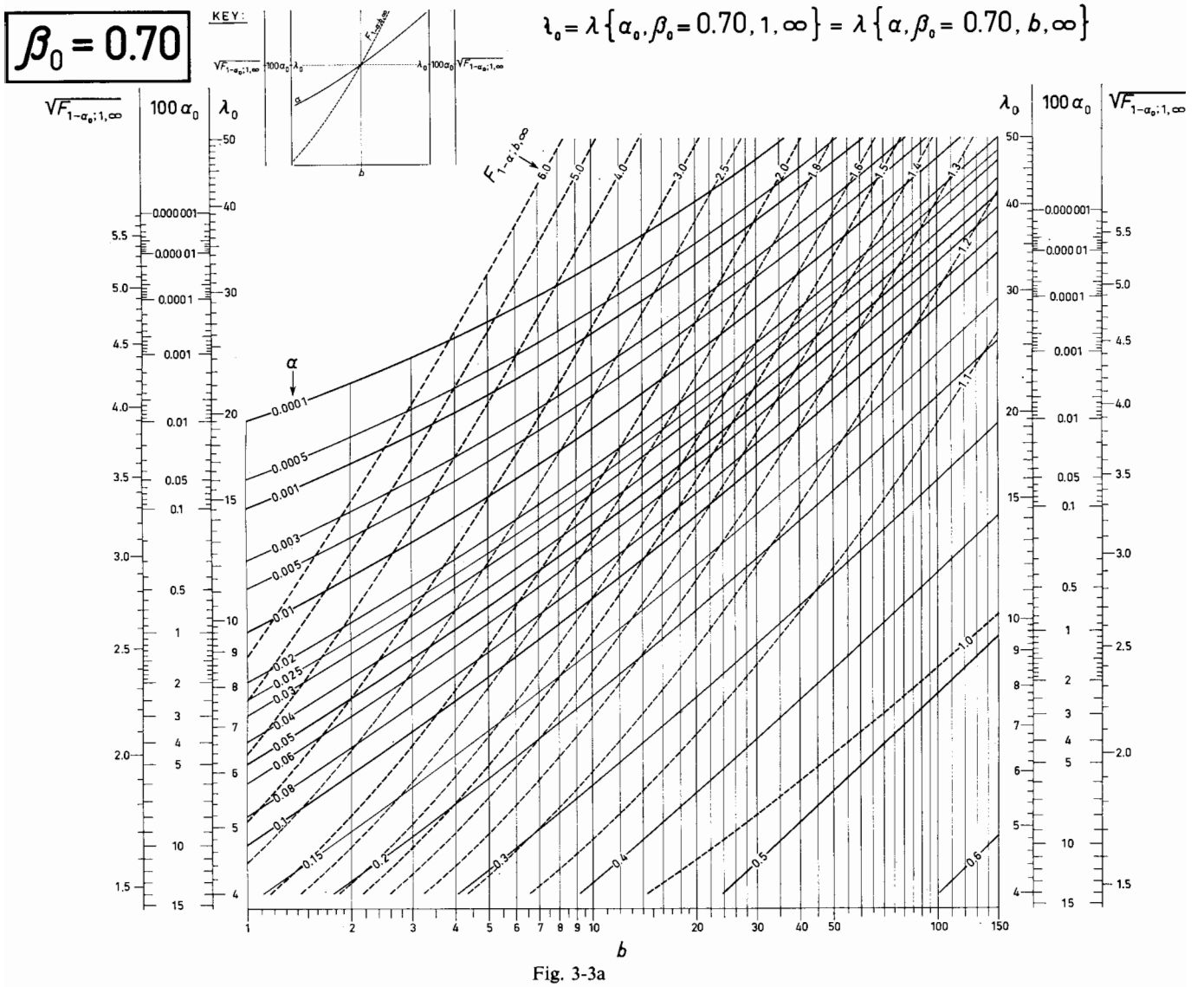

Fig. 3-3a

If one follows the line of thought given by (3.1)–(3.6), it is in all respects an acceptable choice to keep the bounds for β and λ equal to those of (3.5). The tests (3.2) and (3.9) are then equivalent as far as β and λ are concerned, only α remains as a dependent variable. Then:

choice	α_0 , $\beta_{p,0} = \beta_0$
(3.5)	$\lambda_{p,0} = \lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$
(3.10)	$\beta_0 = \beta\{\alpha, \lambda_0, b, \infty\}$
gives	α, $F_{1-\alpha;\,b,\infty}$

. . . . . (3.11)

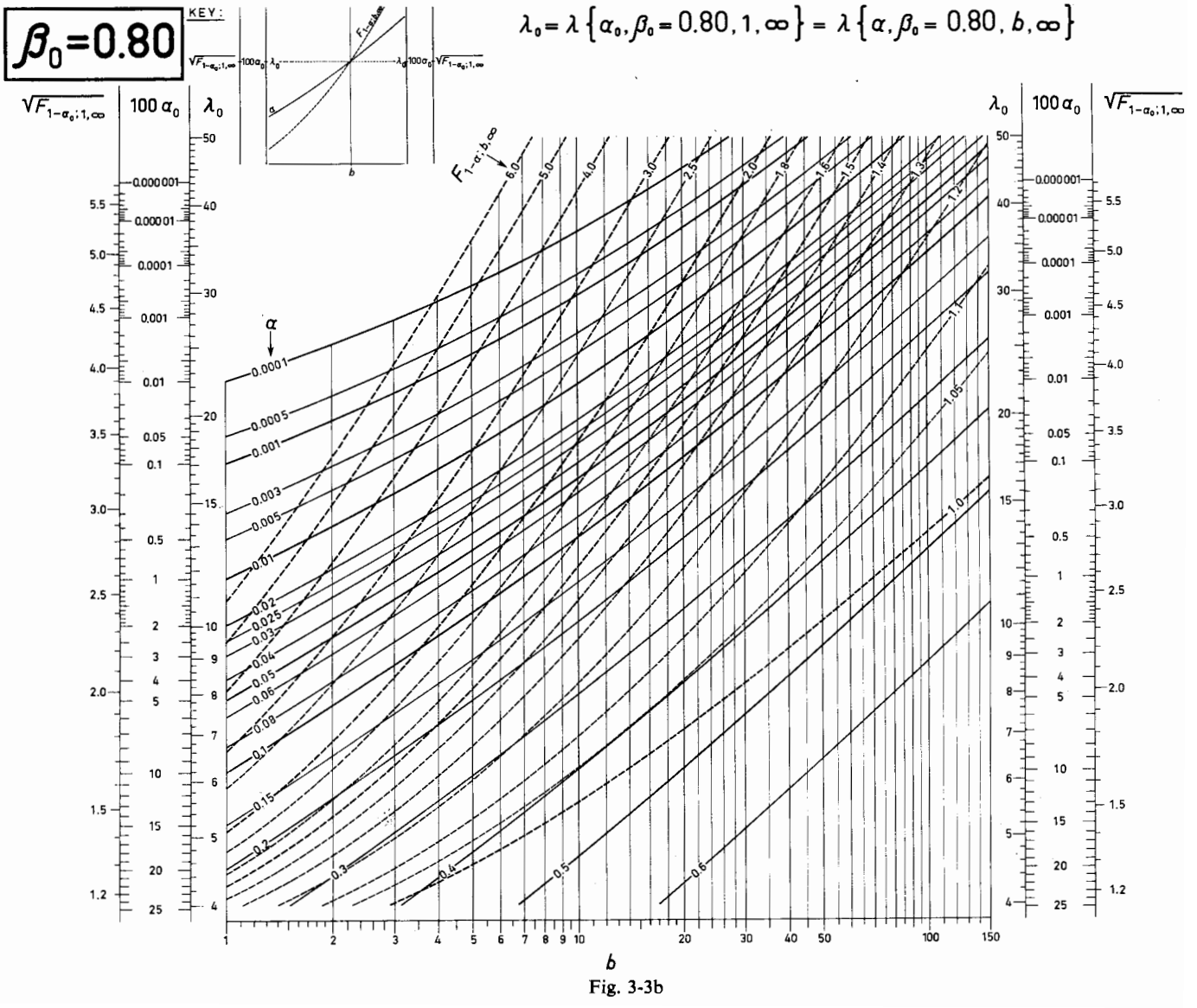

Fig. 3-3b

The complicated computations necessary for (3.11) have been programmed and executed by J. C. P. DE KRUIF with the collaboration of F. W. VAN DER ZWAN, see [11]. A graphical representation in nomograms is given by figures 3-3a, 3-3b, and 3-3c for the cases $\beta_0 = 0.70$, 0.80, and 0.90 respectively. The subdivision of the integer-scale on the b-axis is made with a view to (6.9).

The test (3.9) can consequently be used simultaneously for *all* H_{a_p}, whereas it remains always possible to try out the effects of supplementary alternative hypotheses after the adjustment, for "data-snooping". Then (3.6) remains unaltered.

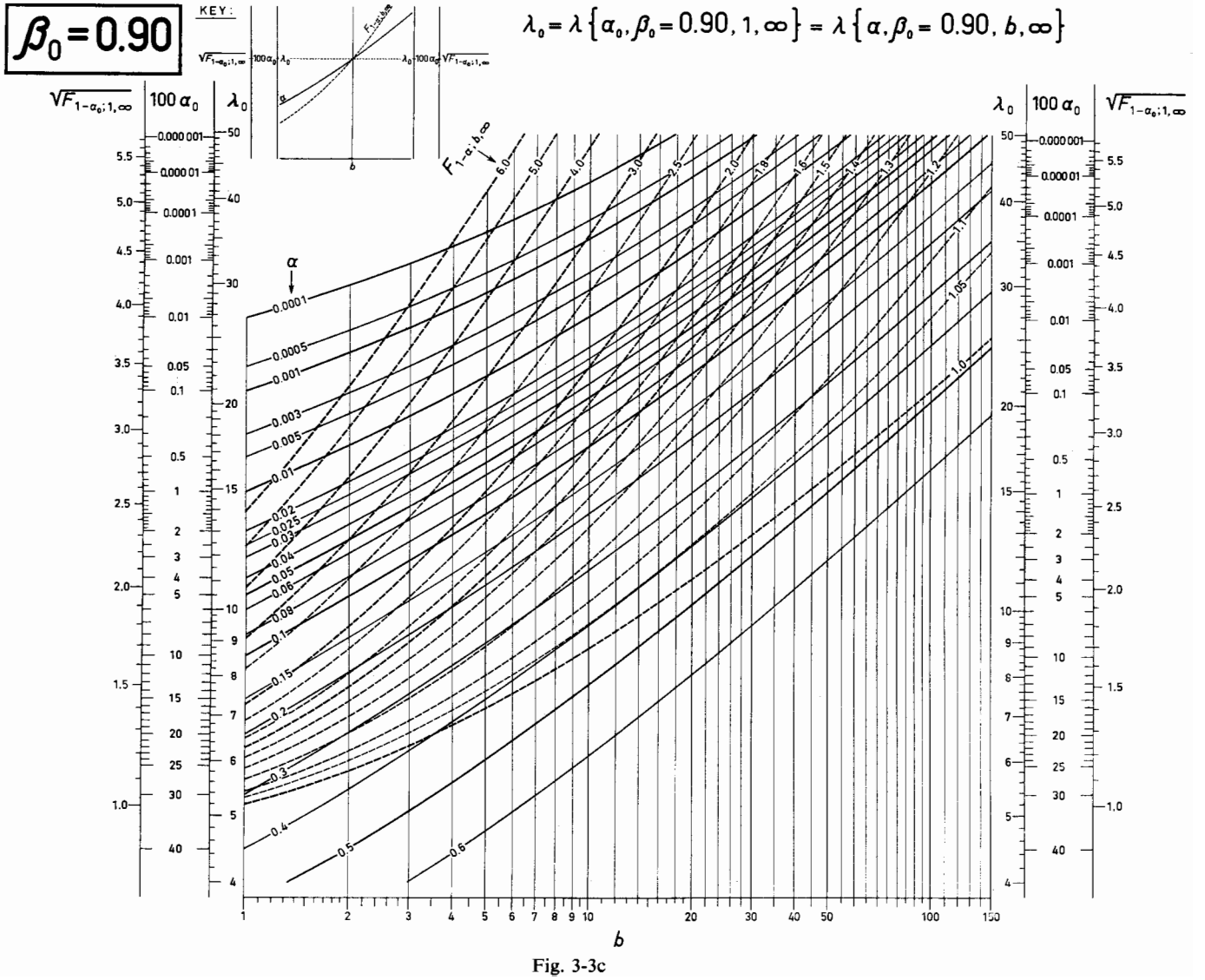

Fig. 3-3c

With (2.9) and (3.2.6) and (3.2.22) in [3] one obtains the different bounds wanted:

$$
\begin{aligned}
|\nabla_{p,0}| &= \sqrt{\frac{\lambda_0}{N_p}} \\
\left(\frac{\tilde{\nabla}_{p,0}x^i}{\sigma}\right) &= (c_p^i)\cdot|\nabla_{p,0}| \\
\left(\frac{\tilde{\nabla}_{p,0}X^R}{\sigma}\right) &= (G^{Rj})(\bar{g}_{ji})(c_p^i)\cdot|\nabla_{p,0}|
\end{aligned}
\qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.12)
$$

Whereas the test (3.9) can replace the totality of the tests (3.2) as far as (3.6) is concerned, it will be clear that $B^{(q)}$ for (3.6) will be different from the $B_p^{(q)}$ for (3.2), so that the tests may give different results regarding the rejection of H_0. This situation has been sketched in figure 3-4 on the basis of figure 3-1. Compare also figure 3-2.

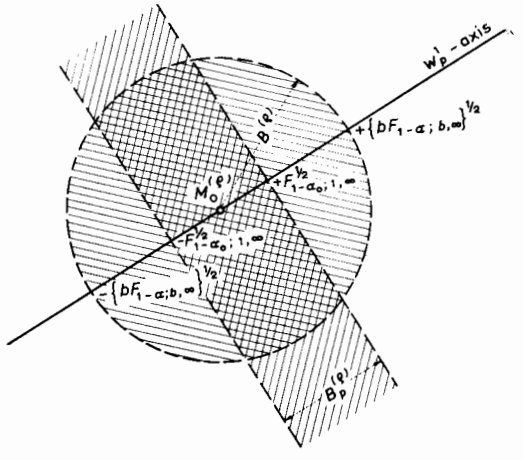

Fig. 3-4

An example is given in (3.13):

b	11				
α_0	0.001	$F_{1-\alpha_0;\,1,\infty}$	10.8	$\sqrt{F_{1-\alpha_0;\,1,\infty}}$	3.29
β_0	0.80	λ_0	16.8		
α	0.05	$F_{1-\alpha;\,11,\infty}$	1.79	$\sqrt{11\cdot F_{1-\alpha;\,11,\infty}}$	4.44
α_0	0.05	$F_{1-\alpha_0;\,1,\infty}$	3.84	$\sqrt{F_{1-\alpha_0;\,1,\infty}}$	1.96

. . . (3.13)

For comparison, there has been given in the bottom row the basis for the well-known test: "rejection based on an acceptance interval of $\pm$ 2 standard deviations".

Although (3.13) gives a reasonably acceptable result {the example was chosen accordingly}, a closer look reveals a practical difficulty.

Among statisticians it is well known that the tails of probability distributions usually provide only a rather poor description of the behaviour of samples. This means that for small values of α {and, consequently, of α_0} the testing procedure becomes dubious, if only it were for this reason. Compare [14], the note at the bottom of page 33.

If α is too large, the probability of rejection without the occurence of an alternative hypothesis becomes too great and consequently the method too costly. This is the reason that in statistical literature we find:

$$\left.\begin{array}{ll} \text{usually:} & 0.05 > \alpha, \alpha_0 > 0.01 \\ \text{sometimes} & 0.10 > \alpha, \alpha_0 > 0.001 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.14)$$

Other examples that indicate how easily the limits given by (3.14) are exceeded, are shown in (3.15). The really very small values of α_0 make the one-dimensional test carried out according to the above mentioned procedure, very dubious, see the text after (8.9)

$\alpha = 0.05; \qquad \beta_0 = 0.80$

b	$F_{0.95;\,b,\infty}$	$\sqrt{b \cdot F_{0.95;\,b,\infty}}$	λ_0	$100\alpha_0$	$F_{1-\alpha_0;\,1,\infty}$	$\sqrt{F_{1-\alpha_0;\,1,\infty}}$
1	3.84	1.96	7.85	5.0	3.84	1.96
10	1.83	4.28	16.2	0.17	9.85	3.14
20	1.57	5.60	21.0	0.025	13.4	3.66
30	1.46	6.62	24.6	0.0042	16.8	4.10
40	1.39	7.47	27.6	0.0010	19.5	4.42
60	1.32	8.89	32.6	0.00011	23.8	4.87
80	1.27	10.09	36.8	0.000017	27.4	5.23
100	1.24	11.15	40.6	0.000003	30.7	5.54

(3.15)

The last examples point towards a practical solution when testing on (3.9). For, instead of the sequence followed in (3.11):

$$\alpha_0, \beta_0 \rightarrow \lambda_0 \rightarrow \alpha, F_{1-\alpha;\,b,\infty}$$

one can also follow the sequence:

$$\alpha, \beta_0 \rightarrow \lambda_0 \rightarrow \alpha_0$$

Finally, β_0 may be altered, although this is not recommended, in view of the effect which may be seen in figure 3-5.

The core of the whole line of thought is the *choice of the level of* λ_0, which is essential for (3.6) and consequently for (3.12).

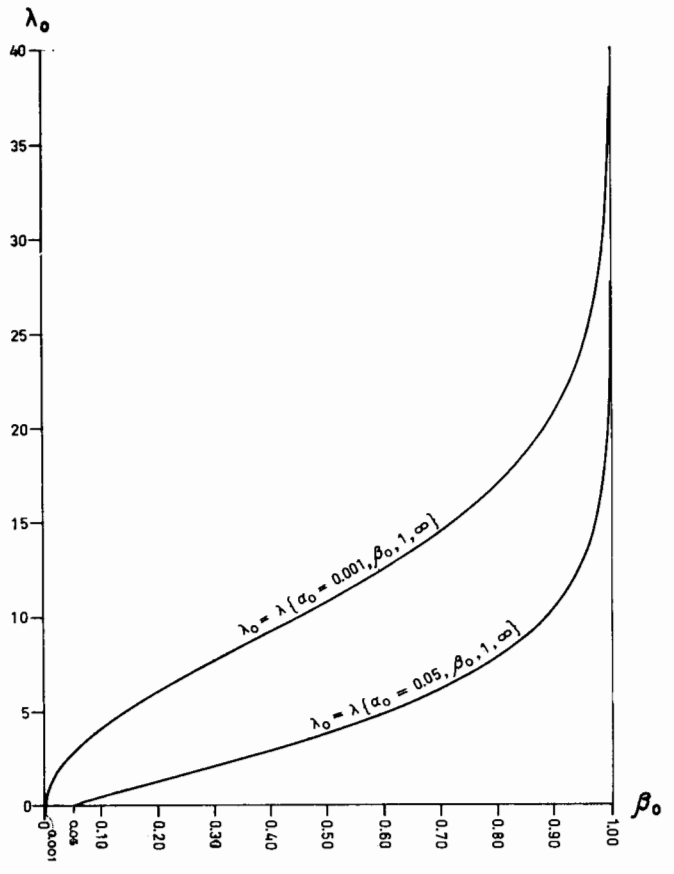

Fig. 3-5

Although α_0 will in many cases only have a theoretical significance, it is important to fix this question of the level, and to arrive at a comparison of tests with different degrees of freedom b. Therefore it is *proposed* that:

Every F-test be characterized by indicating α_0 and β_0	. . . . . (3.16)

In doing so, one is also made conscious of the seriousness of the rejection of the tails in the test (3.9). We shall return to this question when treating tests connected with adjustment in steps.

4 ADJUSTMENT IN ONE STEP DATA-SNOOPING

If after measurement and adjustment the test (3.9) leads to rejection of H_0, the geodesist will certainly try to find the cause of this rejection. This is not specific for geodesists: SEAL [13] claims the same for biologists, SCHEFFÉ, [12] page 80, for statisticians in general.

If we now return to what was said introducing (2.9), viz. that possible alternative hypotheses H_{a_p} do not simultaneously occur, then we can find a method of data-snooping which corresponds to the line of thought followed.

"Data-snooping" can be defined as searching for the observation in which a gross error has been made during measurement {if computational mistakes are left out of consideration, which is justified in present-day automated computation}. Such a gross error in observation results can be described by H_{a_i} in (2.15).

The question is: which $\widetilde{\nabla} x^i$?

Figure 2-2 suggests the solution to compute the m values $\cos \gamma_i$ and to check by remeasurement those observations x^i for which:

$$|\cos \gamma_i| \approx 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (4.1)$$

These computations can be executed according to (2.16):

$$\cos \gamma_i = \frac{1}{\sqrt{N_i \underline{E}}} (\bar{g}_{ii'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i''})(\underline{x}^{i''} - a_0^{i''}) \qquad \{i, i', i'' = 1, \ldots, m\} \quad (4.2)$$

Such computations have indeed been executed and they prove to give a reasonably good indication of possible sources of error.

However, $\cos \underline{\gamma}_i$ is a stochastic quantity, for which usually is found:

$$|\cos \gamma_i| \ll 1$$

so that this method of searching is not very sensitive. Besides, it is very difficult to deduce the probability distribution of (4.2), so that no test is immediately available.

Now (4.2) was derived from (2.16), so why should not (2.16) be used straight away? (3.2) provides a direct testing possibility with the same λ_0 per alternative hypothesis as (3.9). For each H_{a_i} we then obtain the procedure:

DATA-SNOOPING Compute for $i = 1, .., m$:

$$N_i = (\bar{g}_{ii'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})$$

$$w_i^1 = \frac{1}{\sqrt{N_i}}(\bar{g}_{ii'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i''})\left(\frac{x^{i''} - a_0^{i''}}{\sigma}\right) \qquad \ldots\ldots (4.3)$$

For each i for which w_i^1 falls *outside* the interval:

$B_i^{(\varrho)}$ $\quad -F^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty} < \underline{w}_i^1 < F^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty}$ $\quad \alpha_0$ see (3.11)

check at least the observation x^i

In fact we move here in a direction opposite to the course followed in section 3, the difference being, however, that (4.3) is only executed if (3.9) leads to rejection of H_0.

Of course this reasoning is valid for each H_{a_p}; the formulas follow from (2.10), (2.14) and (3.2).

It must be emphasized that the procedure (4.3) does not give certainty, only a *supposition.* For the situation may be that different H_{a_i} {or H_{a_p}} occur simultaneously, and the device (4.3) is not of any help against this. Data-snooping will therefore always be a risky activity. Remeasurement of *all* x^i will always be the safest way, although this will usually be avoided for reasons of economy, and because conceivably remeasurement will include new gross errors.

Arbitrary functions of $(\underline{y}^\varrho)$

As an introduction to the study of adjustment in steps, an arbitrary linear function of $(\underline{y}^\varrho)$ is considered:

$$\underline{\psi} = (c_\varrho)^*(\underline{y}^\varrho) \qquad \ldots\ldots (4.4)$$

Put: $$(c_\varrho)^* = C_0\left(\frac{c^j}{\sqrt{N}}\right)^*(u_j^\tau)^*(\bar{g}_{\tau\varrho})\frac{1}{\sigma}$$

with: $$N = (c^j)^*(u_j^\tau)^*(\bar{g}_{\tau\varrho})(u_i^\varrho)(c^i) \qquad \ldots\ldots (4.5)$$

Then with (2.3) and (4.5):

$$\{\sigma_\psi\}^2 = (c_\varrho)^*(g^{\varrho\tau})(c_\tau)\sigma^2 = \{C_0\}^2$$

hence: $$C_0 = \sigma_\psi \qquad \ldots\ldots (4.6)$$

The vector $\left(\frac{c^j}{\sqrt{N}}\right)^*$ can be determined from the transposed form (4.5), because with (4.6):

$$(u_i^\varrho)\left(\frac{c^i}{\sqrt{N}}\right) = \frac{\sigma}{\sigma_\psi}(g^{\varrho\tau})(c_\tau) = \underset{\text{define}}{(\bar{c}^\varrho)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.7)$$

Complete (u_i^ϱ) to form the non-singular matrix, see (2.2):

$$\begin{pmatrix} u_i^\varrho \\ a_\beta^j \bar{g}_{ji} \end{pmatrix} \equiv (T) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.8')$$

then it follows from (4.7), introducing the parameters:

$$\bar{c}_\beta \qquad \{\beta = b+1, \ldots, m\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.8'')$$

$$\left(\frac{c^i}{\sqrt{N}}\right) = (T)^{-1}\begin{pmatrix} \bar{c}^\varrho \\ \bar{c}_\beta \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.9')$$

or, the vector is determined up to $\{m-b\}$ degrees of freedom.

Denote the solution(s) by:

$$\left(\frac{c_q^i}{\sqrt{N_q}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.9'')$$

Then it follows from (4.4)–(4.9):

$$\underline{\psi} = \sigma_\psi\left(\frac{c_q^j}{\sqrt{N_q}}\right)^*(u_j^\tau)^*(\bar{g}_{\tau\varrho})\left(\frac{\underline{y}^\varrho}{\sigma}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.10)$$

Introducing the variate $\underline{w}^1$ according to (2.12):

$$\underline{w}^1 = \left(\frac{c_q^j}{\sqrt{N_q}}\right)^*(u_j^\tau)^*(\bar{g}_{\tau\varrho})\left(\frac{\underline{y}^\varrho}{\sigma}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.11)$$

it follows from (4.10) and (4.11) that $\underline{\psi}$ can be written as:

$$\boxed{\underline{\psi} = \sigma_\psi \cdot \underline{w}^1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4.12)$$

From (4.12) it follows that any such function $\underline{\psi}$ plays the same rôle as the w-variates, the difference being that the same function $\underline{\psi}$ can be generated by $\{m-b\}^\infty$ alternative hypotheses H_{a_q}. A clear connection with a certain alternative hypothesis H_{a_p} cannot be established, so that in this respect such arbitrary functions $\underline{\psi}$ {and consequently the $\underline{y}^\varrho$ themselves} are less usable for data-snooping than e.g. the variates $\underline{w}_i^1$.

On the other hand it follows from (4.12) that if nevertheless such functions $\underline{\psi}$ are *separately* used one by one for data-snooping, testing like in (4.3) can be applied. One gets:

DATA-SNOOPING Compute for chosen $(c_\varrho)^*$:

$$\underline{\psi} = (c_\varrho)^*(\underline{y}^\varrho)$$

$$(\sigma_\psi)^2 = (c_\varrho)^*(g^{\varrho\tau})(c_\tau)\sigma^2$$

If ψ falls *outside* the interval:

$B_\psi^{(\varrho)}$ $\quad -F^{\ddagger}_{1-\alpha_0;\,1,\infty} < \dfrac{\underline{\psi}}{\sigma_\psi} < +F^{\ddagger}_{1-\alpha_0;\,1,\infty}$ $\quad$ α_0 see (3.11)

or: $\quad -\sigma_\psi \cdot F^{\ddagger}_{1-\alpha_0;\,1,\infty} < \underline{\psi} < +\sigma_\psi \cdot F^{\ddagger}_{1-\alpha_0;\,1,\infty}$

check what? (4.13′)

One may also use the:

$100\{1-\alpha_0\}^0/_0$ - confidence interval for $\tilde{\psi}$

$$\psi - \sigma_\psi \cdot F^{\ddagger}_{1-\alpha_0;\,1,\infty} < \tilde{\psi} < \psi + \sigma_\psi \cdot F^{\ddagger}_{1-\alpha_0;\,1,\infty} \quad \ldots\ldots\ldots\ldots \quad (4.13'')$$

Data-snooping is a very subjective problem-area. Several other ideas have been developed by B. G. K. KRIJGER, and also the foregoing can be seen as having resulted from the discussions between him and the author.

5 RELATED TOPICS IN THE ANALYSIS OF VARIANCE (MODELS WITH "FIXED EFFECTS")

The Analysis of Variance is a highly specialized area of mathematical statistics with a terminology of its own. An important part can be considered as an application of the method of least squares applied to a specific condition model.

In this condition model, means ($\tilde{x}^i$) of observation variates are written as a function of a parameter $\tilde{y}^0$, representing a general mean, and of b parameters ($\tilde{y}^\varrho$) describing a certain pattern of deviations from this mean.*) The condition model in this *first phase* of the computation is consequently the model of the 2nd. standard problem. This part of the whole model is founded on an identity and does not necessarily have a physical significance. It may always be considered "true", even if it is built in such a way that an almost complete orthogonality of the estimators $\underline{Y}^0$ and ($\underline{Y}^\varrho$) is attained, which is often the case. This results in the typical character of the technique of the Analysis of Variance, see e.g. [12].

The model of this first phase is therefore extremely well suited in an investigation which has the purpose to explore vaguely suspected influences. These influences are linked to the model by providing the estimators $\underline{Y}^0$ and ($\underline{Y}^\varrho$) with labels.

Then in a *second phase* one examines to what extent these influences are "significant". For this, one executes what is essentially an adjustment according to the 1st. standard problem:

$$\left.\begin{array}{ll} \text{condition model:} & (\tilde{y}^\varrho) = (0) \\ \text{condition equations:} & (\underline{Y}^\varrho)+(\underline{\varepsilon}^\varrho) = (0) \\ \text{covariance matrix of } (\underline{Y}^\varrho): & \sigma^2(G^{\varrho\tau}) = \sigma^2(\bar{G}_{\tau\varrho})^{-1} \end{array}\right\} \quad \ldots\ldots\ldots \quad (5.1)$$

If one does not consider the possibilities for solving this problem in steps, then the sections 2 to 4 are applicable. Replace the earlier notation in view of (5.1):

$$\left.\begin{array}{lll} (\underline{x}^i)\,,\ (\underline{y}^\varrho) & \text{by } (\underline{Y}^\varrho) & \{\varrho,\tau = 1, \ldots, b\} \\ (g^{ij}),\ (g^{\varrho\tau}) & \text{by } (G^{\varrho\tau}) & \\ (\underline{X}^i) & \text{by } (0) & \\ (G^{ij}) & \text{by } (0) & \\ (\tilde{\nabla}x^i) & \text{by } (\tilde{\nabla}Y^\varrho) & \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (5.2)$$

---

*) *Note:* Only algebraically independent parameters $\tilde{y}^\varrho$ are introduced in our exposition; in the Analysis of Variance, one or more extra parameters are introduced to obtain symmetry, because of which as many algebraic relations between the parameters have to be taken account of.

(2.9)	H_{a_p}	$\left(\frac{\widetilde{\nabla_p Y^\varrho}}{\sigma}\right) = (c_p^\varrho)\cdot\nabla_p$	
(2.7)	$\frac{\underline{E}}{\sigma^2}$	$\left(\frac{\underline{Y}^\tau}{\sigma}\right)^*(\bar{G}_{\tau\varrho})\left(\frac{\underline{Y}^\varrho}{\sigma}\right)$	
(3.9)	$B^{(\varrho)}$	$\frac{\underline{E}}{\sigma^2} < b\cdot F_{1-\alpha;b,\infty}$ {α see (3.11)}	(5.3′)
(2.10)	N_p	$(c_p^\tau)^*(\bar{G}_{\tau\varrho})(c_p^\varrho)$	
(2.11)	$\lvert\widetilde{\nabla w}_p^1\rvert$	$\sqrt{N_p}\cdot\lvert\nabla_p\rvert = \sqrt{\lambda_p}$	
(2.14)	$\underline{w}_p^1$	$\frac{1}{\sqrt{N_p}}(c_p^\tau)^*(\bar{G}_{\tau\varrho})\left(\frac{\underline{Y}^\varrho}{\sigma}\right)$	(5.3″)
(3.2)	$B_p^{(\varrho)}$	$-F^{\frac{1}{2}}_{1-\alpha_0;1,\infty} < \underline{w}_p^1 < +F^{\frac{1}{2}}_{1-\alpha_0;1,\infty}$	

For the special case (2.15), (5.3) becomes:

(2.15)	H_{a_ϱ}	$\frac{\widetilde{\nabla Y^\varrho}}{\sigma} = \nabla_\varrho$	
(2.16)	N_ϱ	$\bar{G}_{\varrho\varrho}$	
(2.16)	$\lvert\widetilde{\nabla w}_\varrho^1\rvert$	$\sqrt{N_\varrho}\cdot\lvert\nabla_\varrho\rvert = \sqrt{\lambda_\varrho}$	(5.4′)
(2.16)	$\underline{w}_\varrho^1$	$\frac{1}{\sqrt{N_\varrho}}(\bar{G}_{\varrho\varrho'})\left(\frac{\underline{Y}^{\varrho'}}{\sigma}\right)$	
(4.3)	$B_\varrho^{(\varrho')}$	$-F^{\frac{1}{2}}_{1-\alpha_0;1,\infty} < \underline{w}_\varrho^1 < +F^{\frac{1}{2}}_{1-\alpha_0;1,\infty}$	
Full orthogonalization of $(\underline{Y}^\varrho)$ results in: $\bar{G}_{\varrho\varrho'} = 0$ for $\varrho' \neq \varrho$; $\underline{w}_\varrho^1 = \sqrt{\frac{1}{\sigma^2}\bar{G}_{\varrho\varrho}}\cdot\underline{Y}^\varrho = \frac{\underline{Y}^\varrho}{\sigma_{Y^\varrho}}$			(5.4″)

Contrasts

Here we have, for the functions $\underline{\psi}$ (4.4):

$$\underline{\psi} = (c_\beta)^*(\underline{Y}^\varrho) \quad \ldots \quad (5.5)$$

In the slightly different meaning used in [13], these functions are called contrasts. If, as usual, one wishes to introduce more algebraically dependent variates $\underline{Y}^\varrho$ {cf. the note on

page 31}, then the relations determined by the model must be taken into account. In the sketch given here we shall not go into this, the reader is referred to [12].

In consequence of the deduction (4.4)–(4.12), it turns out that the following holds unambiguously for (5.5):

$$\left.\begin{aligned} &\boxed{\underline{\psi} = \sigma_{\psi} \cdot \underline{w}^1} \\ &(\sigma_{\psi})^2 = (c_{\varrho})^*(G^{\varrho\tau})(c_{\tau}) \cdot \sigma^2 \\ &\underline{w}^1 = \left(\frac{c^{\tau}}{\sqrt{N}}\right)^*(\bar{G}_{\tau\varrho})\left(\frac{\underline{Y}^{\varrho}}{\sigma}\right) \\ &\left(\frac{c^{\tau}}{\sqrt{N}}\right)^* = \frac{\sigma}{\sigma_{\psi}} \cdot (c_{\varrho})^*(G^{\varrho\tau}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5.6)$$

This implies that the functions $\underline{w}_p^1$ and $\underline{\psi}$ in the sections 2 to 4 can also be called "contrasts".

Data-snooping

The foregoing gives the possibility to apply the reasoning developed in section 4. One can use the contrasts $\underline{w}_p^1$, $\underline{w}^1$ or $\underline{\psi}$.

The S-method of multiple comparison

Methods of "multiple comparison" have grown into a new area of mathematical statistics. The book [14] is exclusively devoted to it. SCHEFFÉ developed his S-method for this kind of investigation, cf. [12] pages 66 ff. Clear applications of this method are given in [13].

As far as the author of this paper has understood the method, the purpose is to judge a great number of contrasts simultaneously with respect to significant deviations. To do this, confidence intervals for these contrasts are derived. The S-method is the most analogous to the method developed in the previous sections, which will be referred to as the "B-method".

SCHEFFÉ starts with the assumption that for the adjustment problem the test (3.9), cf. also (5.3′), has been applied. The acceptance region $B^{(\varrho)}$ has been sketched in figure 3-4.

From (5.6) and (4.12) it follows that contrasts – being functions $\underline{\psi}$, after division by σ_{ψ} – can be considered as variates $\underline{w}^1$.

We consider now an acceptance region $\bar{B}^{(\varrho)}$, as indicated in figure 5-1. If now the $\underline{w}^1$-axis is rotated in all directions, corresponding to as many variates $\underline{\psi}$, the overlap of *all* these regions $\bar{B}^{(\varrho)}$ will coincide with $B^{(\varrho)}$ more and more as the number of variates $\underline{\psi}$ increases. On this ground SCHEFFÉ arrives at the confidence interval of his S-method to which can be opposed the corresponding confidence interval (4.13″) of the B-method:

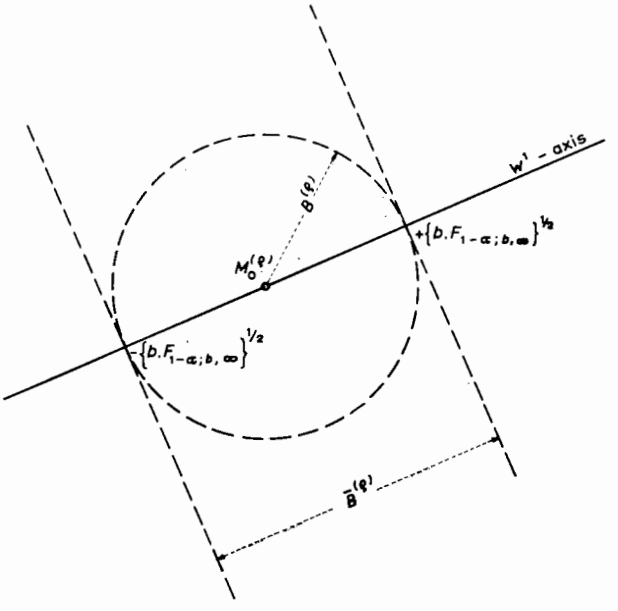

Fig. 5-1

S-method	$100\cdot\{1-\alpha\}\%$ - confidence interval for $\tilde{\psi}$
$\psi-\sigma_\psi\cdot\{b\cdot F_{1-\alpha;\,b,\infty}\}^{\frac{1}{2}} < \tilde{\psi} < \psi+\sigma_\psi\cdot\{b\cdot F_{1-\alpha;\,b,\infty}\}^{\frac{1}{2}}$	. . . . . (5.7)
B-method	$100\cdot\{1-\alpha_0\}\%$ - confidence interval for $\tilde{\psi}$
$\psi-\sigma_\psi\cdot F^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty} < \tilde{\psi} < \psi+\sigma_\psi\cdot F^{\frac{1}{2}}_{1-\alpha_0;\,1,\infty}$	. . . . . (5.8)
with the relation between α_0 and α from (3.11)	

For the example (3.13), the ratio of the lengths of the intervals is:

S-method : B-method = 4.4 : 3.3

Probably, the following is always valid, see (3.14):

$$\text{Interval}_{\text{S-method}} > \text{Interval}_{\text{B-method}} \quad . . . . . \quad (5.9)$$

SCHEFFÉ himself mentions that sometimes the intervals of the S-method are very long and in [12] page 80 he consequently points towards a division of the problem into phases, which makes shorter intervals possible. This reasoning is related to that of (2.4.8) in [3]. The S-method uses the interval (5.7) also for data-snooping.

For the computation of bounds, the S-method uses the same formula (3.6) as the B-method. But the S-method presupposes testing on many *simultaneously* occurring significant deviations of contrasts, and consequently alternative hypotheses H_{a_p} And, as was remarked before, this does not agree with the approach of the line of thought leading from (3.1.9) to (3.1.10) in [3].

This makes the author very hesitative towards the S-method and the methods of multiple comparison in general. But the literature referred to enables the reader to form his own opinion.

Estimable functions

In discussing contrasts, SCHEFFÉ uses the concept "estimable functions" introduced by R. C. BOSE, cf. [12] page 13. To avoid misunderstanding it seems advisable to make a connection with concepts mentioned in [3].

With the parameters $(\tilde{y}^\alpha)$ in the parametric form of the condition model, see (2.2.5) in [3], consider:

$$\left.\begin{array}{ll} \text{condition model:} & (\tilde{x}^i - a_0^i) = (a_\alpha^i)(\tilde{y}^\alpha) \\ \text{"parametric functions":} & (\tilde{x}^R - a_0^R) = (A_\alpha^R)(\tilde{y}^\alpha) \end{array}\right\} \quad \ldots\ldots\ldots (5.10)$$

> Parametric functions $(\tilde{x}^R - a_0^R)$ are called estimable functions if they have unbiased linear estimators, i.e. if a matrix (A_i^R) can be found, such that
>
> $$(\tilde{x}^R - a_0^R) = (A_i^R)(\tilde{x}^i - a_0^i)$$
>
> is fulfilled, irrespective of the values $(\tilde{y}^\alpha)$

$\ldots\ldots\ldots\ldots$ (5.11)

In [3], page 34, is mentioned the theorem that *any* unbiased linear estimator can be written as a pseudo least squares estimator.

According to (3.3.5′) in [3] we have for (5.10):

$$(\underset{\sim}{X}_{(h)}^R - a_0^R) = (H^{Rj})(\bar{h}_{ji})(\underset{\sim}{x}^i - a_0^i)$$

having the expectation:

$$(\tilde{x}^R - a_0^R) = (H^{Rj})(\bar{h}_{ji})(\tilde{x}^i - a_0^i)$$

On comparing with (5.11) we obtain:

$$(A_i^R) = (H^{Rj})(\bar{h}_{ji}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.12)$$

Although this is only meant to be an indication, it appears that the concepts "estimable functions" and "pseudo least squares estimators" correspond to each other.

No wonder that the concept "contrast" found also a place in our considerations.

6 BJERHAMMAR'S "TRACED" STATISTIC

In 1961, BJERHAMMAR published a paper [15] in which a "traced" estimator of the variance factor σ^2 was derived, which in many important applications in triangulation proved to give estimates equal or almost equal to those of the classical formula of FERRERO from 1887. The latter formula was in the last decades considered with some suspicion, because correlation arising from the procedure of measuring directions was neglected. Moreover, today's completely mechanized computation {in the first instance important for the planning-computations before measurement} furnishes without any extra work the estimate $\hat{\sigma}^2$, cf. (2.2.17) in [3], so that some doubt arose with respect to the usefulness of the "traced" estimate.

It was not until the reasoning of this paper was developed that the author realized that the traced estimate could in many cases give an important contribution to the treatment of the problem of testing in large geodetic networks, to which we shall return later. Besides, some fundamental points in BJERHAMMAR's derivation were not clear, for example, in the following summary the start is made at the first equation of (6.1), whereas BJERHAMMAR used the second one. In order to diminish such terminological difficulties of understanding, the basic theory of [15] will be translated into the notation and terminology of [3].

From the first standard problem (2.2.11) in [3]:

$$\left.\begin{aligned}
(u_i^\varrho)(\underline{x}^i - a_0^i) &= (\underline{y}^\varrho) && \{i,j = 1, \ldots, m;\ \varrho,\tau = 1, \ldots, b\} \\
(u_i^\varrho)(\underline{\varepsilon}^i) &= (-\underline{y}^\varrho) && \text{"discrepancies"} \\
\sigma^2(g^{ij}) &= \sigma^2(\bar{g}_{ji})^{-1} && \text{covariance matrix of } (\underline{x}^i) \\
(u_i^\varrho)(g^{ij})(u_j^\tau)^* &= (g^{\varrho\tau}) && \\
(g^{\varrho\tau})(-\underline{K}_\tau) &= (-\underline{y}^\varrho) && \text{"normal equations"}
\end{aligned}\right\} \quad \ldots\ (6.1)$$

The computation of the shifting variate $\underline{E}$ in (2.2.13) in [3] is now modified. This can be seen as the introduction of a matrix of pseudo weight coefficients, the unit matrix:

$$\left.\begin{aligned}
&(\delta^{\varrho\tau}) \text{ instead of } (g^{\varrho\tau}) \\
&(\delta_{\tau\varrho}) \text{ instead of } (\bar{g}_{\tau\varrho}) = (g^{\varrho\tau})^{-1}
\end{aligned}\right\} \quad \ldots\ (6.2)$$

Hence, one obtains a pseudo shifting variate, see (3.3.5) in [3]:

$$\boxed{\underline{E}_{(h)} = (\underline{y}^\tau)^*(\delta_{\tau\varrho})(\underline{y}^\varrho)} \quad \ldots\ (6.3)$$

According to [15] we have to compute the latent roots (or eigenvalues) of the $\{m,m\}$-matrix:

$$(g^{ij})(u_j^\tau)^*(\delta_{\tau\varrho})(u_i^\varrho)$$

If we introduce the following notation for the vector of latent roots of a $\{m,m\}$-matrix A:

$$L\{A\} \equiv (\lambda_1 \dots \lambda_\nu \dots \lambda_m)^* \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad (6.4)$$

then we have, not considering the sequence of the latent roots:

$$L\{(g^{ij})(u_j^\tau)^*(\delta_{\tau\varrho})(u_i^\varrho)\} = L\left\{\begin{pmatrix} g^{\varrho\tau} & 0 \\ 0 & 0 \end{pmatrix}\right\} \quad \dots\dots\dots\dots\dots\dots \quad (6.5)$$

Of the m latent roots, b are real and non-zero, the remaining $\{m-b\}$ latent roots are zero; notation as in (6.4).

(6.5) is not explicitly given in [15], but was no doubt known to BJERHAMMAR. Next follows the computation:

$$\left.\begin{aligned} &\text{trace}\,\{(g^{ij})(u_j^\tau)^*(\delta_{\tau\varrho})(u_i^\varrho)\} = \sum_{\nu=1}^{m} \lambda_\nu \equiv [\lambda_\nu] \\ &\boxed{k = \frac{[\lambda_\nu]^2}{[\lambda_\nu\lambda_\nu]}}\,, \quad [\lambda_\nu\lambda_\nu] \equiv \sum_{\nu=1}^{m} \{\lambda_\nu\}^2 \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (6.6)$$

Introduce now BJERHAMMAR's "traced" statistic $\underline{\tau}_k$ {t is reserved for Student's distribution, B for the beta distribution}, with (6.3) and (6.6):

$$\boxed{\underline{\tau}_k = \frac{\underline{E}_{(h)}}{\sigma^2[\lambda_\nu]}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (6.7)$$

In [15] it is deduced, with [6.6], that under the *null hypothesis* H_0 the following is valid:

$$\left.\begin{aligned} &\text{first moment:} && E\{\underline{\tau}_k\} = 1 \\ &\text{second reduced moment:} && E\{(\underline{\tau}_k - 1)^2\} = \frac{2}{k} \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (6.8)$$

Because the corresponding moments of the $F_{k,\infty}$-distribution are equal to (6.8) – a difference being that k from (6.6) is not necessarily an integer – the following approximation is introduced in [15]:

$$\boxed{\underline{\tau}_k \approx \underline{F}_{k,\infty} \equiv \frac{\underline{\chi}_k^2}{k}} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \quad (6.9')$$

with probability density function:

$$f(\tau_k) \approx \frac{\left(\frac{k}{2}\right)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} (\tau_k)^{\frac{k}{2}-1} e^{-\frac{k}{2}\tau_k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.9'')$$

From (6.8) it follows that an unbiased "traced" estimator of σ^2 can be obtained from:

$$\boxed{\underline{s}^2 = \frac{\underline{E}_{(h)}}{[\lambda_v]}} = \sigma^2 \cdot \underline{\tau}_k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.10)$$

with:

$$\left.\begin{aligned} E\{\underline{s}^2\} &= \sigma^2 \\ E\{(\underline{s}^2-\sigma^2)^2\} &= \sigma^4 \cdot \frac{2}{k} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.11)$$

The estimator $\underline{s}^2$ will for theoretical reasons be less efficient {in the sense of R. A. FISCHER} than the estimator used in the previous sections:

$$\hat{\underline{\sigma}}^2 = \frac{\underline{E}}{b}$$

but by a computed example in [15] page 8, BJERHAMMAR supports his opinion that the difference is of little importance in practice.

Consider now two unconnected networks, or two steps in one adjustment problem, so that two values τ can be computed with the same value for the variance factor σ^2. Then the corresponding statistics are stochastically independent. Denote these by:

$$\left.\begin{aligned} &\underline{\tau}_{k^{\mathrm{I}}},\ \underline{\tau}_{k^{\mathrm{II}}} \text{ stochastically independent} \\ &\text{with same variance factor } \sigma^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.12)$$

Then it follows with some approximation from (6.7)–(6.10) that:

$$\boxed{\frac{\underline{\tau}_{k^{\mathrm{I}}}}{\underline{\tau}_{k^{\mathrm{II}}}} = \frac{\{\underline{s}^{\mathrm{I}}\}^2}{\{\underline{s}^{\mathrm{II}}\}^2} \approx \underline{F}_{k^{\mathrm{I}},k^{\mathrm{II}}}} \quad \boxed{\begin{array}{c}\text{provided}\\ (6.12)\\ \text{is valid}\end{array}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.13)$$

If no further specification of alternative hypotheses can be given, then we have for the analysis of variance considered in [15]:

$100\{1-\alpha\}\%$ acceptance region $B^{(\varrho)}$ (with some approximation)
$F_{\frac{\alpha}{2};k^{\mathrm{I}},k^{\mathrm{II}}} = \dfrac{1}{F_{1-\frac{\alpha}{2};k^{\mathrm{II}},k^{\mathrm{I}}}} < \dfrac{\tau_{k^{\mathrm{I}}}}{\tau_{k^{\mathrm{II}}}} < F_{1-\frac{\alpha}{2};k^{\mathrm{I}},k^{\mathrm{II}}}$

. . . . . (6.14)

For a comparison with a similar analysis, the reader is referred to [16].

For a special case of a geodetic network, critical F-values have been tabulated in [15] page 19, after a transformation to more essential geodetic variables for the arguments.

A tentative appreciation

The method given in [15] derives its power and its significance for geodetic applications from the fact that in a number of geodetic problems $[\lambda_v]$ and $[\lambda_v\lambda_v]$ can be computed according to a simple rule, cf. formula (26) in [15], so that in an elementary way estimates for σ^2 can be obtained from (6.10). An extreme example of the simplicity is FERRERO's formula.

The fact that this implies a limitation of the number of "discrepancies" (6.1) to which (6.10) can be applied, is no drawback and in some cases even an advantage. The same limitation will appear in the discussion on the adjustment in steps in section 8 and particularly in section 9. This supports, at least for larger adjustments, the statement in [15] page 9:

> "Testing after the complete adjustment according to the method of least squares is of little or no value".

From a *theoretical* point of view there are, however, some objections.

1. From (6.5) it follows that in a general case latent roots of $(g^{\varrho\tau})$ must be computed. This is about as much work as the computation of $(\bar{g}_{\tau\varrho})$, so that in that case one might as well compute $\hat{\sigma}^2$ instead of s^2.
2. In testing, one uses the tails of the probability distribution of a statistic. Therefore it is questionable if the approximation leading via (6.8) to (6.9) and (6.13) is acceptable in practice. This is the more important if the reasoning connected to (3.13) – the B-method – is adopted.
3. This section was planned to make possible a comparison with previous sections. There it was reasoned how important it is to consider well-specified alternative hypotheses. From (6.3)–(6.8) follows:

$$E\left\{\frac{\underline{E}_{(h)}}{\sigma^2}\,|\,H_a\right\} = [\lambda_v] + \left(\frac{\widetilde{\nabla}y^{\tau}}{\sigma}\right)^{*}(\delta_{\tau\varrho})\left(\frac{\widetilde{\nabla}y^{\varrho}}{\sigma}\right) \quad . . . . . (6.15')$$

If now in analogy with (2.7) one introduces:

$$\lambda_{(h)} = \left(\frac{\widetilde{\nabla}y^{\tau}}{\sigma}\right)^{*}(\delta_{\tau\varrho})\left(\frac{\widetilde{\nabla}y^{\varrho}}{\sigma}\right) \quad . . . . . (6.15'')$$

is it then justified to introduce, on the analogy of (2.6), besides (6.9) the approximation:

$$\left\{\frac{1}{\sigma^2}\underline{\underline{\mathsf{E}}}_{(h)} \mid H_a\right\} \overset{?}{\approx} [\hat{\lambda}_\nu] \cdot \underline{F}'_{k,\infty,\lambda_{(h)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.16)$$

In the present author's opinion the approximation is here pushed very far, if not too far. For the present he would therefore refrain from using the "traced" statistic for this further analysis.

7 ADJUSTMENT IN STEPS EFFECTS OF ALTERNATIVE HYPOTHESES

Some mathematical considerations must be inserted to be able to interpret some new quantities to be introduced.

Adjustment in steps means a partial orthogonalization of the (y^{ϱ})-variates with respect to their probability distribution. It is nothing but a generalized reduction according to GAUSS. For simplicity, consider only two steps I and II:

Decompose:

$$(\underline{y}^{\varrho}) \text{ into } \begin{pmatrix} \underline{y}^{\varrho_{\mathrm{I}}} \\ \underline{y}^{\varrho_{\mathrm{II}}} \end{pmatrix} \quad \begin{matrix} \{\varrho_{\mathrm{I}}, \tau_{\mathrm{I}} = 1, \ldots, b^{\mathrm{I}}\} \\ \{\varrho_{\mathrm{II}}, \tau_{\mathrm{II}} = b^{\mathrm{I}}+1, \ldots, b\} \\ b = b^{\mathrm{I}}+b^{\mathrm{II}} \end{matrix} \Bigg\} \quad \ldots\ldots\ldots\ldots \quad (7.1')$$

$$\{\text{covariance matrix of } \begin{pmatrix} \underline{y}^{\varrho_{\mathrm{I}}} \\ \underline{y}^{\varrho_{\mathrm{II}}} \end{pmatrix}\} = \sigma^2 \begin{pmatrix} g^{\varrho_{\mathrm{I}}\tau_{\mathrm{I}}} & g^{\varrho_{\mathrm{I}}\tau_{\mathrm{II}}} \\ g^{\varrho_{\mathrm{II}}\tau_{\mathrm{I}}} & g^{\varrho_{\mathrm{II}}\tau_{\mathrm{II}}} \end{pmatrix} = \sigma^2 \begin{pmatrix} \bar{g}_{\tau_{\mathrm{I}}\varrho_{\mathrm{I}}} & \bar{g}_{\tau_{\mathrm{I}}\varrho_{\mathrm{II}}} \\ \bar{g}_{\tau_{\mathrm{II}}\varrho_{\mathrm{I}}} & \bar{g}_{\tau_{\mathrm{II}}\varrho_{\mathrm{II}}} \end{pmatrix}^{-1} \quad (7.1'')$$

Introduce the reduction matrix (R) with:

$$(g^{\varrho_{\mathrm{I}}\tau_{\mathrm{I}}})^{-1} = (\bar{g}_{\tau_{\mathrm{I}}\varrho_{\mathrm{I}}.\mathrm{I}}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.2')$$

$$(R) = \begin{pmatrix} \delta^{\varrho_{\mathrm{II}}}_{\varrho'_{\mathrm{II}}} & 0 \\ -g^{\varrho_{\mathrm{II}}\tau_{\mathrm{I}}} \cdot \bar{g}_{\tau_{\mathrm{I}}\varrho'_{\mathrm{I}}.\mathrm{I}} & \delta^{\varrho_{\mathrm{II}}}_{\varrho'_{\mathrm{II}}} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7.2'')$$

Then the orthogonal partial vectors $(\underline{y}^{\varrho_{\mathrm{I}}})$ and $(\underline{y}^{\varrho_{\mathrm{II}}.\mathrm{I}})$ are:

$$\begin{pmatrix} \underline{y}^{\varrho_{\mathrm{I}}} \\ \underline{y}^{\varrho_{\mathrm{II}}.\mathrm{I}} \end{pmatrix} = (R) \begin{pmatrix} \underline{y}^{\varrho_{\mathrm{I}}} \\ \underline{y}^{\varrho_{\mathrm{II}}} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.3')$$

$$\{\text{covariance matrix of } \begin{pmatrix} \underline{y}^{\varrho_{\mathrm{I}}} \\ \underline{y}^{\varrho_{\mathrm{II}}.\mathrm{I}} \end{pmatrix}\} = \sigma^2 (R) \begin{pmatrix} g^{\varrho_{\mathrm{I}}\tau_{\mathrm{I}}} & g^{\varrho_{\mathrm{I}}\tau_{\mathrm{II}}} \\ g^{\varrho_{\mathrm{II}}\tau_{\mathrm{I}}} & g^{\varrho_{\mathrm{II}}\tau_{\mathrm{II}}} \end{pmatrix} (R)^* =$$

$$= \sigma^2 \begin{pmatrix} g^{\varrho_{\mathrm{I}}\tau_{\mathrm{I}}} & 0 \\ 0 & g^{\varrho_{\mathrm{II}}\tau_{\mathrm{II}}.\mathrm{I}} \end{pmatrix} = \sigma^2 \begin{pmatrix} \bar{g}_{\tau_{\mathrm{I}}\varrho_{\mathrm{I}}.\mathrm{I}} & 0 \\ 0 & \bar{g}_{\tau_{\mathrm{II}}\varrho_{\mathrm{II}}} \end{pmatrix}^{-1} \quad \ldots\ldots\ldots\ldots \quad (7.3'')$$

From (2.7), with (7.3), see (2.3.3) in [3]:

$$\frac{\underline{E}}{\sigma^2} = \frac{1}{\sigma^2}\begin{pmatrix}\underline{y}^{\tau_I}\\ \underline{y}^{\tau_{II}}\end{pmatrix}^* \begin{pmatrix}\bar{g}_{\tau_I\varrho_I} & \bar{g}_{\tau_I\varrho_{II}}\\ \bar{g}_{\tau_{II}\varrho_I} & \bar{g}_{\tau_{II}\varrho_{II}}\end{pmatrix}\begin{pmatrix}\underline{y}^{\varrho_I}\\ \underline{y}^{\varrho_{II}}\end{pmatrix} =$$

$$= \left(\frac{\underline{y}^{\tau_I}}{\sigma}\right)^*(\bar{g}_{\tau_I\varrho_I.I})\left(\frac{\underline{y}^{\varrho_I}}{\sigma}\right) + \left(\frac{\underline{y}^{\tau_{II}.I}}{\sigma}\right)^*(\bar{g}_{\tau_{II}\varrho_{II}})\left(\frac{\underline{y}^{\varrho_{II}.I}}{\sigma}\right) =$$

$$= \frac{\underline{E}^I}{\sigma^2} + \frac{\underline{E}^{II}}{\sigma^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.4)$$

With alternative hypothesis H_a, see (3.1.5) and (3.2.8) in [3]:

$$\lambda = \frac{1}{\sigma^2}\begin{pmatrix}\widetilde{\nabla}y^{\tau_I}\\ \widetilde{\nabla}y^{\tau_{II}}\end{pmatrix}^* \begin{pmatrix}\bar{g}_{\tau_I\varrho_I} & \bar{g}_{\tau_I\varrho_{II}}\\ \bar{g}_{\tau_{II}\varrho_I} & \bar{g}_{\tau_{II}\varrho_{II}}\end{pmatrix}\begin{pmatrix}\widetilde{\nabla}y^{\varrho_I}\\ \widetilde{\nabla}y^{\varrho_{II}}\end{pmatrix} =$$

$$= \left(\frac{\widetilde{\nabla}y^{\tau_I}}{\sigma}\right)^*(\bar{g}_{\tau_I\varrho_I.I})\left(\frac{\widetilde{\nabla}y^{\varrho_I}}{\sigma}\right) + \left(\frac{\widetilde{\nabla}y^{\tau_{II}.I}}{\sigma}\right)^*(\bar{g}_{\tau_{II}\varrho_{II}})\left(\frac{\widetilde{\nabla}y^{\varrho_{II}.I}}{\sigma}\right) = \lambda^I + \lambda^{II} \quad (7.5)$$

Analogous to (2.6):

$\left\{\frac{1}{\sigma^2}\underline{E} \mid H_0\right\} \equiv b\cdot\underline{F}_{b,\infty}$	$\left\{\frac{1}{\sigma^2}\underline{E} \mid H_a\right\} \equiv b\cdot\underline{F}'_{b,\infty,\lambda}$
$\left\{\frac{1}{\sigma^2}\underline{E}^I \mid H_0\right\} \equiv b^I\cdot\underline{F}_{b^I,\infty}$	$\left\{\frac{1}{\sigma^2}\underline{E}^I \mid H_a\right\} \equiv b^I\cdot\underline{F}'_{b,\infty,\lambda^I}$
$\left\{\frac{1}{\sigma^2}\underline{E}^{II} \mid H_0\right\} \equiv b^{II}\cdot\underline{F}_{b^{II},\infty}$	$\left\{\frac{1}{\sigma^2}\underline{E}^{II} \mid H_a\right\} \equiv b^{II}\cdot\underline{F}'_{b_{II},\infty,\lambda^{II}}$

. . . . . (7.6′)

$E\left\{\frac{1}{\sigma^2}\underline{E} \mid H_a\right\} = b+\lambda$	$b = b^I+b^{II}$	$\lambda = \lambda^I+\lambda^{II}$
$E\left\{\frac{1}{\sigma^2}\underline{E}^I \mid H_a\right\} = b^I+\lambda^I$	$\underline{E}^I, \underline{E}^{II}$ stochastically	
$E\left\{\frac{1}{\sigma^2}\underline{E}^{II} \mid H_a\right\} = b^{II}+\lambda^{II}$	independent	

. . . . . (7.6″)

In figure 7-1, this is illustrated by sketching the y^e-subspace of figure 2-1 in more detail, compare figure 2-2.

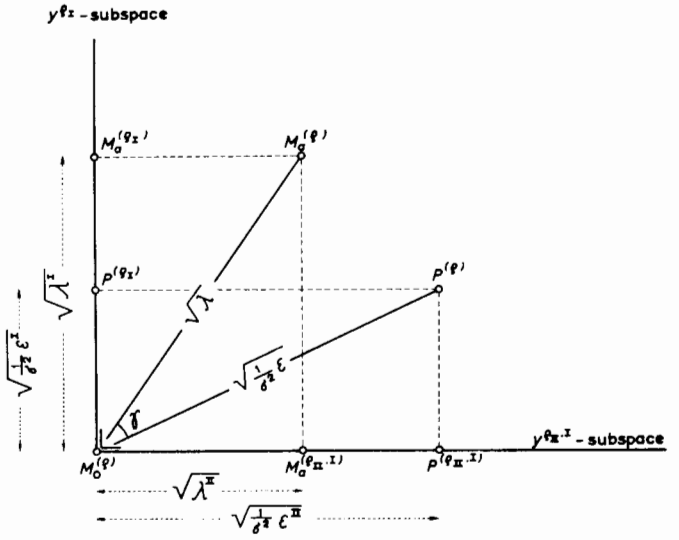

Fig. 7-1

Applying the alternative hypothesis in the form (2.9), for the present dropping the subscript p, we obtain:

$$\begin{pmatrix} \frac{\widetilde{\nabla} y^{\varrho_I}}{\sigma} \\ \\ \frac{\widetilde{\nabla} y^{\varrho_{II}\cdot I}}{\sigma} \end{pmatrix} = (R) \begin{pmatrix} \frac{\widetilde{\nabla} y^{\varrho_I}}{\sigma} \\ \\ \frac{\widetilde{\nabla} y^{\varrho_{II}}}{\sigma} \end{pmatrix} = (R) \begin{pmatrix} u_{i'}^{\varrho_I} \\ \\ u_{i'}^{\varrho_{II}} \end{pmatrix} (g^{i'j})(\bar{g}_{ji})(c^i)\cdot\nabla =$$

$$= (R) \begin{pmatrix} g^{\varrho_I j} \\ \\ g^{\varrho_{II} j} \end{pmatrix} (\bar{g}_{ji})(c^i)\cdot\nabla = \begin{pmatrix} g^{\varrho_I j} \\ \\ g^{\varrho_{II} j\cdot I} \end{pmatrix} (\bar{g}_{ji})(c^i)\cdot\nabla \quad \ldots\ldots\ldots \quad (7.7)$$

With:

$$N = (c^j)^*(\bar{g}_{ji'}) \begin{pmatrix} g^{\tau_I i'} \\ \\ g^{\tau_{II} i'} \end{pmatrix}^* \begin{pmatrix} \bar{g}_{\tau_I \varrho_I} & \bar{g}_{\tau_I \varrho_{II}} \\ \\ \bar{g}_{\tau_{II} \varrho_I} & \bar{g}_{\tau_{II} \varrho_{II}} \end{pmatrix} \begin{pmatrix} g^{\varrho_I j'} \\ \\ g^{\varrho_{II} j'} \end{pmatrix} (\bar{g}_{j'i})(c^i) =$$

$$= (c^j)^*(\bar{g}_{ji'})(g^{i'\tau_I})(\bar{g}_{\tau_I \varrho_I\cdot I})(g^{\varrho_I j'})(\bar{g}_{j'i})(c^i) +$$

$$+ (c^j)^*(\bar{g}_{ji'})(g^{i'\tau_{II}\cdot I})(\bar{g}_{\tau_{II}\varrho_{II}})(g^{\varrho_{II} j'\cdot I})(\bar{g}_{j'i})(c^i) = N^I + N^{II} \quad \ldots \quad (7.8')$$

we then obtain:

$$\boxed{\begin{array}{l} \lambda = N\cdot\nabla^2 \\ \hline \lambda^I = N^I\cdot\nabla^2 \\ \lambda^{II} = N^{II}\cdot\nabla^2 \end{array}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.8'')$$

A further orthogonalization as in (2.11) and (2.12) can then be obtained for each of the subspaces y^{ϱ_I} and $y^{\varrho_{II}\cdot I}$ by the transformation:

$$\begin{pmatrix} \underline{w}^{1.I} \\ \underline{w}^{2.I} \\ \cdot \\ \cdot \\ \cdot \\ \underline{w}^{b^I.I} \\ \hdashline \underline{w}^{b^I+1.I} \\ \underline{w}^{b^I+2.I} \\ \cdot \\ \cdot \\ \cdot \\ \underline{w}^{b.I} \end{pmatrix} = \left(\begin{array}{cc} \frac{1}{\sqrt{N^I}}(c^j)^*(\bar{g}_{ji})(g^{i\tau_I})(\bar{g}_{\tau_I\varrho_I\cdot I}) & 0 \\ (C^{2.I}_{\varrho_I}) & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ (C^{b^I.I}_{\varrho_I}) & 0 \\ \hdashline 0 & \frac{1}{\sqrt{N^{II}}}(c^j)^*(\bar{g}_{ji})(g^{i\tau_{II}\cdot I})(\bar{g}_{\tau_{II}\varrho_{II}}) \\ 0 & (C^{b^I+2.I}_{\varrho_{II}}) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & (C^{b.I}_{\varrho_{II}}) \end{array}\right) \begin{pmatrix} \\ \\ \\ \underline{y}^{\varrho_I} \\ \\ \\ \hdashline \\ \\ \\ \underline{y}^{\varrho_{II}\cdot I} \\ \\ \\ \end{pmatrix} \qquad (7.9')$$

with:

$$\left.\begin{aligned} |\tilde{\nabla}w^{1.I}| &= \sqrt{N^I}\cdot|\nabla| = \sqrt{\lambda^I} \\ |\tilde{\nabla}w^{b^I+1.I}| &= \sqrt{N^{II}}\cdot|\nabla| = \sqrt{\lambda^{II}} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7.9'')$$

But (2.12) can also be transformed directly with (7.3), after which a comparison with (7.9) gives:

$$\begin{aligned} \sqrt{N}\cdot\underline{w}^1 &= (c^j)^*(\bar{g}_{ji})(g^{i\tau_I}\ g^{i\tau_{II}})\begin{pmatrix} \bar{g}_{\tau_I\varrho_I} & \bar{g}_{\tau_I\varrho_{II}} \\ \bar{g}_{\tau_{II}\varrho_I} & \bar{g}_{\tau_{II}\varrho_{II}} \end{pmatrix}\begin{pmatrix} \frac{\underline{y}^{\varrho_I}}{\sigma} \\ \frac{\underline{y}^{\varrho_{II}}}{\sigma} \end{pmatrix} = \\ &= (c^j)^*(\bar{g}_{ji})(g^{i\tau_I})(\bar{g}_{\tau_I\varrho_I\cdot I})\left(\frac{\underline{y}^{\varrho_I}}{\sigma}\right) + (c^j)^*(\bar{g}_{ji})(g^{i\tau_{II}\cdot I})(\bar{g}_{\tau_{II}\varrho_{II}})\left(\frac{\underline{y}^{\varrho_{II}\cdot I}}{\sigma}\right) = \\ &= \sqrt{N^I}\cdot\underline{w}^{1.I} + \sqrt{N^{II}}\cdot\underline{w}^{b^I+1.I} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7.10') \end{aligned}$$

or, $\underline{w}^{b^I+1.I}$ fulfills the rôle of $\underline{w}^1$ in step II and can also be considered as a component of $\underline{w}^1$. Therefore we introduce the notation:

$$\underline{w}^{1.II} \equiv \underline{w}^{b^I+1.I} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.10'')$$

By a derivation as in (2.10) one can then write, see (3.2.7) in [3]:

$$\begin{array}{|l||l|}
\hline
N_p = (c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})(c_p^i) & \lambda_p = N_p \cdot \{\nabla_p\}^2 \\
\hline
N_p^{\mathrm{I}} = (c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'.\mathrm{I}})(\bar{g}_{j'i})(c_p^i) & \lambda_p^{\mathrm{I}} = N_p^{\mathrm{I}} \cdot \{\nabla_p\}^2 \\
N_p^{\mathrm{II}} = (c_p^j)^*(\bar{g}_{ji'})(G^{i'j'.\mathrm{I}} - G^{i'j'})(\bar{g}_{j'i})(c_p^i) & \lambda_p^{\mathrm{II}} = N_p^{\mathrm{II}} \cdot \{\nabla_p\}^2 \\
\hline
\end{array} \qquad (7.11')$$

$$\begin{array}{|l|l|}
\hline
\underline{w}_p^1 = \dfrac{1}{\sqrt{N_p}}(c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})\left(\dfrac{\underline{x}^i - a_0^i}{\sigma}\right) & \\
\quad = \dfrac{1}{\sqrt{N_p}}(c_p^j)^*(\bar{g}_{ji})\left(\dfrac{-\underline{\varepsilon}^i}{\sigma}\right) = \cos\underline{\gamma}_p \cdot \sqrt{\dfrac{1}{\sigma^2}\underline{E}} & \\
\hline
N_p = N_p^{\mathrm{I}} + N_p^{\mathrm{II}} & \sqrt{N_p}\cdot\underline{w}_p^1 = \sqrt{N_p^{\mathrm{I}}}\cdot\underline{w}_p^{1.\mathrm{I}} + \sqrt{N_p^{\mathrm{II}}}\cdot\underline{w}_p^{1.\mathrm{II}} \\
\hline
\underline{w}_p^{1.\mathrm{I}} = \dfrac{1}{\sqrt{N_p^{\mathrm{I}}}}(c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'.\mathrm{I}})(\bar{g}_{j'i})\left(\dfrac{\underline{x}^i - a_0^i}{\sigma}\right) & \\
\quad = \dfrac{1}{\sqrt{N_p^{\mathrm{I}}}}(c_p^j)^*(\bar{g}_{ji})\left(\dfrac{-\underline{\varepsilon}^{i.\mathrm{I}}}{\sigma}\right) = \cos\underline{\gamma}_p^{\mathrm{I}} \cdot \sqrt{\dfrac{1}{\sigma^2}\underline{E}^{\mathrm{I}}} & \\
\hline
\underline{w}_p^{1.\mathrm{II}} = \dfrac{1}{\sqrt{N_p^{\mathrm{II}}}}(c_p^j)^*(\bar{g}_{ji'})(G^{i'j'.\mathrm{I}} - G^{i'j'})(\bar{g}_{j'i})\left(\dfrac{\underline{x}^i - a_0^i}{\sigma}\right) & \\
\quad = \dfrac{1}{\sqrt{N_p^{\mathrm{II}}}}(c_p^j)^*(\bar{g}_{ji})\left(\dfrac{-\underline{\varepsilon}^{i.\mathrm{II}}}{\sigma}\right) = \cos\underline{\gamma}_p^{\mathrm{II}} \cdot \sqrt{\dfrac{1}{\sigma^2}\underline{E}^{\mathrm{II}}} & \\
\hline
\end{array} \qquad \ldots\ldots\ (7.11'')$$

For each step this situation can be illustrated as was done in figure 2-2 for the non-decomposed y^ϱ-subspace.

8 ADJUSTMENT IN STEPS CRITICAL REGIONS FOR F-TESTS

We can now apply the reasoning of section 3. Presupposing one specified H_{a_p}, acceptance regions (3.2) can be found for step I, for step II, and finally for the combination of step I and step II. But we can only *once* make a statement on the basis of α_0 and β_0, if we want to avoid the vagueness implied in the distribution of α_0 and β_0 over the steps, as discussed in [3].

Independently of the index number p of several possible {not simultaneously occurring} alternative hypotheses, the acceptance regions (3.9) can then be used for practical applications, computing α as in (3.11). The acceptance regions mentioned first are better suited for data-snooping, as was explained in section 4.

For the computation of bounds, (3.6) remains valid.

Applying the formulas from section 7 the following summary is obtained:

Computation of networks for all H_{a_p}		
Initial choice α_0, β_0		$\lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$
Step I	α^{I} from	$\beta_0 = \beta\{\alpha^{\mathrm{I}}, \lambda_0, b^{\mathrm{I}}, \infty\}$
	$B^{(\varrho_{\mathrm{I}})}$	$\frac{1}{\sigma^2}\underline{E}^{\mathrm{I}} < b^{\mathrm{I}} \cdot F_{1-\alpha^{\mathrm{I}};\, b^{\mathrm{I}}, \infty}$
	$\lvert\nabla^{\mathrm{I}}_{p,0}\rvert$	$\sqrt{\frac{\lambda_0}{N^{\mathrm{I}}_p}}$; N^{I} see (7.11)
Step II	α^{II} from	$\beta_0 = \beta\{\alpha^{\mathrm{II}}, \lambda_0, b^{\mathrm{II}}, \infty\}$
	$B^{(\varrho_{\mathrm{II}.\mathrm{I}})}$	$\frac{1}{\sigma^2}\underline{E}^{\mathrm{II}} < b^{\mathrm{II}} \cdot F_{1-\alpha^{\mathrm{II}};\, b^{\mathrm{II}}, \infty}$
	$\lvert\nabla^{\mathrm{II}}_{p,0}\rvert$	$\sqrt{\frac{\lambda_0}{N^{\mathrm{II}}_p}}$; N^{II} see (7.11)
Steps I+II	$\underline{E}$	$\underline{E}^{\mathrm{I}}+\underline{E}^{\mathrm{II}}$
	b	$b^{\mathrm{I}}+b^{\mathrm{II}}$
	N_p	$N^{\mathrm{I}}_p+N^{\mathrm{II}}_p$
	α from	$\beta_0 = \beta\{\alpha, \lambda_0, b, \infty\}$
	$B^{(\varrho)}$	$\frac{1}{\sigma^2}\underline{E} < b \cdot F_{1-\alpha;\, b,\, \infty}$
	$\lvert\nabla_{p,0}\rvert$	$\sqrt{\frac{\lambda_0}{N_p}}$

(8.1) (8.2) (8.3)

For data-snooping per H_{a_p}		
Initial choice α_0, β_0		$\lambda_o = \lambda\{\alpha_0, \beta_0, 1, \infty\}$
Step I	$w_p^{1.\mathrm{I}}$	(7.11) (8.4)
	$B_p^{(\varrho_\mathrm{I})}$	$-F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}} < \underline{w}_p^{1.\mathrm{I}} < +F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}}$
Step II	$w_p^{1.\mathrm{II}}$	(7.11) (8.5)
	$B_p^{(\varrho_{\mathrm{II}\cdot\mathrm{I}})}$	$-F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}} < \underline{w}_p^{1.\mathrm{II}} < +F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}}$
Steps I+II	w_p^1	$\frac{1}{\sqrt{N^\mathrm{I}+N^\mathrm{II}}}\{\sqrt{N^\mathrm{I}}\cdot w_p^{1.\mathrm{I}}+\sqrt{N^\mathrm{II}}\cdot w_p^{1.\mathrm{II}}\}$ (8.6)
	$B_p^{(\varrho)}$	$-F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}} < \underline{w}_p^{1} < +F_{1-\alpha_0;\,1,\infty}^{\frac{1}{2}}$

But under *this* condition:

either (8.1), (8.4)
or (8.2), (8.5)
or (8.3), (8.6) } . . . . . (8.7)

To compare the reasoning with the one followed in (2.4.8) in [3], it is mentioned that the following proves to be always valid in (8.1)–(8.3):

$$\alpha > 1-\{1-\alpha^\mathrm{I}\}\{1-\alpha^\mathrm{II}\} \quad . . . . . (8.8')$$

An example illustrates this:

$\beta_0 = 0.80$	$\alpha_0 = 0.00\,00\,2$	$\lambda_0 = 26.1$
$b^\mathrm{I} = 20$	$\alpha^\mathrm{I} = 0.015$	$F_{1-\alpha^\mathrm{I};\,b^\mathrm{I},\infty} = 1.8$
$b^\mathrm{II} = 15$	$\alpha^\mathrm{II} = 0.007$	$F_{1-\alpha^\mathrm{II};\,b^\mathrm{II},\infty} = 2.1$ (8.8'')
$b = 35$	$\alpha = 0.05$	$F_{1-\alpha;\,b,\infty} = 1.42$
	$1-\{1-\alpha^\mathrm{I}\}\{1-\alpha^\mathrm{II}\} = 0.021 < 0.05$	

In looking for a practically usable solution one is faced with three problems:

a. what choice to make from (8.1)–(8.3) in view of (8.7); (8.9a)

b. there is a possibility of different conclusions regarding the rejection of H_0, because the product space of $B^{(\varrho_\mathrm{I})}$ and $B^{(\varrho_{\mathrm{II}\cdot\mathrm{I}})}$ is not equal to $B^{(\varrho)}$; compare the remarks preceding (2.4.6) in [3]; (8.9b)

c. for increasing b: $\alpha \gg \alpha_0$; see the text about (3.13)–(3.16) (8.9c)

To introduce directives for the solution of these problems, the situation of figure 7-1 is sketched for three alternative hypotheses H_{a_p}, $H_{a_{p'}}$, $H_{a_{p''}}$; see figure 8-1 and formulas (7.11).

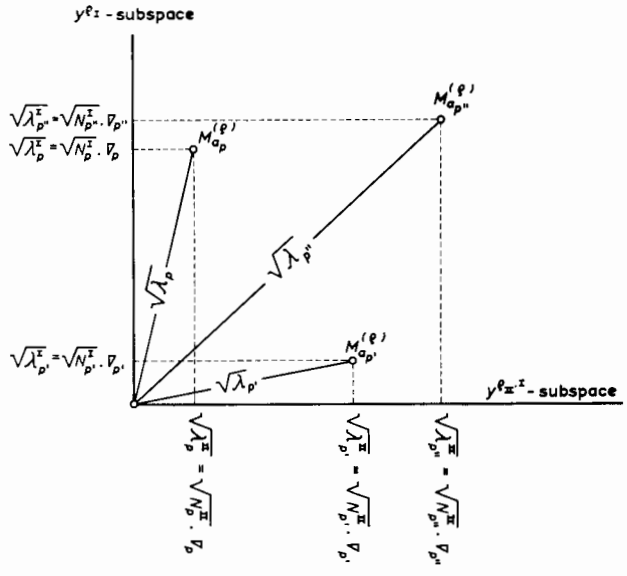

Fig. 8-1

For the situation sketched in figure 8-1 we obtain:

$$\left.\begin{aligned} &N_p^{\mathrm{I}} \gg N_p^{\mathrm{II}} \quad , \text{ hence: } N_p^{\mathrm{I}} \lesssim N_p \\ &N_{p'}^{\mathrm{I}} \ll N_{p'}^{\mathrm{II}} \quad , \text{ hence: } N_{p'}^{\mathrm{II}} \lesssim N_{p'} \\ &N_{p''}^{\mathrm{I}} \approx N_{p''}^{\mathrm{II}} \quad , \text{ hence: } < N_p^{\mathrm{I}}, N_{p'}^{\mathrm{II}}, N_{p''} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (8.10)$$

and hence, according to (8.1)–(8.3):

$$\left.\begin{aligned} &|\nabla_{p,0}| \lesssim |\nabla_{p,0}^{\mathrm{I}}| \ll |\nabla_{p,0}^{\mathrm{II}}| \\ &|\nabla_{p',0}| \lesssim |\nabla_{p',0}^{\mathrm{II}}| \ll |\nabla_{p',0}^{\mathrm{I}}| \\ &|\nabla_{p'',0}| < |\nabla_{p'',0}^{\mathrm{I}}| \approx |\nabla_{p'',0}^{\mathrm{II}}| \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.11)$$

If one takes into account the remarks in [3] page 33, viz. that weight coefficients cannot and need not be given better than within 10% or perhaps even 30% of their value, then something equivalent will be valid for the N's and consequently for the $|\nabla_0|$. For practice it will therefore make little difference {in fact we will be more on the safe side} if we use:

$|\nabla_{p,0}^{\mathrm{I}}|$ instead of $|\nabla_{p,0}|$

$|\nabla_{p',0}^{\mathrm{II}}|$ instead of $|\nabla_{p',0}|$

Then a case like (8.11) would mean, for practical application:

Situation	test	H_a	in step	$\lvert\nabla_0\rvert$
fig. 8.1 with (8.10) and (8.11)	(8.1)	H_{a_p}	I	$\lvert\nabla^{\mathrm{I}}_{p,0}\rvert$
	(8.2)	$H_{a_{p'}}$	II	$\lvert\nabla^{\mathrm{II}}_{p',0}\rvert$
	(8.3)	$H_{a_{p''}}$	I+II	$\lvert\nabla_{p'',0}\rvert$

. . . . . (8.12)

This means the solution for (8.9a); all three tests are applied, but the interpretation is different with regard to the alternative hypotheses.

The problem (8.9b) remains, but this provides perhaps a directive for data-snooping according to (8.4)–(8.6), because there the formulation of alternative hypotheses is the first requirement.

The problem (8.9c) remains also, because the alternative hypothesis $H_{a_{p''}}$ requires the test (8.3). In that case, b is greater than b^{I} or b^{II} and hence in view of (8.8′):

$$\alpha > \alpha^{\mathrm{I}},\ \alpha^{\mathrm{II}} > \alpha_0 \quad . . . . . (8.13)$$

To meet this difficulty, one could split each of the steps I and II again into two, as sketched in figure 8-2:

step	step	step	step
I^1	I^2	II^1	II^2

Fig. 8-2

in such a way that instead of (8.10) we obtain, e.g.:

$$\left.\begin{aligned} N^{\mathrm{I}^1}_p + N^{\mathrm{I}^2}_p &= \{1-\delta^2\}N_p \ ; \quad \delta^2 \approx 0.2? \\ N^{\mathrm{II}^1}_{p'} + N^{\mathrm{II}^2}_{p'} &= \{1-\delta^2\}N_{p'} \ ; \quad \delta^2 \approx 0.2? \\ N^{\mathrm{I}^2}_{p''} + N^{\mathrm{II}^1}_{p''} &= \{1-\delta^2\}N_{p''} \ ; \quad \delta^2 \approx 0.2? \end{aligned}\right\} \quad . . . . . (8.14)$$

Then we obtain instead of (8.12):

Situation	test H_a	in step	$\lvert\nabla_0\rvert$
fig. 8−2 with (8.14)	H_{a_p}	$\mathrm{I}^1+\mathrm{I}^2$	$\lvert\nabla^{\mathrm{I}^1+\mathrm{I}^2}_{p,0}\rvert \equiv \lvert\nabla^{\mathrm{I}}_{p,0}\rvert$
	$H_{a_{p''}}$	$\mathrm{I}^2+\mathrm{II}^1$	$\lvert\nabla^{\mathrm{I}^2+\mathrm{II}^1}_{p'',0}\rvert$
	$H_{a_{p'}}$	$\mathrm{II}^1+\mathrm{II}^2$	$\lvert\nabla^{\mathrm{II}^1+\mathrm{II}^2}_{p',0}\rvert \equiv \lvert\nabla^{\mathrm{II}}_{p',0}\rvert$

(8.15)

These tests are not independent, but in view of (8.7) this is not essential for this approach.

To meet the difficulty (8.9c) one can try to keep the different degrees of freedom b within certain limits. For example:

$$\left.\begin{array}{lll} b^{I^1} + b^{I^2} = b^{I} & < 30 & \text{or?} \\ b^{I^2} + b^{II^1} & < 30 & \text{or?} \\ b^{II^1} + b^{II^2} = b^{II} & < 30 & \text{or?} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8.16)$$

Data-snooping according to section 4 will then remain reasonably realistic.

This idea (8.16) can in any case be applied for the alternative hypotheses H_{a_i}, see (2.15.)

It is well known that the precision of adjusted observations in geodetic networks usually increases only a little if one adds condition equations generated by geodetic observations farther and farther away. This proves to hold for the N_i's too, which promotes (8.16) to the real possibilities.

One might then make a design according to the extreme consequence of (8.15), in which each new condition-equation generated by the proceeding measurement leads to a next step.

One obtains:

Test procedure. Theoretical solution	
orthogonalized misclosure variates	$\underline{y}^1, \underline{y}^{2.1}, \ldots, \underline{y}^{\varrho.\varrho-1}, \ldots, \underline{y}^{b.b-1}$
shifting variates	$\left\{\frac{\underline{y}^1}{\sigma_{y^1}}\right\}^2, \left\{\frac{\underline{y}^{2.1}}{\sigma_{y^{2.1}}}\right\}^2, \ldots, \left\{\frac{\underline{y}^{\varrho.\varrho-1}}{\sigma_{y\varrho.\varrho-1}}\right\}^2, \ldots, \left\{\frac{\underline{y}^{b.b-1}}{\sigma_{y^{b.b-1}}}\right\}^2$
N matrix $\{i = 1, \ldots, m\}$	$N_i^1, N_i^{2.1}, \ldots, N_i^{\varrho.\varrho-1}, \ldots, N_i^{b.b-1}$

For *that* value of i which is the first to make:

$$\sum_{\varrho=1}^{b_i} N_i^{\varrho.\varrho-1} \approx (1-\delta^2)N_i$$

compute α_i: $\quad \beta_0 = \beta\{\alpha_i, \lambda_0, b_i, \infty\}$

and if: $\quad \frac{1}{\sigma^2}\sum_{\varrho=1}^{b_i} E^{\varrho.\varrho-1} > b_i \cdot F_{1-\alpha_i;\, b_i, \infty}$

stop measurements and check observations, especially x^i (8.17)

If: $\quad \frac{1}{\sigma^2}\sum_{\varrho=1}^{b_i} E^{\varrho.\varrho-1} < b_i \cdot F_{1-\alpha_i;\, b_i, \infty}$

repeat procedure for the next i'-value for which the bound $\sum N_{i'}^{\varrho.\varrho-1}$ is reached, possibly omitting the first steps if they only give an unimportant contribution to this sum.

etc.

(8.17) is a too far-reaching theoretical extreme, but a mixture of (8.15) and (8.17) will be quite feasible. The measuring programme of the network will then have to be adapted somewhat, so that from time to time a next group of condition equations can be taken together to form a partial step.

One *loses* then the idea of testing a larger geodetic network when steps are joined to form one whole.

For, in that case one would obtain:

$$\alpha_0 \ll \alpha \to 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (8.18)$$

Further it will be *clear* what is the significance of keeping all α's constant, as is now usually done. For example:

$$\alpha_0, \alpha^{\mathrm{I}}, \alpha^{\mathrm{II}}, \alpha, \ldots = 0.05 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (8.19')$$

then:

$$\left.\begin{array}{l}\text{if } b \text{ increases: rejection region} \\ \qquad\qquad\quad \text{relatively decreases}\end{array}\right\} \quad \ldots \ldots \ldots \quad (8.19'')$$

Hence:

$$\left.\begin{array}{l}\text{either if } b \text{ small: reject too much} \\ \text{or if } b \text{ large: reject too little}\end{array}\right\} \quad \ldots \ldots \ldots \quad (8.19''')$$

This combination of (8.15) and (8.17) makes it possible to follow the measurement of geodetic networks more or less continually, and so check the measurements before towers and signals have been taken down. The compromise implied in this method of checking is in any case better than checks executed years later! The result of this procedure is the possibility to exert control on the $|\nabla_{p,0}|$, so that one may refer to it as the planning of the reliability of geodetic networks.

The situation becomes entirely different if only one test is made on the total adjustment of an extensive geodetic network. If we wish to avoid (8.18) and choose, e.g., α from (8.19') then we obtain (see the end of section 3):

$$\lambda_0 = \lambda\{\alpha, \beta_0, b, \infty\} \quad \text{large} \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (8.20')$$

and hence, in view of the remarks about N_p in this section:

$$|\nabla_{p,0}| \quad \text{large} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (8.20'')$$

The test will therefore be much less sensitive to the occurrence of model errors.

Besides, the question is raised: *what* is actually being tested? It is a multitude of different types of model errors, see [3], and finding them will be very difficult if not practically impossible. Such a net has been measured in a period usually covering many years and would elaborate checking measurements then be executed afterwards?

For comparison, see the remark by BJERHAMMAR [15] page 9, already referred to in section 6.

Application of other F-distributions

In the case of a preceding step in which:

$$N_p^{1^0} = 0 \quad \text{for all values } p \quad . \quad (8.21)$$

the tests described can be based on the statistics $\underline{F}_{m,n}$ mentioned in (2.3.5) in [3]. The dependence of the tests is of no importance in the reasoning followed. With the formula from [3] referred to, some practical difficulties have been mentioned, whereas there are also difficulties connected with the interpretation of a chosen method. To avoid confusing the line of thought, which is complicated enough in itself, we shall not go into the consequences of (8.21).

A good example of (8.21) is the first step in the analysis of variance; see the introduction of section 5.

Previous investigations

The author made his first geodetic applications of the power functions of F-tests in 1956 with the choice (8.19′). Its usefulness was doubted when it was tried to interpret the results intuitively. In [16], B. G. K. KRIJGER drew the consequences of this choice. His ideas can be considered as forerunners of the combined methods of (8.15) and (8.17), although orthogonalization or computation in steps was not applied. The conclusions in [16] pages 23, 24, can be seen as an illustration of (8.19) and (8.20). In 1964, a new analysis was undertaken for the planning of the base extension network in the north of our country, along with an application by KRIJGER of the ideas developed in [16]. Continued dissatisfaction led the author to the present line of thought, which to him seems to be more acceptable, also on intuitive grounds.

9 DATA-SNOOPING BEFORE FINAL COMPUTATION

As an introduction, we start from the considerations on (4.4)–(4.13) for a single function $\underline{\psi}$. Choose now a group of these functions, which are algebraically independent:

$$(\underline{\psi}^r) = (c_\varrho^r)(\underline{y}^\varrho) \qquad \{r,s = 1, \ldots, \bar{b}; \bar{b} \leqq b\} \quad \ldots\ldots\ldots\ldots \quad (9.1')$$

Then they are also functions of misclosure variates which are orthogonalized to some degree, such as in (8.17) and (7.3). Consequently they can fulfill the same task. From (9.1') follows:

$$\{\text{covariance matrix of } (\underline{\psi}^r)\} = (c_\varrho^r)(g^{\varrho\tau})(c_\tau^s)^*\sigma^2 \equiv \sigma^2 \cdot (g^{rs}) = \sigma^2(\bar{g}_{sr})^{-1} \qquad (9.1'')$$

Then the same reasoning as in section 8 is possible, i.e. a procedure of tests with consequences, compare (8.1):

Initial choice α_0, β_0	$\lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$
$\bar{\alpha}$ from	$\beta_0 = \beta\{\bar{\alpha}, \lambda_0, \bar{b}, \infty\}$
$\frac{1}{\sigma^2}\bar{\underline{E}}$	$\left(\frac{\underline{\psi}^s}{\sigma}\right)^*(\bar{g}_{sr})\left(\frac{\underline{\psi}^r}{\sigma}\right)$
$B^{(\psi^r)}$	$\frac{1}{\sigma^2}\bar{\underline{E}} < \bar{b} \cdot F_{1-\bar{\alpha};\,\bar{b},\infty}$
$\lvert\bar{\nabla}_{p,0}\rvert$	$\sqrt{\frac{\lambda_0}{\bar{N}_p}}$; $\bar{N}_p$ not specified

. . . . . (9.2)

(4.13) is a special case of (9.2).

For the situation (9.1') with $\bar{b} < b$, usually even $\bar{b} \ll b$, we have a situation analogous to the adjustment in steps, namely the adjustment in one of the steps.

Now one can consider a number of groups of functions $(\underline{\psi}^r)$ simultaneously and compute (9.2) for each of them. Something which is similar to the mixture of methods (8.15) and (8.17) is then obtained, but *without* a strict computation in steps. Usually the application will be restricted to groups of y^ϱ-variates. In fact, this was done by KRIJGER in his investigations referred to at the end of section 8 {but with the choice (8.19')}.

In geodetic networks a number of condition equations occur whose misclosures can be easily computed in the field. This applies to the angle condition equations in triangulation, and – if distance ratios have been measured – for distance ratio condition equations in

trilateration. In traverse networks, the combined condition equation might be called the {complex} "polygon condition equation", whose {complex} misclosure variate is given by (4.2.31) in [3]. If no distance ratios have been measured, only the angle condition equation remains.

For the latter case, J. VAN MIERLO investigated a regular polygon with a constant side length but with a variable number of sides. In a test on the angle misclosure only he compared $\overline{N}_i$ with N_i in a test on the angle and coordinate misclosures together. The index i pertains then only to direction variates, indicated by kernel letter r.

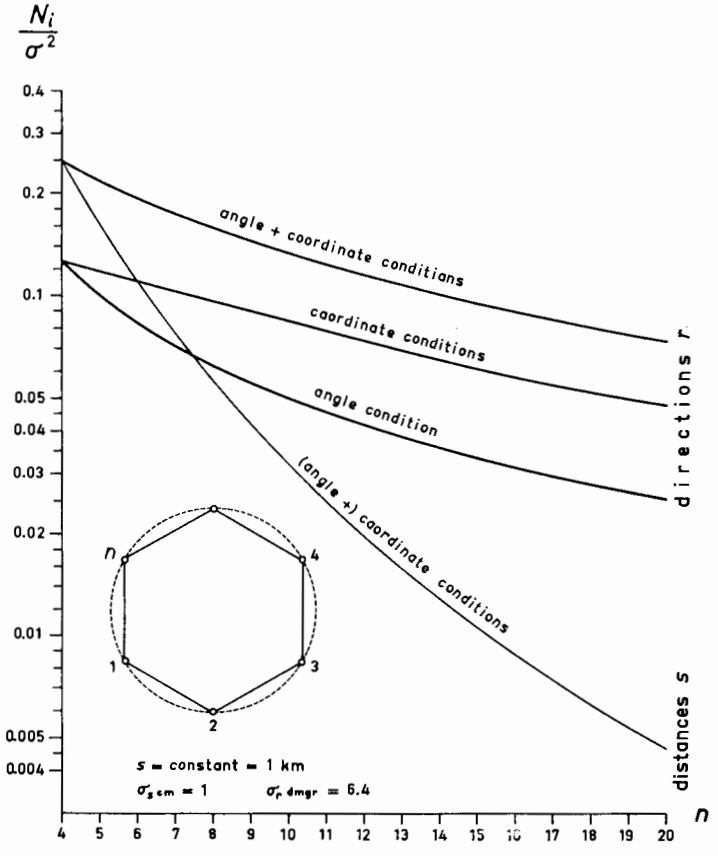

Fig. 9-1

In figure 9-1 the results are pictured. It appears that, for the alternative hypotheses H_{a_i} concerned, the test on the single misclosure of the angle condition equation provides already a reasonable check, see (9.2):

$$|\bar{\nabla}_{i,0}| = \sqrt{\frac{\lambda_0}{\overline{N}_i}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9.3)$$

This means that it is useful to execute provisional tests (9.2) in the field as soon as possible

after measurement in order to check the measurements made. In doing so, one can consider single functions $\underset{\sim}{\psi}$ {i.e. single y^{ϱ}} as in (4.13), but also groups. If one considers groups, the inverse of (9.1'') must be computed for the computation of $\bar{E}$ in (9.2). This practical difficulty is met by using the traced statistic of BJERHAMMAR, as explained in section 6. To do this, the field geodesist must have at his disposal easy formulas for the latent roots λ_{ν}, or use a table of critical values as given in [15] for a certain case.

Final computation is an elastic concept. For example, checks by means of a "free adjustment" of national networks – as advocated by H. WOLF in many publications on the connection of such networks – can be seen as an application of the subject-matter of this section, but with a cautionary reference to the text accompanying (8.20).

10 CONFIDENCE REGIONS FOR MEANS OF $(\underline{x}^r)$ AND ESTIMATORS $(\underline{X}^R)$

Influence of H_{a_p} on estimators

Denote, as was done in (2.2.15) in [3], the estimators $(\underline{Y}^\alpha)$, $(\underline{X}^i)$ and $(\underline{X}^r)$ together by $(\underline{X}^R)$ as functions of $(\underline{y}_\beta)$. Then we consider:

$$\left.\begin{aligned}
&(\underline{x}^i)\ ,\ (E\{\underline{x}^i \mid H_a\}) = (\tilde{x}^i) + (\widetilde{\nabla} x^i)\\
&(\underline{y}^\varrho)\ ,\ (E\{\underline{y}^\varrho \mid H_a\}) = (0) + (\widetilde{\nabla} y^\varrho)\\
&(\underline{y}_\beta)\ ,\ (E\{\underline{y}_\beta \mid H_a\}) = (\tilde{y}_\beta) + (\widetilde{\nabla} y_\beta)\\
&(\underline{X}^R)\ ,\ (E\{\underline{X}^R \mid H_a\}) = (\tilde{x}^R) + (\widetilde{\nabla} X^R)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (10.1')$$

With the alternative hypothesis (2.9), and dropping for simplicity the subscript p, see, i.a. (3.12):

$$\left.\begin{aligned}
\left(\frac{\widetilde{\nabla} x^i}{\sigma}\right) &= (c^i)\cdot\nabla\\
\left(\frac{\widetilde{\nabla} y^\varrho}{\sigma}\right) &= (u_i^\varrho)(c^i)\cdot\nabla\\
\left(\frac{\widetilde{\nabla} y_\beta}{\sigma}\right) &= (a_\beta^j)^*(\bar{g}_{ji})(c^i)\cdot\nabla\\
\left(\frac{\widetilde{\nabla} X^R}{\sigma}\right) &= (G^{Rj})(\bar{g}_{ji})(c^i)\cdot\nabla
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10.1'')$$

Bounds for ∇_p can be computed, or possibly approximated, by the method developed in previous sections. So with (3.6) and further with (10.1) one obtains corresponding bounds for the influence of $|\nabla_{p,0}|$:

$$\boxed{\begin{aligned}
&|\nabla_{p,0}| = \sqrt{\frac{\lambda_0}{N_p}}\ ,\ \text{or approximated}\\
&\left(\frac{\widetilde{\nabla_{p,0} x^i}}{\sigma}\right) = (c_p^i)\cdot|\nabla_{p,0}|\\
&\left(\frac{\widetilde{\nabla_{p,0} y^\varrho}}{\sigma}\right) = (u_i^\varrho)(c_p^i)\cdot|\nabla_{p,0}|\\
&\left(\frac{\widetilde{\nabla_{p,0} y_\beta}}{\sigma}\right) = (a_\beta^j)^*(\bar{g}_{ji})(c_p^i)\cdot|\nabla_{p,0}|\\
&\left(\frac{\widetilde{\nabla_{p,0} X^R}}{\sigma}\right) = (G^{Rj})(\bar{g}_{ji})(c_p^i)\cdot|\nabla_{p,0}|
\end{aligned}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10.2)$$

For the present, no comment is given to these formulas, because another consequence of (2.9) must be investigated first.

For the following, refer also to the text about (2.5) and figure 2-1.

Consider the quadratic forms:

$$\left.\begin{aligned}
\frac{1}{\sigma^2}\underline{E} &= \left(\frac{\underline{y}^{\tau}}{\sigma}\right)^{*}(\bar{g}_{\tau\varrho})\left(\frac{\underline{y}^{\varrho}}{\sigma}\right)\\
\frac{1}{\sigma^2}\underline{E}' &= \left(\frac{\underline{y}_{\alpha}-\tilde{y}_{\alpha}}{\sigma}\right)^{*}(\bar{g}^{\alpha\beta})\left(\frac{\underline{y}_{\beta}-\tilde{y}_{\beta}}{\sigma}\right)\\
\frac{1}{\sigma^2}\underline{E}'' &= \left(\frac{\underline{x}^{j}-\tilde{x}^{j}}{\sigma}\right)^{*}(\bar{g}_{ji})\left(\frac{\underline{x}^{i}-\tilde{x}^{i}}{\sigma}\right)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (10.3')$$

with the influence of (2.9):

$$\left.\begin{aligned}
\lambda &= \left(\frac{\widetilde{\nabla}y^{\tau}}{\sigma}\right)^{*}(\bar{g}_{\tau\varrho})\left(\frac{\widetilde{\nabla}y^{\varrho}}{\sigma}\right)\\
\lambda' &= \left(\frac{\widetilde{\nabla}y_{\alpha}}{\sigma}\right)^{*}(\bar{g}^{\alpha\beta})\left(\frac{\widetilde{\nabla}y_{\beta}}{\sigma}\right)\\
\lambda'' &= \left(\frac{\widetilde{\nabla}x^{j}}{\sigma}\right)^{*}(\bar{g}_{ji})\left(\frac{\widetilde{\nabla}x^{i}}{\sigma}\right)
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (10.3'')$$

The transformation (2.2) causes an orthogonalization of $(\underline{y}^{\varrho})$ with respect to $(\underline{y}_{\beta})$. Working out (10.3') gives:

$$\begin{array}{ccccc}
\frac{1}{\sigma^2}\underline{E} & + & \frac{1}{\sigma^2}\underline{E}' & = & \frac{1}{\sigma^2}\underline{E}'' \\
\{\text{rank } b\} & & \{\text{rank } m-b\} & & \{\text{rank } m\}
\end{array} \quad \ldots\ldots\ldots\ldots \quad (10.4')$$

Working out (10.3'') gives with (10.1) after some derivations indicated earlier:

$\lambda = N \cdot \nabla^2$	$N = (c^{j})^{*}(\bar{g}_{ji'})(g^{i'j'}-G^{i'j'})(\bar{g}_{j'i})(c^{i})$
$\lambda' = N' \cdot \nabla^2$	$N' = (c^{j})^{*}(\bar{g}_{ji'})(G^{i'j'})(\bar{g}_{j'i})(c^{i})$
$\lambda'' = N'' \cdot \nabla^2$	$N'' = (c^{j})^{*}(\bar{g}_{ji})(c^{i})$
$\lambda+\lambda' = \lambda''$	$N+N' = N''$

$\ldots\ldots\ldots$ (10.4'')

A fictitious adjustment in steps

The comparison of (10.4) with corresponding formulas in section 7 suggests an adjustment in steps, compare an analogous procedure in section 5:

$$\begin{array}{llll} & & \text{condition} & \text{model} \\ 1^\circ & \text{step, see (2.1):} & (E\{\underline{y}^e\}) = (\tilde{y}^e) & = (0) \\ 2^\circ & \text{step, see (10.1):} & (E\{\underline{Y}_\beta - \tilde{y}_\beta\}) = (\tilde{y}_\beta - \tilde{y}_\beta) & \equiv (0) \\ 1^\circ+2^\circ & \text{step, see (10.1):} & (E\{\underline{x}^i - \tilde{x}^i\}) = (\tilde{x}^i - \tilde{x}^i) & \equiv (0) \end{array} \quad \Big\} \quad \ldots\ldots \quad (10.5)$$

From (10.5) it is clear that only the first step is real; this is in fact the actual adjustment. But for the sake of the theoretical reasoning, the formula-system is further developed.

This is not entirely unrealistic. For, every later extension or densification of the geodetic network means in fact the addition of a next step. The fictitious adjustment on (10.5) thus closely follows this practical reasoning.

Now apply (8.1)–(8.3), from which the same result $|\nabla_{p,0}|$ must be obtained as from the actual adjustment plus testing:

In fictitious adjustment for all H_{a_p}		
Initial choice α_0, β_0		$\lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$
1st step	α from	$\beta_0 = \beta\{\alpha, \lambda_0, b, \infty\}$
actual adjustment	$B^{(e)}$	$\frac{1}{\sigma^2}\underline{E} < b \cdot F_{1-\alpha;\, b, \infty}$ (10.6)
	$\lvert\nabla_{p,0}\rvert$	$\sqrt{\frac{\lambda_0}{N_p}}$; see (10.3), (10.4)
2nd step	α' from	$\beta_0 = \beta\{\alpha', \lambda_0, m-b, \infty\}$
fictitious	$B_{(\beta)}$	$\frac{1}{\sigma^2}\underline{E}' < (m-b) \cdot F_{1-\alpha';\, m-b, \infty}$ (10.7)
	$\lvert\nabla'_{p,0}\rvert$	$\sqrt{\frac{\lambda_0}{N'_p}}$; see (10.3), (10.4)
1st+2nd steps	α'' from	$\beta_0 = \beta\{\alpha'', \lambda_0, m, \infty\}$
fictitious	$B^{(i)}$	$\frac{1}{\sigma^2}\underline{E}'' < m \cdot F_{1-\alpha'';\, m, \infty}$ (10.8)
	$\lvert\nabla''_{p,0}\rvert$	$\sqrt{\frac{\lambda_0}{N''_p}}$; see (10.3), 10.4)

(8.7) saves the situation, for a requirement is:

$$\left.\begin{array}{l}\text{either (10.6)}\\ \text{or (10.7)}\\ \text{or (10.8)}\end{array}\right\}\ \text{hence (10.6), because } E' \text{ and } E'' \text{ cannot be computed} \qquad (10.9)$$

Nevertheless we continue in theory, first with (10.7), and apply the reasoning of (9.1)–(9.2):

For *each group* of algebraically independent functions of $(\underset{\beta}{y})$:

$$\left.\begin{array}{l}(\underline{X}^R) \qquad \{\text{range } R,S = \bar{n} \leqq m-b\}\\ \{\text{covariance matrix of } (\underline{X}^R)\} = \sigma^2 \cdot (G^{RS}) = \sigma^2 (\bar{G}_{SR})^{-1}\end{array}\right\} \quad \ldots\ldots \quad (10.10')$$

we have:

Initial choice α_0, β_0	$\lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$	$\bar{n} \leqq m-b$
$\bar{\alpha}$ from $B^{(X^R)}$	$\beta_0 = \beta\{\bar{\alpha}, \lambda_0, \bar{n}, \infty\}$ $\left(\frac{\underline{X}^S - \tilde{x}^S}{\sigma}\right)^* (\bar{G}_{SR}) \left(\frac{\underline{X}^R - \tilde{x}^R}{\sigma}\right) < \bar{n} \cdot F_{1-\bar{\alpha};\,\bar{n},\,\infty}$	

(10.10'')

Similarly (10.8) can be followed:

For *each group* of algebraically independent functions of $(\underset{i}{x})$:

$$\left.\begin{array}{l}(\underline{x}^r) \qquad \{\text{range } r,s = \bar{m} \leqq m\}\\ \{\text{covariance matrix of } (\underline{x}^r)\} = \sigma^2 \cdot (g^{rs}) = \sigma^2 \cdot (\bar{g}_{sr})^{-1}\end{array}\right\} \quad \ldots\ldots \quad (10.11')$$

we have:

Initial choice α_0, β_0	$\lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$	$\bar{m} \leqq m$
$\bar{\bar{\alpha}}$ from $B^{(x^r)}$	$\beta_0 = \beta\{\bar{\bar{\alpha}}, \lambda_0, \bar{m}, \infty\}$ $\left(\frac{\underline{x}^s - \tilde{x}^s}{\sigma}\right)^* (\bar{g}_{sr}) \left(\frac{\underline{x}^r - \tilde{x}^r}{\sigma}\right) < \bar{m} \cdot F_{1-\bar{\bar{\alpha}};\,\bar{m},\,\infty}$	

(10.11'')

The acceptance regions $B^{(X^R)}$ and $B^{(x^r)}$ are not determined because $(\tilde{y}_\beta)$ and $(\tilde{x}^i)$ and consequently $(\tilde{x}^R)$ and $(\tilde{x}^r)$ are unknown.

In a representation on an arbitrary (X^R) – respectively (x^r) coordinate system, the limits of the two acceptance regions are represented by {hyper-}ellipsoids. In running coordinates (X_c^R) and (x_c^r), the equations concerned are:

$$\left(\frac{X_c^S - \tilde{x}^S}{\sigma}\right)^* (\bar{G}_{SR}) \left(\frac{X_c^R - \tilde{x}^R}{\sigma}\right) = \bar{n} \cdot F_{1-\bar{\alpha};\,\bar{n},\,\infty} \quad \ldots\ldots\ldots\ldots \quad (10.12)$$

$$\left(\frac{x_c^s - \tilde{x}^s}{\sigma}\right)^* (\bar{g}_{sr}) \left(\frac{x_c^r - \tilde{x}^r}{\sigma}\right) = \bar{m} \cdot F_{1-\bar{\bar{\alpha}};\,\bar{m},\,\infty} \quad \ldots\ldots\ldots\ldots \quad (10.13)$$

Confidence regions

If the ellipsoids (10.12) and (10.13) are translated by making their centres shift from $(\tilde{x}^R)$ and $(\tilde{x}^r)$ to the representations of the derived samples (X^R) and (x^r), then one obtains the:

$$\left.\begin{array}{l} 100\cdot(1-\bar{\alpha})^0/_0 \text{ confidence region for } (\tilde{x}^R): \\ \left(\dfrac{X_c^S-X^S}{\sigma}\right)^*(\bar{G}_{SR})\left(\dfrac{X_c^R-X^R}{\sigma}\right) = \bar{n}\cdot F_{1-\bar{\alpha};\,\bar{n},\,\infty} \end{array}\right\} \quad \ldots\ldots\ldots \quad (10.14)$$

respectively:

$$\left.\begin{array}{l} 100\cdot(1-\bar{\alpha})^0/_0 \text{ confidence region for } (\tilde{x}^r): \\ \left(\dfrac{x_c^s-x^s}{\sigma}\right)^*(\bar{g}_{sr})\left(\dfrac{x_c^r-x^r}{\sigma}\right) = \bar{m}\cdot F_{1-\bar{\alpha};\,\bar{m},\,\infty} \end{array}\right\} \quad \ldots\ldots\ldots \quad (10.15)$$

To eliminate the representation, (10.14) is usually written as:

$$\boxed{\begin{array}{l} 100\cdot(1-\bar{\alpha})^0/_0 \text{ confidence region for } (\tilde{x}^R) \\ \hline \left(\dfrac{\tilde{x}^S-X^S}{\sigma}\right)^*(\bar{G}_{SR})\left(\dfrac{\tilde{x}^R-X^R}{\sigma}\right) < \bar{n}\cdot F_{1-\bar{\alpha};\,\bar{n},\,\infty} \end{array}} \quad \ldots\ldots\ldots \quad (10.16)$$

and (10.15) as:

$$\boxed{\begin{array}{l} 100\cdot(1-\bar{\alpha})^0/_0 \text{ confidence region for } (\tilde{x}^r) \\ \hline \left(\dfrac{\tilde{x}^s-x^s}{\sigma}\right)^*(\bar{g}_{sr})\left(\dfrac{\tilde{x}^r-x^r}{\sigma}\right) < \bar{m}\cdot F_{1-\bar{\alpha};\,\bar{m},\,\infty} \end{array}} \quad \ldots\ldots\ldots \quad (10.17)$$

In view of (10.10'') and (10.11''), one obtains for the one-dimensional case the intervals:

$$\boxed{\begin{array}{l} 100\cdot(1-\alpha_0)^0/_0 \text{ confidence interval for } \tilde{x}^R \\ \hline X^R-\sigma_{X^R}\cdot F^{\frac{1}{2}}_{1-\alpha_0;\,1,\,\infty} < \tilde{x}^R < X^R+\sigma_{X^R}\cdot F^{\frac{1}{2}}_{1-\alpha_0;\,1,\,\infty} \end{array}} \quad \ldots\ldots\ldots \quad (10.18)$$

$$\boxed{\begin{array}{l} 100\cdot(1-\alpha_0)^0/_0 \text{ confidence interval for } \tilde{x}^r \\ \hline x^r-\sigma_{x^r}\cdot F^{\frac{1}{2}}_{1-\alpha_0;\,1,\,\infty} < \tilde{x}^r < x^r+\sigma_{x^r}\cdot F^{\frac{1}{2}}_{1-\alpha_0;\,1,\,\infty} \end{array}} \quad \ldots\ldots\ldots \quad (10.19)$$

Compare the text referring to (5.7), (5.8).

This exposition has been held on a very elementary level. For the theoretical foundation of the concept "confidence region", reference is made to statistical literature. Some further considerations have been given in [9]. It should be realized that confidence regions are constructed on the basis of the null hypothesis H_0.

Discussion

This reasoning, which also for confidence intervals connects the α to the initial choice $\{\alpha_0, \beta_0\}$, and depending on the rank of the quadratic form, causes the problem (8.9c) to come into the picture again.

The reasoning of section 8 can here be interpreted as follows:

> There is little practical use in taking a quadratic form whose rank is great, i.e. in considering a large number of means of X^R- and x^r-variates (10.20)

From a practical point of view this seems to be acceptable too. From [3] it is apparent how many types of model errors may distort a geodetic network after adjustment, be it that the methods of section 8 {and, perhaps, section 9} may counteract a serious local distortion.

Because the use of a geodetic network in society mostly bears a rather local character, the social purpose of the coordinate system of net points – always somewhat distorted, see (10.2) – will as a consequence not be endangered.

It is really very curious that the reasoning of this section leads to the construction of confidence regions {e.g. for means of computed coordinates of net points} on the basis of the same initial choice $\{\alpha_0, \beta_0\}$ that was used for the testing in the adjustment {and possible data-snooping} of the net itself.

However, this has far-reaching consequences. Extension or densification of the network means keeping fixed $\{\alpha_0, \beta_0\}$ in the different steps. Two networks, initially independent, are joined in a next step, hence $\{\alpha_0, \beta_0\}$ for both networks plus for the connecting step. And because all networks on earth will be connected within a certain space of time, one arrives at the extension of (3.16):

> Theoretically, one should use in all tests of geodetic measurements the same initial choice $\{\alpha_0, \beta_0\}$. (10.21)

In practice this seems now to be difficult to realize. Perhaps it is not necessary either.

One could, as is done in practice, divide geodetic networks into systems of higher and lower order, in such a way that the effects of undetected model errors {see (10.2)} in the network of higher order are of no practical significance for networks of lower order.*)

*) To give a rough idea: 1. distances between points 1000 km, 2. distances 20–50 km, 3. distances 1–10 km.

Coordinates once accepted and fixed will then be left unchanged, they are introduced as *errorless* quantities into later computations of new networks. It is questionable how far this is admissible for networks of the same order, however this is a point of great practical significance.

An advantage of this is, that when testing measurements in new networks, the effects of (10.2) on the coordinates – considered errorless – of earlier computed points, need not to be checked. It is sufficient to check for disturbance of the signals. Compare also [3] pages

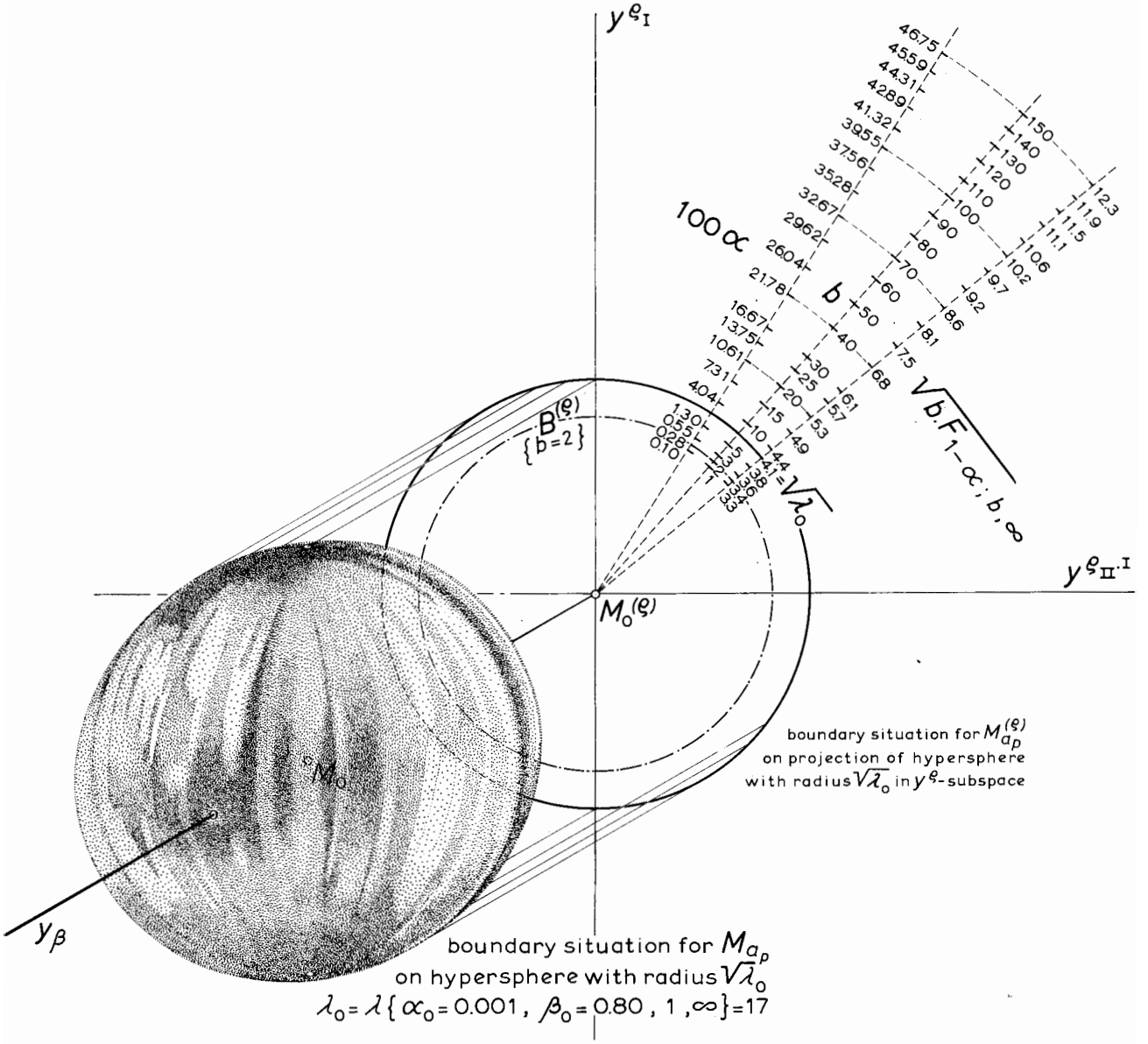

Fig. 10-1

30–32. But then a requirement is that the results (10.2) for these "errorless" coordinates are sufficiently small {how small?}.

This introduction of coordinates as being errorless once they have been determined, finds support from an entirely different side. For practical reasons there had already been developed a theory probabilistically describing local distortions of geodetic networks. In this theory, there proved to be no room for corrections to given coordinates caused by the measurement of later densification networks. Provisionally, refer to [3], page 36. This description of local parts of a net is consequently connected to the statements mentioned after (10.20). This again gives the author more confidence in a possible practical applicability of the designed method of testing.

Addendum

The method developed is characterized by the fixing of a constant bound λ_0 for λ, as a function of chosen parameters α_0 and β_0.

The points M_{a_p} are the mid-points of the probability distributions under the alternative hypotheses H_{a_p}, see figure 2-1. After this section it will be clear that the fixing of λ_0 implies a boundary situation for these points which is a hypersphere with radius $\sqrt{\lambda_0}$, centred on the mid-point M_0 of the probability distribution under the null hypothesis H_0.

In figure 10-1, an example is pictured, using:

$$\alpha_0 = 0.001, \quad \beta_0 = 0.80, \quad \text{hence } \lambda_0 = 17$$

This figure can be compared with figures 2-1 and 7-1. Besides the hypersphere with centre M_0 and radius $\sqrt{\lambda_0}$, the projection on the $y^{(q)}$-subspace has been drawn. This projection, which is again a hypersphere with radius $\sqrt{\lambda_0}$, now centred in $M_0^{(q)}$, represents the boundary situation of the points $M_{a_p}^{(q)}$ as projections of the points M_{a_p}.

For comparison, the acceptance region $B^{(q)}$ for $b = 2$ has been drawn, compare figure 3-4. For $b > 2$ the corresponding $B^{(q)}$ are hyperspheres with radius $\sqrt{b \cdot F_{1-\alpha;b,\infty}}$, in which α is determined by (3.11). In figure 10-1 there has been drawn a triple scale to show the increase of α and of the radius $\sqrt{b \cdot F_{1-\alpha;b,\infty}}$ with b.

11 RELIABILITY OF GEODETIC NETWORKS

A definition of the reliability of geodetic networks will in any case have to pertain to variates, either observation variates or estimators such as coordinate-variates obtained from an adjustment procedure. To avoid possible confusion between the definitions of reliability pertaining to these two groups, we shall use the terms "internal reliability" and „external reliability" respectively. There is a close connection between the two; the terminology must be seen as a provisional one.

In section 10 it has been indicated how statements about the precision of the two groups of variates can be made, using confidence regions or intervals based on the null hypothesis H_0. Possible requirements regarding the precision fall within the scope of problem (1.1), which at present cannot be analysed in a satisfactory way.

Apart from these statements on precision, there is the influence of model errors of an order of magnitude that can just be detected with probability β_0 by the test procedure. Bounds for the values of these errors were given in (10.2). The definition of reliability will be based on these bounds.

Internal reliability of a network

As long as the problem (1.1) cannot be sharper analysed, one will have to start from the fact that the shape of networks, especially high-order networks, will now and perhaps always be determined by the limited availability of time, money or material means.

Starting with a net-figure, planned according to these possibilities, plus a designed measuring programme one can design the procedure of adjustment and testing, from which follows the possibility to compute (10.2). The question arises what requirements have to be made to the results of (10.2). To begin with, consider only the $(\nabla_{p,0}\tilde{x}^i)$.

In (10.2) occurs as a partial scale parameter the value:

$$|\nabla_{p,0}| \quad \text{per } H_{a_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11.1)$$

Now the most essential point in the method designed is that in one-dimensional tests always the "Type I error probability" – see [3] page 28 –:

$$\alpha_0 = \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11.2)$$

whereas always bounds for the influences of just detectable model errors are computed on the basis of the bound for the power for each alternative hypothesis H_{a_p}:

$$\beta_{p,0} = \text{constant} = \beta_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11.3)$$

For b-dimensional tests we have the "Type I error probability":

$$\left.\begin{array}{ll} \alpha \text{ from:} & \beta_0 = \beta\{\alpha, \lambda_0, b, \infty\} \\ \text{with:} & \lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.4)$$

Hence, this implies a constant value for λ_0. From (3.6) or (10.2) follows then:

$$|\nabla_{p,0}| = \sqrt{\frac{\lambda_0}{N_p}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11.5)$$

so that in fact the task of the partial scale parameter (11.1) is taken over by, see (2.10):

$$N_p = (c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})(c_p^i) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.6')$$

From (10.2) it follows that a more general parameter σ occurs, which is determined by the planning of the precision. If so desired this parameter can be united with N_p to give:

$$\frac{1}{\sigma^2} N_p = \frac{1}{\sigma^2}(c_p^j)^*(\bar{g}_{ji'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i})(c_p^i) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.6'')$$

The quantities (11.6) depend on the precision of the observation variates and the adjusted observation variates, or on the precision of the correction variates ($\underset{\sim}{\varepsilon}^i$):

$$\overline{(\underset{\sim}{\varepsilon}^i), (\underset{\sim}{\varepsilon}^j)^*} = (g^{ij}) - (G^{ij}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11.7)$$

and consequently they depend on the shape and the measuring procedure of the network. By adding more or fewer measured elements to the network, keeping the same measuring procedure, or by modification of the measuring procedure, the results (11.6) can be made smaller or greater.

However, in view of the remarks on (8.16)–(8.17), this can only be done within certain limits. Also, it must be taken into account that addition of more measured elements also possibly introduces new errors and hence new alternative hypotheses.

Now one may ask what is the use of observations if there is no possibility to check the possible occurrence of alternative hypotheses, or to keep their influence within certain limits. An observation will have to have a function in the network, for it costs money and that money must be justified.

This means that in the first instance it is important to consider, see (10.2):

$$\left(\frac{\nabla_{p,0}\underset{\sim}{x}^i}{\sigma}\right) \text{ or } (\nabla_{p,0}\underset{\sim}{x}^i) \text{ per } H_{a_p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11.8)$$

As stated already in [3] and in section 2, the formulation of the more general form (2.9) for an alternative hypothesis is difficult if not impossible. Therefore it seems to be best to accept the CONVENTION that the study is limited to the form (2.15):

$$p \to i: \quad \frac{\nabla_0^{\sim} x^i}{\sigma} \text{ or } \nabla_0^{\sim} x^i \text{ per } H_{a_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.9)$$

with for (11.6), respectively:

$$N_i = (\bar{g}_{ii'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.10')$$

$$\frac{1}{\sigma^2} N_i = \frac{1}{\sigma^2}(\bar{g}_{ii'})(g^{i'j'} - G^{i'j'})(\bar{g}_{j'i}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.10'')$$

Now we may consider "equivalent *) observation variates", by which we mean direction variates – in the sense of (4.2.6) in [3] – between approximately equidistant points, or distance variates – in the sense of (4.2.14) in [3] – also between approximately equidistant points. In the case of such equivalent observation variates one might require:

$$\text{for equivalent } (\underset{\sim}{x}^i): \quad |\nabla_0^{\sim} x^i| \approx \text{constant} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.11)$$

Then equivalent observations are "equally well checked", a first requirement for internal reliability of the network. This implies that a practically usable measure for judging the "internal reliability of a network" might be the ratio of the {absolute} values (11.9).

For (11.9) we have, with (2.15), see (11.5) and with N_i from (11.10):

$$\left|\frac{\nabla_o^{\sim} x^i}{\sigma}\right| = |\nabla_{i,o}| = \sqrt{\frac{\lambda_0}{N_i}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.12')$$

$$|\nabla_0^{\sim} x^i| = \sqrt{\frac{\lambda_0}{\frac{1}{\sigma^2} N_i}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11.12'')$$

Combining (11.12) with (11.11) we obtain:

	Measure for *internal reliability* of network
(11.12)	ratios of $\lvert\nabla_{i,o}\rvert$ or of $\lvert\nabla_0^{\sim} x^i\rvert$, $\{i = 1, \ldots, m\}$

$\ldots\ldots$ (11.13)

	Relative requirement for *internal reliability*
(11.10)	for equivalent $(\underset{\sim}{x}^i)$: $N_i \approx$ constant

$\ldots\ldots$ (11.14)

*) The word equivalent is here used in its general meaning, not in the special meaning of the "equivalent observations" of classical adjustment theory.

(11.10) can always be computed, so that the relative requirement (11.14) can be applied in practice.

To obtain an impression of the order of magnitude, the effect of the formulas (11.10) with (11.7) and (11.12) can be examined for the case that the covariance matrix of $(\underline{x}^i)$ is a diagonal matrix. Analogous formulas have earlier been used in [16].

$\sigma^2 \cdot (g^{ij})$ diagonal matrix	
$\sqrt{\frac{1}{\sigma^2} N_i} = \sqrt{\frac{1}{\sigma^2} \cdot \frac{1}{g^{ii}} \left\{ 1 - \frac{G^{ii}}{g^{ii}} \right\}} = \frac{1}{\sigma_{x^i}} \sqrt{1 - \frac{\{\sigma_{X^i}\}^2}{\{\sigma_{x^i}\}^2}} = \frac{1}{\sigma_{x^i}} \cdot \frac{\sigma_{\varepsilon^i}}{\sigma_{x^i}} \leqq \frac{1}{\sigma_{x^i}}$	(11.15′)
$\lvert \nabla_0 \tilde{x}^i \rvert = \sqrt{\frac{\lambda_0}{\frac{1}{\sigma^2} N_i}} \geqq \sigma_{x^i} \cdot \sqrt{\lambda_0}$	(11.15″)

If the covariance matrix of $(\underline{x}^i)$ is not a diagonal matrix, one may make an appraisal by using the latent roots of $\sigma^2 \cdot (g^{ij})$. If needed, such appraisals can be used in studying the problem of choosing the λ_0-level.

An entirely different question is, which observation variates may be considered as "equivalent" ones. In the case (4.2.26″) in [3], direction variates and distance variates are even equivalent to each other, but in practice this situation will not often occur. With electronic distance measuring instruments one can often put in (4.2.15) in [3]:

$$\lambda_i^a = \text{constant} = \lambda^a \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (11.16)$$

and in this case the "equivalence" between direction- and distance variates can only be established after considering equivalent influences on coordinate variates, so that a connection with the problem of the external reliability is established.

The λ_0-level

In the formulation (11.13), a statement about the values (11.12) themselves is avoided. If one wishes to make such a statement, then the value of λ_0 must be introduced. From the formulas (11.12), with similarly modified formulas (10.2), it appears that λ_0 clearly bears the character of a general parameter in these computations of bounds, just like σ^2 in the case of computations concerning the precision. One might even speak about a λ_0-*level.* According to (11.4), this level is entirely determined by the initial choice $\{\alpha_0, \beta_0\}$. A statement on the value λ_0 implies, consequently, a statement on the values $\{\alpha_0, \beta_0\}$. Also in view of what was said in connection with (10.21), it seems for the present difficult to give practically acceptable directives for this. This has been one of the reasons for the relativity in the character of (11.13) and (11.14).

Nevertheless, the possibility to influence the λ_0-level via (11.12) cannot be excluded; the key to this will be experience concerning the {gross} errors that are possibly made in measurement and computation. Consider the case of a distance measuring instrument about which

it is known that an error of 25 cm in the observation x^i occurs rather frequently because of a combination of mistakes in the execution of the measuring procedure. Then it is of great practical importance that the value $|\nabla_0\tilde{x}^i|$ is smaller than 25 cm, so that there is a reasonably high probability β that such a "gross" error will be signalized by the testing procedure. Apart from influencing N_i by modification of the net figure, this may also be attained by a sufficiently low value λ_0, provided that one is not obliged to use a particular λ_0-level because of considerations in a wider context.

This means, consequently, that the values $|\nabla_0\tilde{x}^i|$ are viewed in connection with the gross errors expected in x^i, which in "equivalent observations" always have the same order of magnitude. This reasoning has led to the start of an attempt at registration and classification of instrumental and human gross errors. In this field there appears to have been done little research – especially honest research – because usually research is directed towards elimination of these errors. It is questionable if elimination is always possible, for in the course of the attempt mentioned, more errors were discovered than were ever thought possible. And these were even only errors made *in spite of* the greatest possible care in the execution of the measuring procedure.

Where this line of thought will lead to can only be guessed. The greater part of the figures in the examples of section 12 is still waiting for an interpretation in the sense of the considerations given here.

External reliability of a network

For the users of a geodetic network, great importance must be attached to the distortion of computed coordinates by model errors which have not been detected by the testing procedure. In general, this means the influence of these errors on estimators $(\underline{X}^R)$. A bound for this influence is given in (10.2), to which we must again remark that in practice the formulation of the alternative hypothesis in (2.9) will have to be replaced by that in (2.15). This is an emergency solution, but it is necessary for the formulation of a measure for the *external reliability* of the network.

With (10.2) and (11.2) one can put, respectively:

Measure for *external reliability* of network in testing procedure with initial choice $\{\alpha_0, \beta_0\}$ hence λ_0-level: $\lambda_0 = \lambda\{\alpha_0, \beta_0, 1, \infty\}$	
$\left(\frac{\nabla_{i,0}\tilde{X}^R}{\sigma}\right) = (G^{Rj})(\bar{g}_{ji})\cdot\sqrt{\frac{\lambda_0}{N_i}},\ \{i = 1, \ldots, m\}$	. . . . . (11.17′)
$(\nabla_{i,0}\tilde{X}^R) = (G^{Rj})(\bar{g}_{ji})\cdot\sqrt{\frac{\lambda_0}{\frac{1}{\sigma^2}N_i}},\ \{i = 1, \ldots, m\}$	. . . . . (11.17″)

The question which requirements must be fulfilled by (11.17) cannot be answered yet. The problem (1.1) concerning statements on the precision and consequently, i.a., on (G^{Rj}), must be solved first. Then we still have the problem to translate the requirements made to the coordinates of a network. The problem to be solved might be formulated as follows:

> What is in society the purpose of geodetic networks of high order, and how should the requirements following from this purpose be quantified? (11.18)

Discussion

One should be conscious of the significance of a decomposition into "internal" and "external" reliability. It implies that, by the requirement (11.14), certain restrictions are imposed on the influencing of N_i in the analysis of (11.17).

If one geodetic network has to serve many different purposes, this restriction seems to be acceptable. In many cases (11.17) must then be seen as a result of (11.13) with (11.14). A good example is a triangulation network in which only directions have been measured and whose sides have approximately the same length; (11.17) then shows the consequences of (11.14).

The matter is entirely different if there is only one clearly defined purpose. An example is a base extension network, where the length of an invar base is transferred to the distance between two points of a fundamental geodetic network. One can try to establish the figure of this net and the measuring procedure in such a way that (11.17) results in the smallest possible values.

If one uses the well-known method of SCHREIBER, possibly modernized, for the construction of base extension networks, this means that the relative requirement (11.14) is left out of consideration. Maintaining (11.14) would here definitely mean a restriction. We shall not further discuss the question if just this restriction would not be desirable with a view to the checking of the observations.

The connection with the procedure customary in geodesy

Up to now, geodetic networks were almost exclusively judged according to requirements on the precision of direct or derived observation variates. There is room for difference of opinion on the way this is done in practice, see the formulation of problem (1.1). But it is clear that the procedure is definitely aimed at a purpose of geodetic measurement and computation.

This implies that the matrix of weight coefficients of estimators $(\underline{X}^i)$, $(\underline{Y}^\alpha)$ and $(\underline{X}^r)$ {united by the notation $(\underline{X}^R)$, as indicated in (2.2.15) in [3]} is determined by requirements on the precision. Consequently this is valid for the matrix:

$$\sigma^2 \cdot \overline{\begin{pmatrix} x^i \\ X^R \end{pmatrix}, \begin{pmatrix} x^j \\ X^S \end{pmatrix}}^{*} = \sigma^2 \cdot \begin{pmatrix} g^{ij} & g^{iS} \\ g^{Rj} & G^{RS} \end{pmatrix} \quad \ldots\ldots\ldots\ldots \quad (11.19)$$

and hence for:

$$(G^{Rj})(\bar{g}_{ji}) \quad \ldots \ldots \quad (11.20)$$

The introduction of requirements concerning the internal reliability of a network can at most be a slight increase of the burden already imposed by the precision requirements, but if the precision requirements are soundly formulated it can never be a lightening. The introduction of (11.14) will therefore, as a second factor, influence (11.19) and to some extent (11.20). According to (11.10), N_i is thereby fixed.

If now (11.17) is considered, the "measure for external reliability of a network" is entirely determined by:

a. requirements concerning precision
d. requirements concerning internal reliability
c. the λ_0-level, hence the choice of $\{\alpha_0, \beta_0\}$, if possible also corresponding to the order of magnitude of gross errors indicated by experience

(11.21)

Consequently, if for one reason or another one wishes to formulate requirements on (11.17) concerning the external reliability, then one is not free any more in formulating requirements on the items mentioned under (11.21).

Hence, the line of thought developed gives in this context a fair insight into the possibilities and limitations for the formulation of a set of requirements, as an extension of the already customary precision requirements to be aimed at.

Possible connection with decision theory

The choice in favour of a test procedure with constant λ_0, hence for the initial choice $\{\alpha_0, \beta_0\}$ is essentially a decision. The advantage of this decision is that a test procedure, simple in principle, is available for many possible situations in geodetic problems. The Decision Theory approach might then possibly provide arguments for the choice of $\{\alpha_0, \beta_0\}$, because this choice controls, on the one hand, the rejection of observations, and on the other hand, the distortion of computed coordinates by model errors which were not signalized. The underlying problem is essentially one of costs.

In opposition to this, the question may be asked if the methods designed are themselves acceptable from the Decision Theory point of view.

These problems are well worth studying, the more so now points of contact can be found in mathematical statistics. Reference can be made, among others, to the very readable book by R. L. Ackoff [17], which gives an excellent survey of many interconnected theories. A short quotation from pages 292, 293 may illustrate this:

> "Before turning to the Neyman-Pearson framework it is important to observe that the use of Bayes strategies for testing problems is extremely rare. Even enthusiasts for such strategies assert that they are cumbersome to apply. In addi-

> tion, they "involve the giving of explicit loss functions, and still more restrictively, require the existence and knowledge of the *a priori* probability".
>
> . . . . . .
>
> Some knowledge or assumptions about the *a priori* probabilities are always present in formulating a testing problem. We will consider how this knowledge can be used in the discussion that follows, in which we will use the Neyman-Pearson framework, but the problem of testing will be formulated in a way that is equivalent to that just presented."

Remark

The design of a measuring plan, taking account of model errors, may of course be made according to entirely different criteria. This depends mainly on the presumed availability of advance knowledge about alternative hypotheses. An example is the publication [18] of J. BÖHM. This also describes testing procedures aimed at the detection of possibly occurring model errors. The many examples contain a treasure of experimental material. However, it is difficult to make a comparison with the reasoning developed in the foregoing, because BÖHM does not use the power function of his tests. A comparative study of the two lines of thought is no doubt recommendable.

12 EXAMPLES *)

The first three examples treated in this section were made before the theory given here was developed. Some computations have been adapted to the new line of thought. The choice of the λ_0-level, see section 11, was made in accordance with the new theory, by computing λ_0 starting from α and β_0:

$$\lambda_0 = \lambda\{\alpha = 0.05,\ \beta_0 = 0.80,\ b,\ \infty\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12.1)$$

from which follows α_0; this is the procedure of (3.11) reversed. Essentially the same applies to the tables (3.13) and (3.14).

Example IV was added later on.

Example I

This concerns a base extension network, measured in 1965 in the northern part of The Netherlands, see figure 12-1. Here we have a clearly defined purpose of the network: transferring the distance $s_{7,9}$, measured by invar wires, to the required distance $s_{1,3}$ between two points of the national fundamental network.

The analysis of the direction measurements was completed, experiments using distance measurements by geodimeter are being made, but the latter will be left out of consideration. Directions are denoted by the kernel letter r with as subscripts the point numbers of station- and target point.

Data or results under H_0

$$\left.\begin{array}{ll} (\underline{x}^i)\text{:} & \text{direction variates } \underline{r}_{..} \\ \underline{X}^R\text{:} & \ln \underline{s}_{1,3} - \ln \underline{s}_{7,9} \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (12.2')$$

$$\left.\begin{array}{l} \sigma_r \text{ dmgr} = \text{constant} = 1.00;\ \text{no correlation} \\ \sigma_{\{\ln s_{1,3} - \ln s_{7,9}\}} \text{ dmgr} = 0.95 \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (12.2'')$$

$$\left.\begin{array}{l} \text{Number of condition equations} \quad b = 11 \\ (12.1)\text{:} \quad \alpha = 0.05\ ,\ \lambda_0 = 16.8 \\ (3.13)\text{:} \quad \alpha_0 = 0.001,\ F^{\frac{1}{2}}_{1-\alpha_0;\,1,\,\infty} = 3.29 \end{array}\right\} \quad (12.3)$$

---

*) Example I was computed by B. G. K. KRIJGER, II and IV by J. C. P. DE KRUIF and J. VAN MIERLO, III by J. VAN MIERLO.

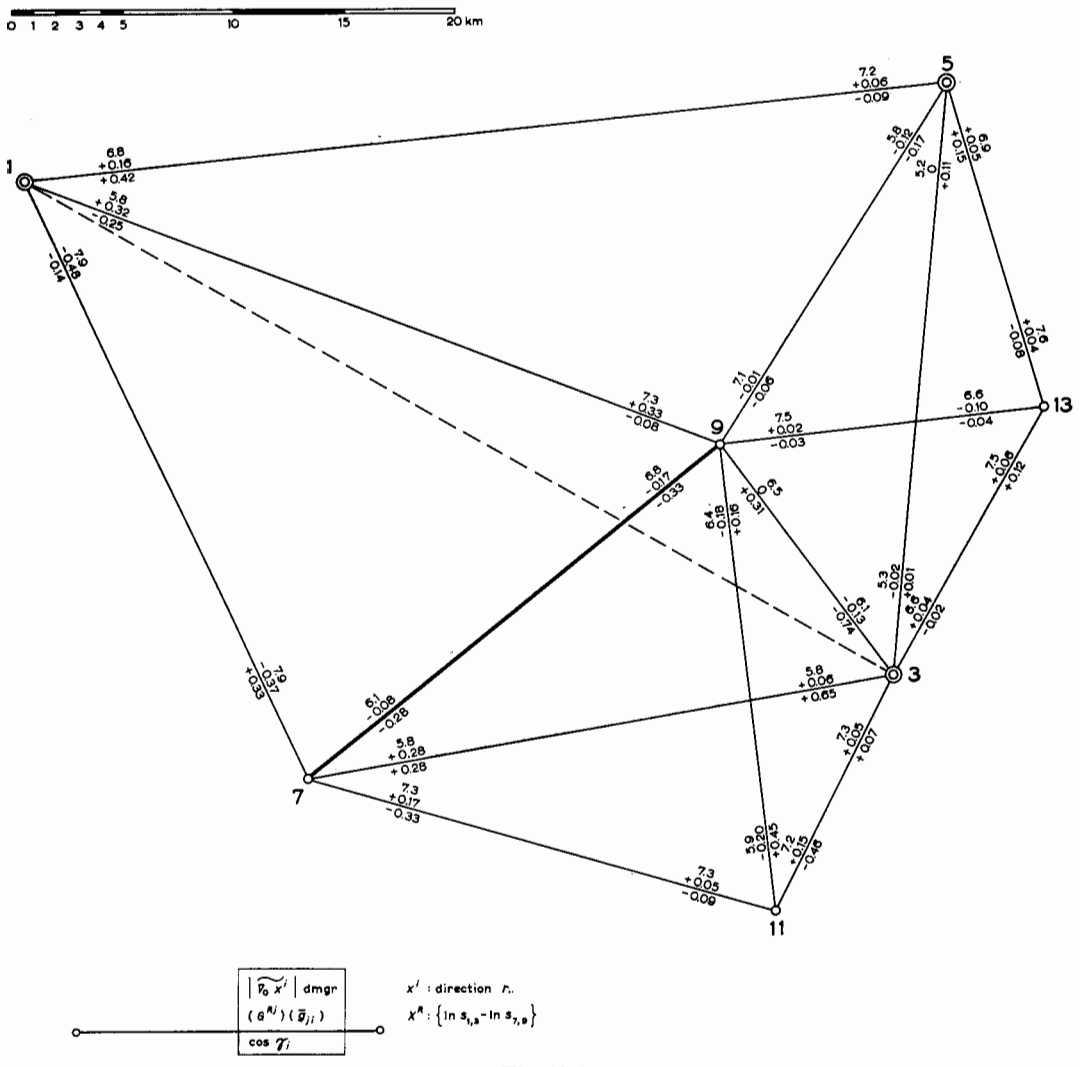

Fig. 12-1

acceptance region (3.9):

$$B^{(\varrho)}: \qquad \frac{\hat{\sigma}^2}{\sigma^2} = \frac{1}{b} \cdot \frac{E}{\sigma^2} < F_{0.95;\,11,\,\infty} = 1.79 \tag{12.4}$$

Bounds under H_{a_i} (2.15), (11.9)

(11.12): $|\widetilde{\nabla_0 x^i}|$ see figure 12.1

$5 < |\widetilde{\nabla_0 x^i}|$ dmgr < 8, hence in view of

(11.14): network complies reasonably with the requirement for internal reliability; compare (12.6)

(12.5)

(11.15): $|\widetilde{\nabla_0 x^i}|$ dmgr $\geqq \sigma_r \cdot \sqrt{\lambda_0} \approx 4$

λ_0-level seems acceptable (12.6)

(11.17): $(G^{Rj})(\bar{g}_{ji})$ see figure 12.1

$\sigma^2 = 1$: $(\widetilde{\nabla_{i,0} X^R}) = (G^{Rj})(\bar{g}_{ji}) \cdot |\widetilde{\nabla_0 x^i}|$

$|\widetilde{\nabla_{i,0} X^R}|$ dmgr < 3.8, cf σ_{X^R} in (12.2)

network complies reasonably with the requirement for external reliability concerning purpose (12.7)

Testing on adjustment

(12.4): $\dfrac{\hat{\sigma}^2}{\sigma^2} = 2.61 > 1.79$, hence

H_0 or (x^i) not accepted (12.8)

Data-snooping

(2.16): $\cos \gamma_i$ see figure 12-1

acceptance region per H_{a_i}, (4.3) with (12.3):

$B_i^{(\varrho)}$: $-3.29 < \underline{w}_i^1 < F^{\frac{1}{2}}_{1-\alpha_0;\,1,\,\infty} = 3.29$ (12.9′)

(2.16): $w_i^1 = \cos\gamma_i \cdot \sqrt{b \dfrac{\hat{\sigma}^2}{\sigma^2}} = 5.36 \cos\gamma_i$

$\cos \gamma_i = \dfrac{w_i^1}{5.36}$, hence instead of $B_i^{(\varrho)}$:

check at least x^i if: $|\cos\gamma_i| > \dfrac{3.29}{5.36} = 0.61$ (12.9″)

From figure 12-1 it appears that directions $r_{3,7}$ and $r_{3,9}$ give values for $\cos \gamma_i$ which lead to further checking.

Check measurements indeed gave indications of a possible influence of refraction in the vicinity of point 3. The localization of errors thus did not prove to be sharp, but the indication was sufficient. The check measurements are being continued.

Example II

A first field reconnaissance of a lower-order network resulted in the configuration of figure 12-2. The purpose was the establishment of ground control for photogrammetric mapping, but a sharp quantification of the requirements following from this purpose could not be given on account of the unsatisfactory solution of problem (1.1). The survey planned would consist of direction- and distance measurements according to (11.16). The latter would consequently be the same measuring procedure as the one on which figure 9-1 is based.

Test computations have been executed concerning the precision and also concerning the {internal} reliability of the network. The condition equations relating to the connection to the known coordinates of higher-order points have in these computations been left out of consideration. The network has: 38 points, 101 directions, 33 distances, 31 condition equations.

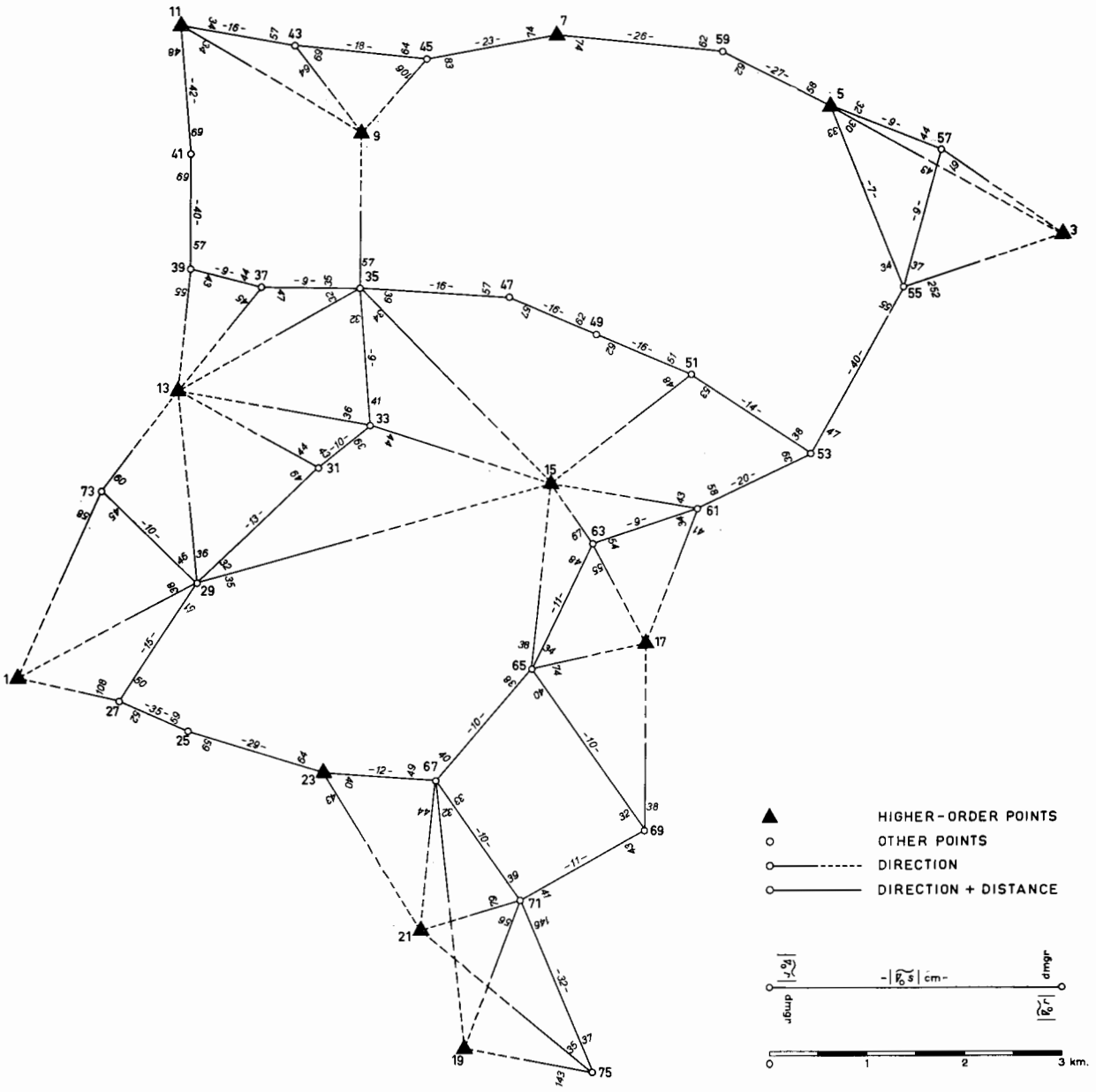

Fig. 12-2

Data under H_0

$(\underline{x}^i)$: direction variates $\underline{r}_{..}$
distance variates $\underline{s}_{..}$ (12.10′)

σ_r dmgr = constant = 4.5
σ_s cm $= \sqrt{s\ \text{km}}$ } no correlation (12.10″)

number of condition equations $b = 31$
(12.1): $\alpha = 0.05$, $\lambda_0 = 24.6$
(3.13): $100\alpha_0 = 0.004$, $F^{\ddagger}_{1-\alpha_0;\,1,\,\infty} = 4.1$ (12.11)

acceptance region (3.9):

$$B^{(0)}: \quad \frac{\hat{\sigma}^2}{\sigma^2} = \frac{1}{b}\cdot\frac{\underline{E}}{\sigma^2} < F_{0.95\,1;31,\,\infty} = 1.45 \qquad (12.12)$$

Bounds under H_{a_i} (2.15), (11.9)

(11.12): $|\widetilde{\nabla_0 x^i}|$ see figure 12.2
$30 < |\widetilde{\nabla_0 r}|$ dmgr < 250
$7 < |\widetilde{\nabla_0 s}|$ cm < 42 , hence in view of
(11.14): network definitely does not comply with the requirement for internal reliability; compare (12.14). A new reconnaissance is necessary (12.13)

(11.15): $|\widetilde{\nabla_0 r}|$ dmgr $\geqq \sigma_r\cdot\sqrt{\lambda_0} = 22$
$|\widetilde{\nabla_0 s}|$ cm $\geqq \sigma_s\cdot\sqrt{\lambda_0} = 5\sqrt{s\ \text{km}}$
λ_0-level does not seem to be unacceptable for this purpose, unless data-snooping becomes less realistic because of the rather large
$F^{\ddagger}_{1-\alpha_0;\,1,\,\infty}$-value, see (12.15) (12.14)

(11.17): not computed, on account of the uncertainty in the quantification of the requirements of the purpose.

Data-snooping

acceptance region per H_{a_i} (4.3) with (12.11):

$$B_i^{(0)}: \quad -4.1 < \underline{w}_i^1 < F^{\ddagger}_{1-\alpha_0;\,1,\,\infty} = 4.1 \qquad (12.15)$$

Illustration of (8.11) *ff.*

This example gives also a good insight in the question to what extent the matter treated in connection with (8.11) ff. occurs in practice. As an introduction to example III, a certainly unsatisfactory quadrilateral from the network of figure 12-2 {bottom left} is considered as a first step of the total adjustment. In this step I, there is one condition equation, hence $b^{\mathrm{I}} = 1$. In (12.16) are printed side by side the values $\{\lambda_0$ see (12.11)$\}$:

total adjustment, $b = 31 : |\widetilde{\nabla_0 x^i}|$

step I, $b^{\mathrm{I}} = 1 : |\widetilde{\nabla_0^{\mathrm{I}} x^i}|$

Quadrilateral 1–27–29–73				
	From point to point	$b = 31$	$b^{\mathrm{I}} = 1$	
Direction r	27– 1	109	111	
	27–29	50	111	
	73–29	45	60	
	73– 1	59	60	$\lvert\widetilde{\nabla_0 r}\rvert$ dmgr
	29–27	51	77	
	29– 1	38	39	
	29–73	46	78	
Distance s	29–27	15	21	$\lvert\widetilde{\nabla_0 s}\rvert$ cm
	29–73	10	21	

(12.16)

From (12.16) it appears that for some $\underset{\sim}{x}^i$ there is hardly a difference, so that the results of step I already give sufficient indication; for other $\underset{\sim}{x}^i$, however, the extension of the first step to more condition equations is necessary.

Example III

The network of figure 12-2 was chosen as an example where the internal reliability is unacceptable, see (12.13).

As was said after (11.7), N_p, and hence N_i, can be increased by adding further elements to the network, of course within the possibilities of the terrain and the economy of the survey. This may be combined with the reasoning of section 8 in order to improve locally the internal reliability.

Consider now the quadrilateral of (12.16), but add successive new elements. In (12.17), the results of this computation have been given; for λ_0 see (12.11). The first column gives the same results as step I in (12.16), the next columns show the results after addition of a practically justifiable number of new measured elements.

Quadrilateral 1–27–29–73
Different combinations of possible observations
* not observed

	From point to point	$b = 1$	$b = 2$	$b = 2$	$b = 3$	$b = 2$	$b = 4$	
direction r	27– 1	111	107	103	98	111	98	$\|\tilde{\nabla}_0 r\|$ dmgr
	27–73	*	37	*	37	*	36	
	27–29	111	38	103	38	111	38	
	73–29	60	58	37	37	60	36	
	73–27	*	*	36	35	*	35	
	73– 1	60	58	56	54	60	54	
	29–27	77	61	43	41	54	39	
	29– 1	39	37	36	34	39	34	
	29–73	78	47	64	44	55	41	
distance s	29–27	21	17	17	14	12	10	$\|\tilde{\nabla}_0 s\|$ cm
	29–73	21	17	17	14	11	10	
	27–73	*	*	*	*	11	10	

(12.17)

In this example it is remarkable that the mere addition of the direction r_{27-73} already gives a much more satisfactory solution {2nd column}.

Although (12.17) provides a nice example of the possibilities for local improvement, the conclusion in (12.13) on the network as a whole remains negative. This means that a new reconnaissance must be made, with, among other improvements, another choice of points. The actual measurement consequently should have been made in a modified network.

Remarks

1. In a computing procedure which forms part of an adjustment algorithm, the judgement of the internal reliability of a network is best executed by using the quantities N_i (11.10′), which can be furnished by the algorithm without much extra work. For a better understanding, the comparison of the N_i's among themselves has in the examples been transformed into a comparison of the $|\tilde{\nabla}_0 x^i|$'s.
2. In agreement with accepted practice, the acceptance regions (3.9) have in the examples been transformed to the form (2.4.5) in [3].
3. In the situation of example III, keeping λ_0 and β_0 constant when the number of condition equations increases, means that the *same* value α_0 is maintained but *not* the same value α and hence not the same value $F_{1-\alpha;b,\infty}$. If one keeps for α the value of (12.1), then the value λ_0, and consequently α_0, are altered if β_0 is kept the same but b increases or decreases. This shows how much more pure the reasoning becomes if it is based on the initial choice $\{\alpha_0, \beta_0\}$.

Example IV

In the summer of 1967, another reconnaissance was made, concerning a lower-order network having the same purpose as the network in example II. The result is pictured in figure 12-3, and, although the network proves not yet to satisfy the criteria for internal reliability, the measurements have been made according to this plan. This example of the measurement of a network has been used to illustrate the method of testing developed in section 8.

For comparison with figure 12-2, figure 12-3 indicates the values $|\widetilde{\nabla_0 x^i}|$ when using one acceptance region on the analogy of (12.12), but this time based on:

$$\alpha_0 = 0.001, \quad \beta_0 = 0.80, \quad \text{hence:} \quad \lambda_0 = 17 \tag{12.18}$$

These values $|\widetilde{\nabla_0 x^i}|$ will have to be increased by roughly 10% when working according to section 8, but usually they provide already sufficient indications.

If, also in this example, the known coordinates of higher-order points are left out of consideration, the network has: 52 points, 136 directions, 50 distances, 41 condition equations.

Data under H_0

$(\underline{x}^i)$ as indicated in connection with (12.10′)

$$\left.\begin{aligned} \sigma_r \text{ dmgr} &= \text{constant} = 7 \\ \sigma_s \text{ cm} &= \text{constant} = 1.5 \end{aligned}\right\} \text{ no correlation} \quad \ldots\ldots\ldots\ldots \tag{12.19}$$

Testing procedure for adjustment in steps

In order to obtain different possibilities for the combination of steps with a view to testing according to section 8, the adjustment was executed in a greater number of steps than practice would require. A partial orthogonalization according to (7.3) in 17 steps was chosen. Table 12.20 shows the choice made:

(Table 12.20, see page 81)

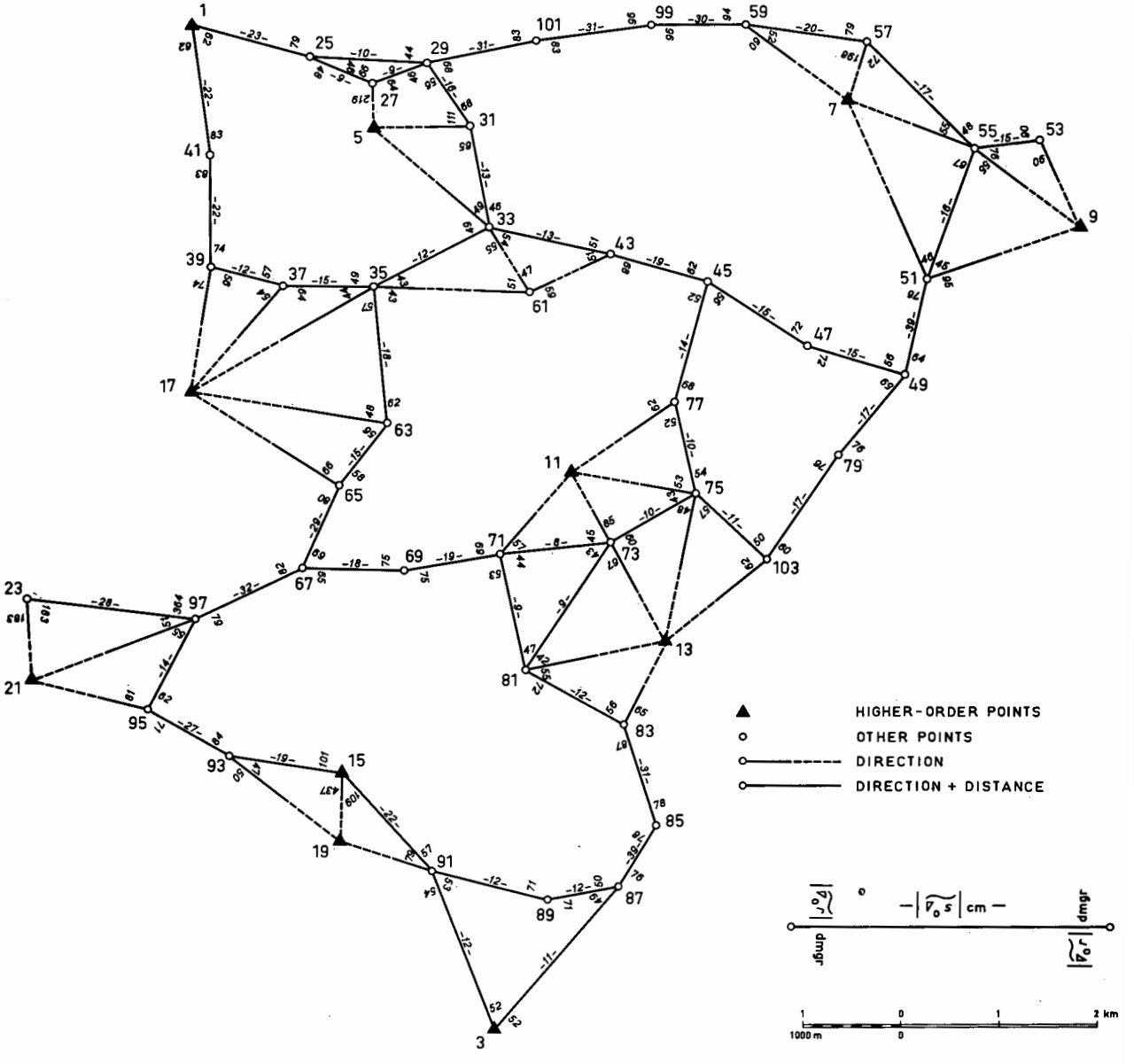
HIGHER-ORDER POINTS
OTHER POINTS
DIRECTION
DIRECTION + DISTANCE
cm
dmgr
dmgr
1000 m
2 km

Fig. 12-3

Step	b = Number of condition equations	Description
1	3	loop 1, 41, 39, 37, 35, 33, 31, 29, 25
2	3	loop 25, 27, 29
3	1	intersection point 5
4	3	loop 29, 31, 33, 43, 45, 49, 51, 55, 57, 59, 99, 101
5	2	intersection point 7
6	1	intersection point 9
7	3	loop 45, 77, 75, 103, 79, 49, 47
8	3	loop 33, 35, 63, 65, 67, 69, 71, 73, 75, 77, 45, 43
9	3	intersection point 17
10	3	intersection point 61
11	2	intersection point 11
12	3	loop 71, 81, 73
13	3	loop 69, 67, 97, 95, 93, 15, 91, 89, 87, 85, 83, 81, 71
14	3	intersection point 13
15	3	loop 3, 87, 89, 91
16	1	intersection point 21
17	1	intersection point 19

(12.20)

The computations were executed with the value of the variance factor:

$$\sigma^2 = 1 \quad \quad (12.21)$$

A number of conventional alternative hypotheses H_{a_i} (2.15) are considered, relating to 10 observation variates $\underline{r}$ and 5 variates $\underline{\ln s}$ as indicated in (12.22). The decomposition of N_i resulting from the orthogonalization according to (12.20) is given for each H_{a_i} in a separate column of (12.22), *see page 82.*

Three testing procedures A, B, C, in accordance with the principle of (8.15), with, see (8.14):

$$\delta^2 \leqq 0.2 \quad \quad (12.23)$$

are represented in the tables (12.24), (12.25) and (12.26). In the right-hand column of each table have been indicated the alternative hypotheses which in the relevant combination of steps obtain a sum of partial N-values from (12.22) satisfying (8.14) with (12.23). The direction- and distance-variates have been indicated by their initial- and terminal-points, the relation being shown by figure 12-3. The practical conclusion to be drawn from this is that the observations concerned will be checked.

Step	Decomposition of N_i according to steps of the adjustment															
	$r_{29,31}$	$r_{29,27}$	$r_{29,25}$	$r_{31,33}$	$r_{31,5}$	$r_{31,29}$	$r_{39,17}$	$r_{39,37}$	$r_{75,11}$	$r_{71,73}$	$\ln s_{35,63}$	$\ln s_{47,45}$	$\ln s_{47,49}$	$\ln s_{73,75}$	$\ln s_{71,81}$
1	0.109	0	0.109	0.101	0	0.101	0	0.115	0	0	0	0	0	0	0
2	0.010	0.298	0.224	0.006	0	0.006	0	0.001	0	0	0	0	0	0	0
3	0.046	0.021	0.005	0.028	0.055	0.004	0	0	0	0	0	0	0	0	0
4	0.048	0.002	0.008	0.024	0.001	0.030	0	0	0	0	0	0.026	0.018	0	0
5	0.002	0	0	0.001	0	0.001	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0.080	0.060	0	0
8	0	0.002	0.006	0.001	0	0.001	0	0.001	0	0.075	0.032	0.005	0.006	0.019	0
9	0	0	0	0	0	0.001	0.123	0.097	0	0.002	0.059	0	0	0	0
10	0.003	0.001	0.002	0.002	0	0.005	0.001	0.001	0	0	0.003	0.001	0	0.002	0
11	0	0	0	0	0	0	0	0	0.235	0.175	0.002	0	0	0.061	0
12	0	0	0	0	0	0	0	0	0.004	0.092	0.001	0	0	0	0.147
13	0	0	0	0	0	0	0.001	0	0.001	0	0.001	0	0	0	0.007
14	0	0	0	0	0	0	0	0	0.005	0.008	0.001	0	0	0.082	0.006
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Sum	0.219	0.324	0.354	0.164	0.056	0.149	0.125	0.217	0.245	0.353	0.098	0.112	0.085	0.164	0.160

(12.22)

Testing procedure A			
Test No.	b	Steps	Check on observations r and ln s between the points
A 1	7	1, 2, 3	39, 41, 1, 25, 27, 5, 29
A 2	6	4, 5, 6	29, 101, 99, 59, 57, 55, 53, 51, 49, 7, 9
A 3	7	1, 3, 4	29, 31, 5, 33
A 4	6	4, 7	45, 47, 49
A 5	9	1, 8, 9	39, 37, 35, 63, 65, 17
A 6	9	1, 8, 10	35, 33, 61
A 7	9	4, 8, 10	33, 43, 61
A 8	8	7, 8, 11	45, 77, 75, 49, 79, 103, 11
A 9	8	8, 11, 12	71, 73, 11
A 10	6	8, 13	65, 67, 69, 71
A 11	6	7, 14	75, 103, 13
A 12	8	8, 11, 14	73, 75, 13
A 13	6	12,14	81, 73, 13
A 14	6	13, 14	81, 83, 13
A 15	8	13, 15, 16, 17	67, 97, 23, 21, 95, 93, 15, 19, 91, 3, 87, 89, 85, 83

(12.24)

Testing procedure B			
Test No.	b	Steps	Check on observations r and ln s between the points
B 1	13	1, 2, 3, 4, 5, 6	39, 41, 1, 25, 27, 5, 29, 31, 33, 101, 99, 57, 59 7, 55, 53, 9, 51, 49
B 2	12	4, 7, 8, 10	33, 61, 43, 45, 47, 49
B 3	12	1, 8, 9, 10	39, 37, 35, 17, 33, 61, 63, 65
B 4	14	7, 8, 11, 12, 14	45, 77, 75, 103, 79, 49, 71, 73, 75, 81, 11, 13
B 5	12	8, 12, 13, 14	65, 67, 69, 71, 81
B 6	11	13, 14, 15, 16, 17	67, 97, 23, 21, 95, 93, 15, 19, 91, 3, 87, 89, 85, 83, 81

(12.25)

Testing procedure C			
Test No.	b	Steps	Check on observations r and ln s between the points
C 1	25	1, 2, 3, 4, 5, 6, 7, 8, 9, 10	17, 65, 63, 35, 37, 39, 41, 1, 25, 27, 29, 5, 31, 33, 61, 43, 45, 47, 49, 51, 53, 55, 9, 7, 57, 59, 99, 101
C 2	22	7, 8, 11, 12, 13, 14, 15, 16, 17	65, 67, 97, 23, 95, 21, 93, 15, 19, 91, 3, 89, 87, 85, 83, 81, 73, 75, 103, 79, 49, 13, 11, 77, 71, 69

(12.26)

Testing on adjustment

From the orthogonalization (12.20) follows the decomposition of the sample value of the shifting variate *after* the measurement, as indicated in table (12.27).

Step	b	E
1	3	13.23
2	3	10.50
3	1	2.36
4	3	2.39
5	2	0.35
6	1	1.31
7	3	6.69
8	3	3.86
9	3	7.37
10	3	2.55
11	2	1.51
12	3	0.34
13	3	1.02
14	3	2.11
15	3	0.52
16	1	0.45
17	1	0.23
Sum	41	56.77

(12.27)

To the testing procedures A, B and C correspond acceptance regions as indicated in (8.1)–(8.3).

Taking (12.21) into account, the acceptance region for e.g. the test A1 will be, on the analogy of (12.4):

$$B^{(A1)}: \quad \frac{E^{(A1)}}{b^{(A1)}} < F_{1-\alpha^{(A1)};\, b^{(A1)},\, \infty} = 2.2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12.28)$$

For $\alpha^{(A1)}$ see figure 3-3b, with α_0 from (12.18).

Like was done for the N's, the partial values E from (12.27) are then compounded in accordance with the description of the tests A1–A15, B1–B6, C1–C2, see (12.24)–(12.26). The result of these computations is shown in tables (12.29)–(12.31), where furthermore are stated the critical F-values as in (12.28) and the conclusion from the test executed.

Test No.	E	b	E/b	$F_{1-\alpha_b;b,\infty}$	action
A 1	26.09	7	3.7	2,2	reject
A 2	4.05	6	0.7	2.4	accept
A 3	17.98	7	2.6	2.2	reject
A 4	9.08	6	1.5	2.4	accept
A 5	24.46	9	2.7	1.9	reject
A 6	19.64	9	2.2	1.9	„
A 7	8.80	9	1.0	1.9	accept
A 8	12.06	8	1.5	2.0	„
A 9	5.71	8	0.7	2.0	„
A 10	4.88	6	0.8	2.4	„
A 11	8.80	6	1.5	2.4	„
A 12	7.48	8	0.9	2.0	„
A 13	2.45	6	0.4	2.4	„
A 14	3.13	6	0.5	2.4	„
A 15	2.22	8	0.3	2.0	„

(12.29)

Test No.	E	b	E/b	$F_{1-\alpha_b;b,\infty}$	action
B 1	30.14	13	2.3	1.7	reject
B 2	15.49	12	1.3	1.8	accept
B 3	27.01	12	2.2	1.8	reject
B 4	14.51	14	1.0	1.6	accept
B 5	7.33	12	0.6	1.8	„
B 6	4.33	11	0.4	1.8	„

(12.30)

Test No.	E	b	E/b	$F_{1-\alpha_b;b,\infty}$	action
C 1	50.61	25	2.0	1.3	reject
C 2	16.73	22	0.8	1.4	accept

(12.31)

Data-snooping

The action in the tables (12.29)–(12.31) is now followed by an attempt to locate errors.

For each alternative hypothesis H_{a_i} one might then establish the acceptance region (4.3), which, however, must now be modified according to (8.4)–(8.5). The statistics $\underline{w}_i$ {the superscript 1 can further be omitted without objection} are computed for each test. We have chosen here the computation by way of partial values of corrections ε^i per step and subsequent addition – like was done for the values for N and E – per test according to the description in (12.24)–(12.26).

Thus, we can put beside (12.28) the acceptance region per H_{a_i}, with σ^2 from (12.21) and test A1 composed of steps 1, 2, 3:

$$\left.\begin{aligned} w_i^{(A1)} &= \frac{-\bar{g}_{ii}\{\varepsilon^{i.1}+\varepsilon^{i.2}+\varepsilon^{i.3}\}}{\sqrt{N_i^{(A1)}}} \\ B_i^{(A1)}: \quad & |w_i^{(A1)}| < F^{\frac{1}{2}}_{1-\alpha_0;\,1,\,\infty} = 3.29 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (12.32)$$

The H_{a_i} pertain then to the observation variates indicated in the right-hand column of (12.24) beside test A1. The test A1 resulted in "rejection", and this implies consequently the w-computation and testing according to (12.32) for this limited number of alternative hypotheses H_{a_i}.

Now the conclusion from (12.29), respectively (12.30), respectively (12.31), is that several tests lead to rejection. This implies the execution of (12.32) for the tests concerned.

Lack of space forces us to make a choice in this example. This choice was to execute (12.32) for the H_{a_i}'s which were also used in (12.22). (12.33) is an auxiliary table which provides a possibility to follow the way in which the $|w|$ in table (12.34) have been computed.

As a curiosity, there has been included in (12.34) a similar table for data-snooping following the test with one acceptance region as in (12.4), if no decomposition of the adjustment into steps is made.

Extremely interesting is the conclusion, valid for *all* the four testing procedures, that the observations $r_{29,31}$ and $r_{31,29}$ should be checked in the first place by remeasurement.

Step	Decomposition of $\bar{g}_{ii}\varepsilon^i$ according to steps of the adjustment														
	$r_{29,31}$	$r_{29,27}$	$r_{29,25}$	$r_{31,33}$	$r_{31,5}$	$r_{31,29}$	$r_{39,17}$	$r_{39,37}$	$r_{75,11}$	$r_{71,73}$	$\ln s_{35,63}$	$\ln s_{47,45}$	$\ln s_{47,49}$	$\ln s_{73,75}$	$\ln s_{71,81}$
1	−1.13	0	1.13	−1.10	0	1.10	0	−0.46	0	0	0	0	0	0	0
2	−0.32	1.43	−1.11	−0.24	0	0.24	0	−0.06	0	0	0	0	0	0	0
3	−0.33	0.22	0.11	0.26	−0.36	0.10	0	0	0	0	0	0	0	0	0
4	−0.21	−0.04	−0.10	−0.19	−0.04	0.23	0	−0.01	0	0	0	−0.18	−0.13	0	0
5	−0.03	−0.01	−0.01	−0.02	0	0.02	0	0	0	0	0	0	0.01	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0.01	0	0	0
7	0.01	0	−0.01	0	−0.01	0.01	0	0	0	0	0	−0.55	−0.58	0	0
8	0.01	−0.05	−0.09	0.01	0.02	−0.03	0	−0.06	0	−0.16	−0.22	−0.03	−0.03	0.15	0
9	−0.03	0	0	−0.02	0	0.02	0.93	−0.79	0	−0.06	−0.55	−0.01	−0.02	0.03	0
10	−0.03	0.03	0.05	−0.03	−0.02	0.04	0.05	0.04	0	0.02	−0.08	0.03	0.01	0.07	0
11	0	0	0.01	0	0	0	0	0	−0.48	0.45	−0.03	0.01	0.01	−0.24	0
12	0	0	0	0	0	0	0	0	−0.03	−0.14	−0.01	0	0	−0.01	0.104
13	0	0	0	0	0	0	−0.01	0.01	−0.01	0	0.01	0	0	0	0.07
14	0	0	−0.01	−0.01	0	0.01	0.01	−0.01	−0.08	0.03	0.04	−0.01	−0.01	−0.23	0.07
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−0.01
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.01
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Sum	−2.05	1.59	−0.03	−1.33	−0.40	1.72	0.97	−1.34	−0.59	0.13	−0.85	−0.73	−0.73	−0.23	0.23

(12.33)

Direction	Data-snooping subsequent to: Without decomposition into steps				Procedure A				Procedure B				Procedure C			
	N	$\bar{g}_{ii}\varepsilon^i$	$\lvert w\rvert$	action	Test No.	N	$\lvert w\rvert$	action	Test No.	N	$\lvert w\rvert$	action	Test No.	N	$\lvert w\rvert$	action
$r_{29,31}$	0.219	−2.05	4.4	reject	A 3	0.203	3.7	reject	B 1	0.215	4.3	reject	C 1	0.219	4.4	reject
$r_{29,27}$	0.324	1.59	2.8	accept	A 1	0.319	2.9	accept	B 1	0.321	2.8	accept	C 1	0.324	2.8	accept
$r_{29,25}$	0.354	−0.03	0.1	,,	A 1	0.338	0.2	,,	B 1	0.346	0.03	,,	C 1	0.354	0.1	,,
$r_{31,33}$	0.164	−1.33	3.2	,,	A 3	0.153	2.6	,,	B 1	0.160	3.2	,,	C 1	0.164	3.2	,,
$r_{31,5}$	0.056	−0.40	1.7	,,	A 3	0.056	1.7	,,	B 1	0.056	1.7	,,	C 1	0.056	1.7	,,
$r_{31,29}$	0.149	1.72	4.5	reject	A 3	0.135	3.9	reject	B 1	0.142	4.5	reject	C 1	0.149	4.5	reject
$r_{39,17}$	0.125	0.97	2.7	accept	A 5	0.123	2.7	accept	B 3	0.124	2.8	accept	C 1	0.124	2.7	accept
$r_{39,37}$	0.217	−1.34	2.9	,,	A 5	0.213	2.9	,,	B 3	0.214	2.8	,,	C 1	0.217	2.9	,,
$r_{75,11}$	0.245	−0.59	1.2	,,	A 8	0.235	1.0	,,	B 4	0.244	1.2	,,	C 2	0.245	1.2	,,
$r_{71,73}$	0.353	0.13	0.2	,,	A 9	0.342	0.3	,,	B 4	0.350	0.3	,,	C 2	0.351	0.3	,,
$\ln s_{35,63}$	0.098	−0.85	2.7	,,	A 5	0.091	2.6	,,	B 3	0.094	2.8	,,	C 1	0.093	2.8	,,
$\ln s_{47,45}$	0.112	−0.73	2.2	,,	A 4	0.106	2.2	,,	B 2	0.112	2.2	,,	C 1	0.112	2.2	,,
$\ln s_{47,49}$	0.085	−0.73	2.5	,,	A 4	0.078	2.5	,,	B 2	0.084	2.5	,,	C 1	0.085	2.5	,,
$\ln s_{73,75}$	0.164	−0.23	0.6	,,	A 12	0.162	0.8	,,	B 4	0.162	0.8	,,	C 2	0.162	0.8	,,
$\ln s_{71,81}$	0.160	0.23	0.6	,,	A 13	0.153	0.4	,,	B 5	0.160	0.6	,,	C 2	0.160	0.6	,,

(12.34)

Testing after remeasurement

In agreement with the conclusion from (12.34), the series of directions in the points 29 and 31 have been remeasured.

Table (12.35) gives the revised results replacing (12.27).
Tables (12.36–(12.38) give the revised results replacing (12.29)–(12.31).
Table (12.39) gives the revised results replacing (12.34).

Step	b	E
1	3	1.16
2	3	5.06
3	1	0.57
4	3	0.91
5	2	0.11
6	1	1.33
7	3	6.58
8	3	4.98
9	3	7.04
10	3	2.02
11	2	1.56
12	3	0.38
13	3	1.05
14	3	2.01
15	3	0.51
16	1	0.45
17	1	0.23
Sum	41	35.94

(12.35)

Test No.	E	b	E/b	$F_{1-\alpha_b;b,\infty}$	action
A 1	6.79	7	1.0	2.2	accept
A 2	2.35	6	0.4	2.4	,,
A 3	2.64	7	0.4	2.2	,,
A 4	7.49	6	1.2	2.4	,,
A 5	13.18	9	1.4	1.9	,,
A 6	8.16	9	0.9	1.9	,,
A 7	7.91	9	0.9	1.9	,,
A 8	13.12	8	1.6	2.0	,,
A 9	6.92	8	0.9	2.0	,,
A 10	6.03	6	1.0	2.4	,,
A 11	8.59	6	1.4	2.4	,,
A 12	8.55	8	1.1	2.0	,,
A 13	2.39	6	0.4	2.4	,,
A 14	3.06	6	0.5	2.4	,,
A 15	2.24	8	0.3	2.0	,,

(12.36)

Test No.	E	b	E/b	$F_{1-\alpha_b;b,\infty}$	action
B 1	9.14	13	0.7	1.7	accept
B 2	14.49	12	1.2	1.8	,,
B 3	15.20	12	1.3	1.8	,,
B 4	15.51	14	1.1	1.6	,,
B 5	8.42	12	0.7	1.8	,,
B 6	4.25	11	0.4	1.8	,,

(12.37)

Test No.	E	b	E/b	$F_{1-\alpha_b;b,\infty}$	action
C 1	29.76	25	1.2	1.3	accept
C 2	17.75	22	0.8	1.4	,,

(12.38)

	Data-snooping subsequent to:											
Direction	Without decomposition into steps		Procedure A			Procedure B			Procedure C			
	$\|w\|$	action	Test No.	$\|w\|$	action	Test No.	$\|w\|$	action	Test No.	$\|w\|$	action	
$r_{29,31}$	0.25	accept	A 3	0.4	accept	B 1	0.3	accept	C 1	0.3	accept	
$r_{29,27}$	1.48	,,	A 1	1.4	,,	B 1	1.4	,,	C 1	1.3	,,	
$r_{29,25}$	1.65	,,	A 1	1.8	,,	B 1	1.8	,,	C 1	1.7	,,	
$r_{31,33}$	0.44	,,	A 3	0.3	,,	B 1	0.4	,,	C 1	0.4	,,	
$r_{31,5}$	0.70	,,	A 3	0.8	,,	B 1	0.8	,,	C 1	0.7	,,	
$r_{31,29}$	0.03	,,	A 3	0.0	,,	B 1	0.0	,,	C 1	0.0	,,	
$r_{39,17}$	2.65	,,	A 5	2.6	,,	B 3	2.7	,,	C 1	2.7	,,	
$r_{39,37}$	2.57	,,	A 5	2.7	,,	B 3	2.7	,,	C 1	2.6	,,	
$r_{75,11}$	1.21	,,	A 8	1.0	,,	B 4	1.2	,,	C 2	1.2	,,	
$r_{71,73}$	0.27	,,	A 9	0.3	,,	B 4	0.3	,,	C 2	0.3	,,	
$\ln s_{35,63}$	2.63	,,	A 5	2.6	,,	B 3	2.7	,,	C 1	2.7	,,	
$\ln s_{47,45}$	1.94	,,	A 4	2.0	,,	B 2	2.0	,,	C 1	2.0	,,	
$\ln s_{47,49}$	2.38	,,	A 4	2.5	,,	B 2	2.4	,,	C 1	2.3	,,	
$\ln s_{73,75}$	0.51	,,	A 12	0.7	,,	B 4	0.7	,,	C 2	0.7	,,	
$\ln s_{71,81}$	0.56	,,	A 13	0.4	,,	B 5	0.6	,,	C 2	0.6	,,	

(12.39)

From these results it is apparant that the developed method of data-snooping works {at least in this case}.

Remarks

1. Each of the testing procedures A, B and C was designed to make it possible to follow the measurement of the network by consecutive tests. Compare the remarks after (8.19). In the example given here this was not yet possible because the algorithm was not available at the time of measurement. However, the example demonstrates now that errors in a first step need not always affect the E-values of following steps.

2. In figure 12-4, a number of standard ellipses are shown, so that an impression of the point precision and relative precision can be gained. Here, too, the known coordinates of higher-order points have been left out of consideration. Two systems are shown, one referring to the base points P_{69} and P_{71}, the other referring to the base points P_1 and P_3. For the interpretation of these systems the reader is referred to [2].

A comparison of the data regarding the internal reliability in figure 12-3 with the data regarding precision in figure 12-4, reveals that the two concepts need not be directly connected.

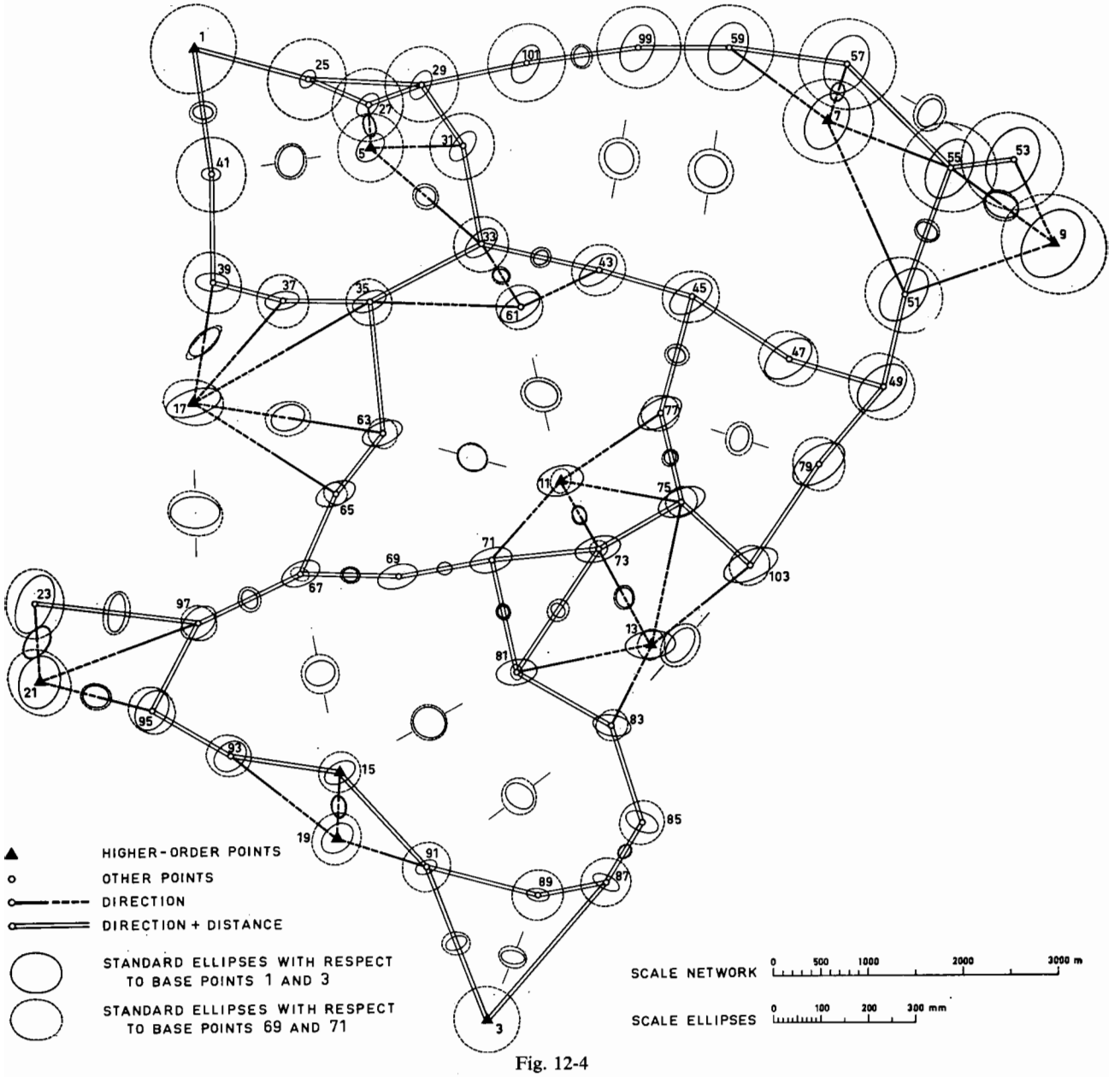

Fig. 12-4

13 INTRODUCTION OF SUBSTITUTE VARIATES WHEN ADJUSTING IN STEPS

There is a hidden difficulty in the examples of section 12. For, these examples are preceded by another step, viz. the station adjustment. In this section we shall discuss the technical treatment of such situations; derivations will be avoided wherever possible. Examples of these derivations are given in section 7.

We shall discuss an adjustment in three steps, but the reasoning can be connected to [3] pages 30–32, by there uniting the first two steps to step I.

A more general case of estimators $(\underline{X}^R)$

So far, we have assumed a *functional relationship* between variates $(\underline{x}^R)$ and $(\underline{x}^i)$. Introducing temporarily the notation $(\underline{x}^r)$ for this kind of $(\underline{x}^R)$, we have:

$$(\underline{x}^r - a_0^r) = (\Lambda_i^r)(\underline{x}^i - a_0^i) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.1)$$

Simplifications of the adjustment procedure are often possible when no functional relationship but a, more general, *stochastic dependency* is assumed. Many examples are known where without this assumption it would not be possible to compute correction-variates, although often in practice no attention is paid to this situations. These variates we denote by $(\underline{x}^R)$. The only information is then:

$$\overline{(x^R), (x^j)^\bullet} = (g^{Rj}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.2)$$

If the $(\underline{x}^i)$ are involved in the adjustment model, the theory of the method of least squares shows that for the estimators of $(\tilde{x}^R)$ we have:

$$\begin{aligned}
(\underline{X}^R) &= (\underline{x}^R) + (\underline{\varepsilon}^R) \\
(\underline{\varepsilon}^R) &= (g^{Rj})(\bar{g}_{ji})(\underline{\varepsilon}^i) \\
(\underline{\varepsilon}^i) &= -\overline{(\varepsilon^i), (\varepsilon^j)^\bullet}(\bar{g}_{ji'})(\underline{x}^{i'} - a_0^{i'}) \\
\overline{(\varepsilon^i), (\varepsilon^j)^\bullet} &= (g^{ij}) - (G^{ij})
\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.3)$$

(13.3) results in simpler formulas for the special case (13.1). From (13.1) follows:

$$(g^{rj}) = (\Lambda_i^r)(g^{ij})$$

hence:

$$(\Lambda_i^r) = (g^{rj})(\bar{g}_{ji}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.4)$$

(13.3) with (13.4) and then (13.1) give:

$$\begin{aligned}(\underline{X}^r - a_0^r) &= (\underline{x}^r - a_0^r) - (\Lambda_i^r)(g^{ij} - G^{ij})(\bar{g}_{ji'})(\underline{x}^{i'} - a_0^{i'}) \\ &= (\Lambda_i^r)(G^{ij})(\bar{g}_{ji'})(\underline{x}^{i'} - a_0^{i'}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.5)\end{aligned}$$

From (13.1) follows for the estimators:

$$(\underline{X}^r - a_0^r) = (\Lambda_i^r)(\underline{X}^i - a_0^i)$$

hence:

$$(G^{rj}) = (\Lambda_i^r)(G^{ij}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.6)$$

Substituting (13.6) into (13.5) gives the already much-used formula:

$$\boxed{(\underline{X}^r - a_0^r) = (G^{rj})(\bar{g}_{ji})(\underline{x}^i - a_0^i)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.7)$$

Assume now a decomposition into steps I, II, III, then we obtain:

$$\left.\begin{aligned}(\underline{\varepsilon}^R) &= (\underline{\varepsilon}^{R.I}) + (\underline{\varepsilon}^{R.II}) + (\underline{\varepsilon}^{R.III}) \\ \left.\begin{aligned}(\underline{X}^{R.I}) &= (\underline{x}^R) + (\underline{\varepsilon}^{R.I}) \\ (\underline{X}^{R.II}) &= (\underline{X}^{R.I}) + (\underline{\varepsilon}^{R.II}) \\ (\underline{X}^R) &= (\underline{X}^{R.II}) + (\underline{\varepsilon}^{R.III})\end{aligned}\right\} & \\ \begin{pmatrix}\underline{\varepsilon}^{R.I} \\ \underline{\varepsilon}^{R.II} \\ \underline{\varepsilon}^{R.III}\end{pmatrix} &= (g^{Rj})(\bar{g}_{ji}) \begin{pmatrix}\underline{\varepsilon}^{i.I} \\ \underline{\varepsilon}^{i.II} \\ \underline{\varepsilon}^{i.III}\end{pmatrix}\end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots \quad (13.8)$$

Introduction of substitute variates

The hidden difficulty mentioned in the introduction to this section is brought out by using a somewhat misleading notation.

For, in fact one will e.g. introduce into step II the group of variates:

$$(\underline{X}^{k.I} - a_0^k) = (t_i^k)(\underline{X}^{i.I} - a_0^i) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.9)$$

to which corresponds, according to the reasoning in [3] page 11, in step I:

$$(\underline{x}^k - a_0^k) = (t_i^k)(\underline{x}^i - a_0^i) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.10)$$

But this last relation does not occur in the situation under discussion, so that in step II in (13.9) the left-hand member of (13.10) is chosen for the notation. The notation of weight coefficients and weights corresponds to this notation.

Considering now the three steps, we find the {partly orthogonalized} misclosure variates from:

$$\left.\begin{aligned} &\text{I:} && (u_i^{\varrho_{\text{I}}})(\underline{x}^i - a_0^i) && = (\underline{y}^{\varrho_{\text{I}}}) \\ &\text{II:} && (u_i^{\varrho_{\text{II}}})(\underline{X}^{i.\text{I}} - a_0^i) && = (\underline{y}^{\varrho_{\text{II}}.\text{I}}) \\ &\text{III:} && (u_i^{\varrho_{\text{III}}})(\underline{X}^{i.\text{II}} - a_0^i) && = (\underline{y}^{\varrho_{\text{III}}.\text{II}}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (13.11)$$

Step I

$$\begin{aligned} (g^{Rj}) &= \overline{(x^R),(x^j)^*} \qquad \text{given} \\ (g^{ij}) &= \overline{(x^i),(x^j)^*} \qquad \text{given} \\ (\bar{g}_{ji}) &= (g^{ij})^{-1} \\ \begin{pmatrix} \underline{\varepsilon}^{R.\text{I}} \\ \\ \underline{\varepsilon}^{i.\text{I}} \end{pmatrix} &= \begin{pmatrix} -g^{Rj}\bar{g}_{ji'} \\ \\ -\delta^i_{i'} \end{pmatrix} \overline{(\varepsilon^{i'.\text{I}}),(\varepsilon^{j'.\text{I}})^*}(\bar{g}_{j'i})(\underline{x}^i - a_0^i) \qquad \ldots\ldots\ldots\ldots \quad (13.12) \end{aligned}$$

Introduce derived variates, viz. an as small as possible number of algebraically independent functions $(\underline{x}^k)$:

$$\boxed{(\underline{x}^k - a_0^k) = (t_i^k)(\underline{X}^{i.\text{I}} - a_0^i)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.13)$$

hence: $(\underline{X}^{k.\text{I}} - a_0^k) = (t_i^k)(\underline{X}^{i.\text{II}} - a_0^i)$

by regrouping the condition equations in step II and step III, e.g.:

$$\begin{aligned} &\text{II:} && (u_k^{\varrho_{\text{II}}})(\underline{x}^k - a_0^k) = (u_k^{\varrho_{\text{II}}})(t_i^k)(\underline{X}^{i.\text{I}} - a_0^i) = (\underline{y}^{\varrho_{\text{II}}.\text{I}}) \\ &\text{III:} && (u_k^{\varrho_{\text{III}}})(\underline{x}^k - a_o^k) = (u_k^{\varrho_{\text{III}}})(t_i^k)(\underline{X}^{i.\text{I}} - a_0^i) = (\underline{y}^{\varrho_{\text{III}}.\text{I}}) \end{aligned}$$

hence with (13.11):

$$\begin{aligned} (u_k^{\varrho_{\text{II}}})(t_i^k) &= (u_i^{\varrho_{\text{II}}}) \\ (u_k^{\varrho_{\text{III}}})(t_i^k) &= (u_i^{\varrho_{\text{III}}}) \end{aligned}$$

Step II

Compute:

$$\begin{aligned} (g^{Rl}) &= \overline{(x^R),(x^l)^*} \\ (g^{il}) &= \overline{(x^i),(x^l)^*} \\ (g^{kl}) &= \overline{(x^k),(x^l)^*} \\ (\bar{g}_{lk}) &= (g^{kl})^{-1} \end{aligned}$$

$$\begin{pmatrix} \underset{\sim}{\varepsilon}^{R.II} \\ \underset{\sim}{\varepsilon}^{i.II} \\ \underset{\sim}{\varepsilon}^{k.I} \end{pmatrix} = \begin{pmatrix} -g^{Rl}\bar{g}_{lk'} \\ -g^{il}\bar{g}_{lk'} \\ -\delta^{k}_{k'} \end{pmatrix} \overline{(\varepsilon^{k'.I}), (\varepsilon^{l'.I})^*}(\bar{g}_{l'k})(\underset{\sim}{x}^{k} - a^{k}_{0}) \quad \ldots\ldots\ldots\ldots \quad (13.14)$$

Introduce derived variates, viz. an again restricted number of independent functions $(\underset{\sim}{x}^{m})$:

$$\boxed{(\underset{\sim}{x}^{m} - a^{m}_{0}) = (t^{m}_{k})(\underset{\sim}{X}^{k.I} - a^{k}_{0})} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13.15)$$

again such, that in step III:

$$\text{III:} \quad (u^{e_{III}}_{m})(\underset{\sim}{x}^{m} - a^{m}_{0}) = (u^{e_{III}}_{m})(t^{m}_{k})(\underset{\sim}{X}^{k.I} - a^{k}_{0}) = (\underset{\sim}{y}^{e_{III}.II})$$

hence with (3.11):

$$(u^{e_{III}}_{m})(t^{m}_{k}) = (u^{e_{III}}_{k})$$

Step III

Compute:

$$(g^{Rn}) = \overline{(x^{R}), (x^{n})^*}$$
$$(g^{in}) = \overline{(x^{i}), (x^{n})^*}$$
$$(g^{kn}) = \overline{(x^{k}), (x^{n})^*}$$
$$(g^{mn}) = \overline{(x^{m}), (x^{n})^*}$$
$$(\bar{g}_{nm}) = (g^{mn})^{-1}$$
$$\begin{pmatrix} \underset{\sim}{\varepsilon}^{R.III} \\ \underset{\sim}{\varepsilon}^{i.III} \\ \underset{\sim}{\varepsilon}^{k.II} \\ \underset{\sim}{\varepsilon}^{m} \end{pmatrix} = \begin{pmatrix} -g^{Rn}\bar{g}_{nm'} \\ -g^{in}\bar{g}_{nm'} \\ -g^{kn}\bar{g}_{nm'} \\ -\delta^{m}_{m'} \end{pmatrix} \overline{(\varepsilon^{m'}), (\varepsilon^{n'})^*}(\bar{g}_{n'm})(\underset{\sim}{x}^{m} - a^{m}_{0}) \quad \ldots\ldots\ldots\ldots \quad (13.16)$$

Now the most important formulas have been obtained, of course much had to be left out. In a well-designed adjustment algorithm, the whole computation does not give any particular difficulty.

From the notation used it appears that as far as $(\underset{\sim}{x}^{k})$ is concerned, the adjustment starts in the original step II, as far as $(\underset{\sim}{x}^{m})$ is concerned it does not start until step III. However this is only apparent, because the whole adjustment in steps is executed, although the names of the variates are changed.

Thus we have for the example in [3] page 31, 32:

$(\underset{\sim}{x}^{k})$ results from station adjustment
$(\underset{\sim}{x}^{m})$ coordinate variates derived from angle- and distance variates.

The first two steps of this example have been combined into one step.

Influence of model errors

From (13.8) follows:

$$\left(\frac{\widetilde{\nabla X}^R}{\sigma}\right) = \left(\frac{\widetilde{\nabla x}^R}{\sigma}\right) + \left(\frac{\widetilde{\nabla \varepsilon}^{R.I}}{\sigma}\right) + \left(\frac{\widetilde{\nabla \varepsilon}^{R.II}}{\sigma}\right) + \left(\frac{\widetilde{\nabla \varepsilon}^{R.III}}{\sigma}\right) \quad \ldots\ldots\ldots \quad (13.17)$$

with further elaboration according to (13.12), (13.14), (13.16). Consequently, in principle with:

$$\left.\begin{array}{ll} & (\widetilde{\nabla x}^i) = (\widetilde{\nabla x}^i) \\ (13.13): & (\widetilde{\nabla x}^k) = (t_i^k)(\widetilde{\nabla X}^{i.I}) = (\ldots)(\widetilde{\nabla x}^i) \\ (13.15): & (\widetilde{\nabla x}^m) = (t_l^m)(\widetilde{\nabla X}^{k.I}) = (\ldots)(\widetilde{\nabla x}^i) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (13.18)$$

$$\left.\begin{array}{ll} (13.1): & (\widetilde{\nabla x}^r) = (\Lambda_i^r)(\widetilde{\nabla x}^i) \\ (13.2): & (\widetilde{\nabla x}^R) = (\widetilde{\nabla x}^R) \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (13.19)$$

Now the reasoning of section 8 is followed, however in the situation as indicated in [3] pages 30–32. For this, it is assumed that $(\widetilde{\nabla_p x}^i)$ according to the alternative hypothesis H_{a_p} in (2.9) is again composed of as many parts as there are steps, three in this case. The first part gives $\widetilde{\nabla x}^i$ with $N_i^{\mathrm{I}} \neq 0$, the second part generates $\widetilde{\nabla x}^k$ with $N_k^{\mathrm{II}} \neq 0$, the third part generates $\widetilde{\nabla x}^m$ with $N_m^{\mathrm{III}} \neq 0$, so that per step is assumed an alternative hypothesis according to (2.15).

Compute then bounds from, using a notation for N corresponding to that previously used:

step I	$\lvert\nabla_{i,0}\rvert = \sqrt{\dfrac{\lambda_0}{N_i^{\mathrm{I}}}}$	$N_i^{\mathrm{I}} \equiv N_i^{\mathrm{I}}$
step II	$\lvert\nabla_{k,0}\rvert = \sqrt{\dfrac{\lambda_0}{N_k^{\mathrm{I}}}}$	$N_k^{\mathrm{I}} \equiv N_k^{\mathrm{II}}$
step III	$\lvert\nabla_{m,0}\rvert = \sqrt{\dfrac{\lambda_0}{N_m}}$	$N_m \equiv N_m^{\mathrm{III}}$

$\ldots\ldots\ldots$ (13.20)

In this case the relations (13.18) do not apply. From (13.19) only the first equation is used, whereas also functions of $(\widetilde{\nabla x}^k)$ and $(\widetilde{\nabla x}^m)$, respectively, are possible under this assumption of alternative hypotheses.

In the case (13.2) one can never obtain an insight about $(\widetilde{\nabla x}^R)$ from tests on the adjustment problem. Hence:

$$(13.2): \quad |\widetilde{\nabla_0 x}^R| \text{ undetermined} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13.21)$$

The development of (13.17) for the different bounds is then, with (13.20) and (13.21):

$$\left(\frac{\tilde{\nabla}_{i,0}\varepsilon^R}{\sigma}\right) = -(g^{Rj})(\bar{g}_{ji'})\overline{(\varepsilon^{i'}),(\varepsilon^{j'})^*}(\bar{g}_{j'i})\cdot|\nabla_{i,0}|$$
$$\overline{(\varepsilon^i),(\varepsilon^j)^*} = \overline{(\varepsilon^{i.\mathrm{I}}),(\varepsilon^{j.\mathrm{I}})^*} + \overline{(\varepsilon^{i.\mathrm{II}}),(\varepsilon^{j.\mathrm{II}})^*} + \overline{(\varepsilon^{i.\mathrm{III}}),(\varepsilon^{j.\mathrm{III}})^*}$$

$$\left(\frac{\tilde{\nabla}_{k,0}\varepsilon^R}{\sigma}\right) = -(g^{Rl})(\bar{g}_{lk'})\overline{(\varepsilon^{k'}),(\varepsilon^{l'})^*}(\bar{g}_{l'k})\cdot|\nabla_{k,0}| \qquad (13.22)$$
$$\overline{(\varepsilon^k),(\varepsilon^l)^*} = \overline{(\varepsilon^{k.\mathrm{I}}),(\varepsilon^{l.\mathrm{I}})^*} + \overline{(\varepsilon^{k.\mathrm{II}}),(\varepsilon^{l.\mathrm{II}})^*}$$

$$\left(\frac{\tilde{\nabla}_{m,0}\varepsilon^R}{\sigma}\right) = -(g^{Rn})(\bar{g}_{nm'})\overline{(\varepsilon^{m'}),(\varepsilon^{n'})^*}(\bar{g}_{n'm})\cdot|\nabla_{m,0}|$$

In the case (13.1) this becomes:

$$\left(\frac{\tilde{\nabla}_{i,0}X^r}{\sigma}\right) = (G^{rj})(\bar{g}_{ji})\cdot|\nabla_{i,0}| \qquad (G^{rj}) = \overline{(X^r),(X^j)^*}$$
$$\left(\frac{\tilde{\nabla}_{k,0}X^r}{\sigma}\right) = (G^{rl})(\bar{g}_{lk})\cdot|\nabla_{k,0}| \qquad (G^{rl}) = \overline{(X^r),(X^l)^*} \qquad (13.23)$$
$$\left(\frac{\tilde{\nabla}_{m,0}X^r}{\sigma}\right) = (G^{rn})(\bar{g}_{nm})\cdot|\nabla_{m,0}| \qquad (G^{rn}) = \overline{(X^r),(X^n)^*}$$

Hence (13.23) can also be applied for $r \to i$ and $r \to k$ respectively, although in practice one usually is less interested in this.

Of course one can also compute intermediate results (13.22) and (13.23), either after step I or after step II. This concerns:

After step I:

$$\left(\frac{\tilde{\nabla}_{p,0}\varepsilon^{R.\mathrm{I}}}{\sigma}\right) \quad \text{resp.} \quad \left(\frac{\tilde{\nabla}_{p,0}X^{r.\mathrm{I}}}{\sigma}\right)$$

After step II:

$$\left(\frac{\tilde{\nabla}_{p,0}\varepsilon^{R.(\mathrm{I}+\mathrm{II})}}{\sigma}\right) \quad \text{resp.} \quad \left(\frac{\tilde{\nabla}_{p,0}X^{r.\mathrm{II}}}{\sigma}\right) \qquad \ldots\ldots\ldots\ldots \quad (13.24)$$

The considerations of section 11 are now again applicable. The examples of section 12 refer to one or more steps, often partial steps. The conclusions can now more easily be placed in the framework of a more extensive adjustment problem.

The line of thought of this section is of great practical importance. For, results of measurement and computation executed in a usually limited space of time will always have to be considered as the results of a step, however extensive this step may seem to be. Even if one wishes in practice to finish an adjustment from time to time, – see the remarks after (10.21) – theoretically one will always have the possibility to study afterwards a combination of "finished" adjustments, to analyse differences in results that may be difficult to explain.

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LIST OF ERRATA

W. BAARDA – Statistical Concepts in Geodesy – Netherlands Geodetic Commission, Publications on Geodesy, New Series, Vol. 2, No. 4, Delft, 1967

Page	Instead of	Read
23, line 19	(2.3.16)	(2.2.16)
25, formula (3.2.9)	$\lvert\nabla_0\rvert = \frac{\lambda_0}{N}$	$\lvert\nabla_0\rvert = \sqrt{\frac{\lambda_0}{N}}$
27, formula (3.2.16)	,,	,,
30, formula (3.2.24)	,,	,,