Optimized Zero and First Order Design of Micro Geodetic Networks

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Abstract

Precision is one of the main elements that control the quality of a geodetic network, which defines as the measure of the network efficiency in propagation of random errors.

This research aims to solve ZOD and FOD problems for a geodetic network using *Rosenbrock* Method to optimize the geodetic networks by using MATLAB programming language, to find the optimal design of geodetic network with high precision.

ZOD problem was applied to a case study network consists of 19 points and 58 designed distances with a priori deviation equal to 5*mm*, to determine the best points in the network to consider as control points. The results showed that P55 and P73 having the minimum ellipse of error and considered as control points.

FOD problem was applied to three cases of selected network to analyzed using the objective function of A-Optimality and D-Optimality, with selected range of movement of 300*m* to each point in each direction. The first case was a free network, the second case was with P55 and P73 as control points, and the third case was with P42 and P44 as control points. The results showed that the third case was the optimal design with high precision.

Key Words: Micro – Geodetic networks – Optimization – Precision – Zero order design – First order design – Rosenbrock method – A-optimality – D-optimality.

الخلاصة

تعتبر الدقة من العوامل الرئيسية التي تتحكم في جودة أو نوعية الشبكات الجيوديسية، وتعد الدقة قياسا لفاعلية الشبكة في توزيع الأخطاء العشوائية.

تهدف هذه الدراسة إلى حل مسألة التصميم من الرتبة الصفرية ومسألة التصميم من الرتبة الأولى لشبكة جيوديسية باستخدام طريقة Rosenbrock لإيجاد التصميم الأمثل للشبكة الجيوديسية باستخدام لغة MATLAB البرمجية. حيث تم تطبيق مسألة التصميم من الرتبة الصفرية على شبكة الدراسة التي تتألف من 19 نقطة و58 مسافة تصميمية وبانحراف معياري مقداره 5 ملم، لتحديد أفضل نقاط الشبكة التي يمكن اعتبارها كنقاط ضبط. فأظهرت النتائج أن النقطتين P55 وP73 تمتلك اصغر منحني خطأ وبالتالي يمكن اعتبارها كنقاط ضبط.

ثم تم تحليل ثلاث حالات من شبكة الدراسة من خلال تطبيق مسألة التصميم من الرتبة الأولى وباستخدام دالتي الهدف A-Optimality و D-Optimality لتحديد التصميم الأمثل لشبكة الدراسة الذي يحقق أعلى دقة، وتم تحديد حيز الحركة لنقاط الشبكة بمقدار 300 متر لكل نقطة وفي كل اتجاه. الحالة الأولى هي باعتبار الشبكة هي شبكة حرة (بدون نقاط ضبط)، الحالة الثانية هي اعتبار الشبكة بنقطتي ضبط P55 وP73، والحالة الثالثة هي اعتبار الشبكة بنقطتي ضبط P42 وP44.

وأظهرت نتائج تطبيق مسالة التصميم من الرتبة الأولى على الحالات الثلاثة، بان الحالة الثالثة من شبكة الدراسة تعتبر التصميم الأمثل الذي يحقق أعلى دقة.

Introduction

A geodetic network provides a frame work for setting out the main elements of (dams, bridges, power plants, tunnels ports, etc.) and for monitoring the position and deformation of these elements after construction, they can also be used to monitor crystal deformation, these networks are called local or micro networks, which are generally designed for a limited and specific purposes, but when used for multi-purposes, they are defined as national control networks, or reference networks which are play an important role when preparing the coverage maps of the country. The goal of designing, computing, measuring and adjusting procedure is to find the proper and optimal design of a geodetic network then establishing the points of the geodetic network by expressing their positions by (E, N) coordinates. To achieve the best network design, there are different techniques which are triangulation, trilateration, and combined control survey. The optimization problems of geodetic networks are classified into, Zero Order Design problem, ZOD, which define as searching for an optimal datum. First Order Design problem, FOD, which is define as the determination of the optimal position of the network points. Second Order Design problem, SOD, which is defined as weight problem, and Third Order Design problem, TOD, which is the optimal improvement of an existing network or an existing design. Sometimes FOD and SOD problems can be solved simultaneously; in this case the design problem is called a combined design problem, COMD. This research aims to solve ZOD problem and FOD problem of geodetic networks in order to find the optimal design of geodetic network in sense of high precision using least squares adjustment and Rosenbrock Method which solves nonlinear optimization problems, and using the objective function of A-Optimality and D-Optimality, with a constraint of the limits of movement for each point in each direction.

Preparation of Geodetic Network

Any geodetic project involved three steps which are: design (precision and reliability), adjustment (determination of geometry) and testing (validation of geometry), **Teunissen, P. J. G., 2006,** and the preparation of geodetic network contain design procedure, planning for survey, and optimization procedure.

Design Procedure

The approximate positions for stations must be determined to be included in the survey. These positions can be determined from topographic maps, photo measurements, or previous survey data. The approximate locations of the control stations should be dictated by their desired locations, the surrounding terrain, vegetation, soils, sight-line obstructions, and so on. Field reconnaissance at this phase of the design process is generally worthwhile to verify sight lines and accessibility of stations. Moving a station short distance from the original design location may greatly enhance the inter visibility between stations, but not change the geometry of the network significantly. By using topographic maps in the process, clearances of sight-line can be checked by constructing profiles between stations. In the design process, considerations should be given to the abilities of the field personal, quality of the equipment, and observational procedures. After the design is completed, specifications for field crews can be written based on these parameters. These specifications should include the type of the used instrument, number of turnings for angle observations, accuracy of instrument leveling and centering, miscolosure requirements, and many other items. When approximate station coordinates are determined, a stochastic model for the observational system can be designed. Once the stochastic model is designed, simulated observations are computed from the station coordinates, and a least squares adjustment of the observations is to be done. Since actual observations have not been made, their values are computed from the station coordinates. When the adjustment has been completed, the network can be checked for geometrically weak areas, unacceptable error ellipse sizes or shapes and so on. This inspection may dictate the need of any or all of the following, **Ghilani, C. D.,** and **Wolf, P. R., 2006:**

1. More observations.
2. Different observational procedure.
3. Different equipment.
4. More stations.
5. Different network geometry and so on.

Planning for Survey

In past, the planning of survey based on empirical standards, experience and intuition. This procedure depends on its validity on the pattern of new work conforming closely to that of the past experience. Any deviation in design can only be assessed intuitively and may have adverse effects on the precision of survey. By classical methods, the observational precision may be assessed by examining triangle and other miscloses. The size of the corrections to the observations after an adjustment is also of assistance in this assessment. The actual implicit requirement of the survey is that stations be coordinated to some specified positional accuracy. In order to assess the positional accuracy, error ellipses may be calculated for each station and their dimensions to compare with the specification, related to the adjustment reveal any weaknesses, the field part must revisit the scheme to take additional measurements, and this is an expensive proposition. Some recent articles suggested the use of simulated observations to analyze a proposed scheme of survey. All these requirements to analyze the scheme need the knowledge of three components:

1. The approximate coordinates of stations.
2. The type of the proposed observations (Distances, Directions, Azimuths, etc.) and their position in the network.
3. The anticipated precision for each type of observation.

With this information, it is possible to calculate the error ellipses for the stations, and verify the desired accuracy can be attained. The effect of additional or fewer measurements and changes in observational accuracy can also be assessed, by repeating the above procedure varying requirements 2 and 3. In this way, the most economical method to achieve the desired accuracy may be determined before any field measurements have been taken. In other words, the network has been "Optimized", **Allan, J. S., and Hoar, G. J., 1973.**

Optimization Procedure

The optimization and design of a geodetic control network play an important role in network analysis**.** Optimization means maximizing or minimizing an objective function which represents the criteria adopted to define Quality of a Geodetic Network. Generally, the quality of a geodetic network is characterized by precision, reliability, and economy. Precision may be expressed by the covariance matrix of the coordinates, displacement or deformation parameters, etc. So, the precision is the measure of the network characteristics in propagation of random errors, while reliability describes the ability of the redundant observation to check observation errors. Finally, economy is expressed in terms of the observation program, e.g., the cost of observation, transport….etc, **Kuang, S. L., 1991.**

A network can be designed in such way that, **Schmitt, G., 1982:**

- The postulated precision of the network elements, e.g. coordinates or displacement of points, can be realized,
- It becomes sensitive against statistical testing carried in measurements, and it resists against undetected gross errors, and
- The construction of the points and the performance of the measurements satisfy some cost criteria.

In mathematical terms the quality of geodetic network means, **Amiri-Simkooei, A. R., 1998:**

$$\alpha_p(\text{precision})+\alpha_r(\text{Reliability})+\alpha_c(\text{cost})^{-1}\rightarrow \max \quad (1)$$

where:

α_p, α_r and α_c are the weight coefficients for precision, reliability and cost, respectively.

There are two methods that can be used to solve the design problem:

1. Trial and error method: which can be summarize in the following steps, **Amiri-Simkooei, A. R., 2007**:
   a. Specify precision and reliability criteria (e.g. ellipse of error).
   b. Select an observation scheme (stations, observations and their precision).
   c. Compute the values of the quantities specified as precision and reliability criteria (e.g. covariance matrix and redundancy numbers).
   d. If the computed criteria are close to those specified in (a), then go to the next stage; otherwise alter the observation scheme (by adding the observations or increasing the weight if they are not satisfied, or by removing the observations or decreasing the weights if they are too optimistic) and return to (c).
   e. Compute the cost of the network and restart from (b) with completely different scheme (e.g. Trilateration instead of triangulation).
   f. Stop when it is believed that the optimum (minimum cost) network has been found.

In other words – in trial and error method, the objective function is computed with a suggested solution for the problem, and if the suggested solution does not satisfy the objective function, the solution is changed a bit and the objective function is computed again. This process is repeated until the requirement is satisfied. The solution in the trial and error method depends on the experience of the designer. However, in some cases the solution might not even be found, but what might it interesting is its ease of use and lack of complicated mathematical models in its solution, **Sahabi, et al., 2006.**

2. Analytical method: which offer specific algorithms for the solution of particular design, which are used to describe a method that solves a particular design problem by a unique series of mathematical steps, **Kuang, S. L., 1991.** The analytical approach gives some advantages rather than other existing methods for network optimization, as follows:
   - Any type of geodetic observable can be considered.
   - Any condition or constraint can be considered.
   - All the criteria of precision, reliability and cost can be considered simultaneously in the optimal design.
   - The optimization procedure can be performed in the sense of FOD and SOD separately or simultaneously.
   - This methodology can be used for the optimal design of one, two, or three dimensional networks.

The main disadvantage of the analytical method is: the proper formulation of the mathematical model can be difficult. This also holds for solution of the problem in an analytical way, **Simkooei, A. R., 2007.**

Adjustment Computations by Least Squares

The development of the theory of least squares adjustment is based on the variance law for independent observations. The mathematical concept of weights is presented as a function of the variance (squares of standard deviations), while the use of weights, in conjunction with the variance law, leads to the idea of the variance factor. The square root of the variance factor is usually referred to as the standard deviation of an observation having unit weight, **Gale, L. A., 1965**.

The principle and the equations of least squares and adjustment with constraints are explained in, **Mikhail, E. M., 1976.**

Variance - Covariance Matrix

Variance – covariance matrix of coordinates of the analyzed network is a basis for calculations of the relative positional errors. They allow the calculations of the following quantities that are frequently used in describing the positional accuracies which is called *post analysis process*, **Moffitt, F. H.,** and **Bourchard, H., 1987**. The following symbols will be used:

(σ_E , σ_N) = standard deviation of coordinates.

$(\sigma_{\max} , \sigma_{\min})$ = semi – major and semi – minor axes of standard error ellipse.

$(\sigma_d , \sigma_a , \sigma_A)$ = standard deviations of observed distance, directions (azimuths) and angles.

The covariance of a pair of coordinate values belonging to one or two points denoted by $(\sigma_{E_iN_i} ,\sigma_{E_jN_j})$ is a measure of statistical dependence of the two values, if the two coordinates or two observation values are uncorrelated then their covariance equals zero. The algorithm for computing the variance – covariance matrix is as follows:

$$\sigma_0^2 = \frac{v^t wv}{r} \qquad (2)$$

$$\textstyle\sum_{xx} = \sigma_o^2 (B^t\, w\, B)^{-1} \qquad (3)$$

where:

$\sum_{xx}$ = variance-covariance matrix of adjusted unknowns,

σ_o^2 = variance of unit weight,

B= the coefficients matrix,

w = the weight matrix, and

r = redundancy.

It is important to know that $\sum_{xx}$ is square and symmetrical. For instant, for two points i and j in a network, $\sum_{xx}$ will be a matrix of the form:

$$\textstyle\sum_{xx} = \begin{bmatrix} \sigma_{E_i}^2 & \sigma_{E_iN_i} & \sigma_{E_iE_j} & \sigma_{E_iN_j} \\ \sigma_{E_iN_i} & \sigma_{N_i}^2 & \sigma_{N_iE_j} & \sigma_{N_iN_j} \\ \sigma_{E_iE_j} & \sigma_{N_iE_j} & \sigma_{E_j}^2 & \sigma_{E_jN_j} \\ \sigma_{E_iN_j} & \sigma_{N_iN_j} & \sigma_{E_jN_j} & \sigma_{N_j}^2 \end{bmatrix} \qquad (4)$$

Ellipse of Error

The variance or standard deviation are measures of precision for the one dimensional case like an angle or a distance, but in the case of two dimensional problems, such as the horizontal position of a point, error ellipses may be established around the point to designate precision regions for a different probabilities. The orientation of the ellipse relative to the E, N axes system, **Fig. 1**, depends on the correlation between E and N. If they are uncorrelated, the ellipse axes will be parallel to E and N. If the two coordinates and of equal precision, or $\sigma_E = \sigma_N$, the ellipse becomes a circle. Considering the general case where the covariance matrix for the position of point p is given as, **Davis, et al., 1981**:

$$\Sigma = \begin{bmatrix} \sigma_E^2 & \sigma_{EN} \\ \sigma_{EN} & \sigma_N^2 \end{bmatrix} \tag{5}$$

The semi major and semi minor axes of the corresponding ellipse are computed as in the following:

$$\sigma_{\max}^2 = \frac{1}{2}\left(\sigma_N^2 + \sigma_E^2 + \sqrt{\left(\sigma_N^2 - \sigma_E^2\right)^2 + 4\sigma_{EN}^2}\right) \tag{6}$$

$$\sigma_{\min}^2 = \frac{1}{2}\left(\sigma_N^2 + \sigma_E^2 - \sqrt{\left(\sigma_N^2 - \sigma_E^2\right)^2 + 4\sigma_{EN}^2}\right) \tag{7}$$

The orientation of the ellipse is determined by computing (θ) between the E axis and the semi major axis from:

$$\tan 2\theta = \frac{2\sigma_{EN}}{\sigma_E^2 - \sigma_N^2} \tag{8}$$

where:

θ = is laid off counter clockwise from the positive E-axis.

Mathematical Model in Adjusting Horizontal Geodetic Network

The adjustment of geodetic networks still one of the most important problems in surveying for many applications such as designed for monitoring. With advance development of computers, the method of adjusting the geodetic networks by "variation of coordinates" technique become the most preferred one, **Jabiry, J. M., 2007**. The variation of coordinates is the most common method to adjusting the traditional geodetic networks such as triangulation networks, trilateration networks, intersection, resection, hybrid networks and traversing as well as in engineering applications such as displacement computation of heavy structures like dams and towers. The reason for the prevalence of using this method is its suitability for computer programming because it taken specific pattern depending on the type of observations and their relation to the fixed and unknown points, and it is based on assumption coordinates for unknown points, calculated by lengths and observed directions by error does not exceed 1/4000 for lengths and 1 minute for directions to be solve with minimum iteration, **AL-Joboori, B. S., 2010.** The following equation is the final linearized observation equations for an observed distance see **Fig. 2**, where coordinates of *i, j* are known.

$$\begin{aligned} v_s = & \left(\frac{E_i^\circ - E_j^\circ}{S_{ij}}\right) \times \delta E_i + \left(\frac{N_j^\circ - N_i^\circ}{S_{ij}}\right) \times \delta N_i \\ & + \left(\frac{E_i^\circ - E_j^\circ}{S_{ij}}\right) \times \delta E_j - \left(\frac{N_i^\circ - N_j^\circ}{S_{ij}}\right) \times \delta N_j \\ & + \left(S_{comp} - S_{obs}\right) \end{aligned} \tag{9}$$

Free Net Adjustment or Inner Constraints

The datum problem is very important in geodetic networks and there are various methods for defining a datum. The definition of the best datum for a geodetic network can be called as the ZOD. There exist three methods for obtaining the adjusted coordinates of net points. If the datum is not defined for a geodetic network, one can say that there exists infinite number of solution for the coordinates of the net points, in other word, the points can get infinite values as coordinates. In order to restrict the values, some constraints should be included in our adjustment process, so that they convert our infinite number of solutions to finite number. These constraints should be defined so that the network becomes translation-free, rotation-free, and free of expansion or contraction. These characteristics can be included in adjustment process using minimum constraint, inner constraint, or even over constraint least squares adjustment. There are various literatures for mathematical description of these methods like (Teunissen, P.J.G., 1982**), Eshagh, M., 2006.**

In a free network adjustment the system of normal equation is singular because of the rank defect (no external datum). Different approaches for the solution of a free net are available, but the solution is always chosen where the trace of the covariance matrix of the estimated parameters is a minimum. The geometric interpretation of the minimum trace is that there should not be any translation, rotation or scaling changes from the given approximate values of the unknown parameters. One of the advantages of free net adjustment is the fact that it can better identify the existence of certain unmodeled systematic error in the system, as the solution is not influenced by external factors, **Remondino, F., 2006.** If a square, symmetric matrix, such as *N*, is of full rank, then all of its eigenvalues are nonzero, and it's eigenvectors from an orthogonal basis for the row space. If it is not full rank, it has order u, rank ($h < u$), and therefore defect (u-h). Such as matrix will have (u-h) zero eigenvalues. The h eigenvectors associated with nonzero eigenvalues will form an orthogonal basis for the row space, and the (u-h) eigenvectors associated with the zero eigenvalues will form an orthogonal basis for the null space. The locus of solutions to the rank – deficient equations and the intersection point of the solution space and the row space are shown schematically **Fig. 3.** This strategy will have the following characteristics:

1. It will resolve the deficiency from the datum defect and it will therefore permit a unique solution to the system of equations.
2. Of all the possible solutions to the rank – deficient system, it will select the one with minimum magnitude and minimum variance.

This solution is known in the geodetic and photogrammetric literature as the inner constraint solution, or sometimes as the free net solution.

This presentation is useful for understanding the geometry of the problem, but there are easier ways to construct the needed constraint matrix. The eigenvectors just described provide a basis of the null space of the rank – deficient matrix. But, as with any vector space, there are many (an infinite number of) such bases. In the adjustment of geodetic nets, the relationships between the observations and the point coordinate parameters are expressed by condition equations: for example, the angle or the distance condition equations in geodetic applications. These equations all contribute relative information rather than absolute information about the point positions. The usual procedure to introduce absolute information is by constraining (fixing) certain point coordinate components. These points are referred to as control points. Without such control points, or other constraints, the system of normal equations would be rank deficient and hence not uniquely solvable. The rank deficiency is equal to the minimum number of constraints that would be needed to bring the system to full rank. In the case of a horizontal network with only angle observations, the rank deficiency is four. In the case of a horizontal network with at least one distance observation, the rank deficiency is three. The rank deficiencies are referred to as datum defect, since the presence of the necessary control points would define the datum. Of course, there may be other causes of rank deficiency, such as insufficient observations to define a point, these are another matter altogether, and are referred to as configuration defects. In one introduces just enough constraint equations to satisfy the datum defect, and then these are known as minimal constraints. Different sets of minimal constraints have the interesting property that, although the point coordinate estimate may vary, the observation residuals are invariant. Thus, for residual analysis only any minimal set of control points is as good as another. However, in practice the choice of control point is very important. The new network must be consistent with existing networks at shared points. There are many mathematical techniques that could be used to solve for the generalized inverse, they are:

1. Orthogonal bordering.
2. Pseudo inverse.
3. Singular value decomposition (SVD).
4. Orthogonalization (QR).
5. There is another common approach is, **AL-Joboori, B. S., 2010**:

$$N^{+} = \left(B^{t} wB + C^{t} C\right)^{-1} - C^{t}\left(CC^{t} CC^{t}\right)^{-1} C \qquad (10)$$

For the purpose of illustration, consider the geodetic horizontal triangle network in **Fig. 4**, as an example. If only the three angle observations shown are made and no control points are introduced, then the resulting normal equations, of size $[6 \times 6]$ have rank two and rank deficiency of four. The fixing of two control points, or four point coordinates, would resolve this datum defect. These constraints can be implemented very simply by eliminating the effect by replacing the unknowns with numerical constant. However, for generality we assume that the constraints are implemented by the general method of bordering, (Helmert Method), where:

$$\begin{bmatrix} -N & C^{t} \\ C & 0 \end{bmatrix} \begin{bmatrix} \Delta \\ k_{c} \end{bmatrix} = \begin{bmatrix} -t \\ g \end{bmatrix} \qquad (11)$$

In which N and t pertain to the normal equations, where $N=B^{t}\, w\, B$ and $t=B^{t}\, w\, f$.

For the horizontal 2-D network in **Fig. 4**, with no distance observations, the following constraint matrix will have the same effect as the (harder to compute) eigenvectors:

$$C=\begin{bmatrix}1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1\\ y_i^o & -x_i^o & y_j^o & -x_j^o & y_k^o & -x_k^o\\ x_i^o & y_i^o & x_j^o & y_j^o & x_k^o & y_k^o\end{bmatrix} \quad (12)$$

in which approximate coordinates x°, y° of three points are used. If distance observation were present then the datum defect would be one less, and the fourth row of the matrix in eq. (3) will be canceled. In order to demonstrate the plausibity of this solution, it will be shown that the rows of the matrix in eq. (3) are orthogonal to the coefficients of the angle condition equation. This condition equation represents the one most widely used in two dimensional, horizontal triangulation networks. For the clockwise angle at point (*i*), in **Fig. 3**, from point (*j*) to point (*k*), the following row vector represents the coefficients of the linearized angle condition equation, **Mikhail, et al., 2000**.

$$B=\begin{bmatrix}\frac{\partial F_\theta}{\partial x_i} & \frac{\partial F_\theta}{\partial y_i} & \frac{\partial F_\theta}{\partial x_j} & \frac{\partial F_\theta}{\partial y_j} & \frac{\partial F_\theta}{\partial x_k} & \frac{\partial F_\theta}{\partial y_k}\end{bmatrix} \quad (13)$$

If one takes the inner product of B with the rows of C, the result is a vector of zeros. In other words, the rows of C are orthogonal to B.

$$BC^t=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \quad (14)$$

The development will summarize for the two dimensional case, a simple extension can be to make the adjusted coordinate the approximate coordinates, X^o.

$$X_a=T+(1+k)R_\alpha .X^o \quad (15)$$

in which T is the translation vector, (1+*k*) is the scale factor, and R_α is the matrix of a small angle. Written as:

$$\begin{bmatrix}x_a\\ y_a\end{bmatrix}=\begin{bmatrix}t_x\\ t_y\end{bmatrix}+(1+k)\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}x^o\\ y^o\end{bmatrix} \quad (16)$$

Assuming a small angle and scale vector near unity, and assuming that products of small quantities may be disregarded, we obtain:

$$\begin{bmatrix}x_a\\ y_a\end{bmatrix}=\begin{bmatrix}t_x\\ t_y\end{bmatrix}+\begin{bmatrix}x_o\\ y_o\end{bmatrix}+\alpha\begin{bmatrix}y^o\\ -x^o\end{bmatrix}+k\begin{bmatrix}x^o\\ y^o\end{bmatrix} \quad (17)$$

So this represents a step in the iterative solution.

$$\begin{bmatrix}x_a\\ y_a\end{bmatrix}=\begin{bmatrix}x^o+dx\\ y^o+dy\end{bmatrix} \quad (18)$$

The last two equations can be combining to obtain:

$$\begin{bmatrix}dx\\ dy\end{bmatrix}=\begin{bmatrix}t_x\\ t_y\end{bmatrix}+\alpha\begin{bmatrix}y^o\\ -x^o\end{bmatrix}+k\begin{bmatrix}x^o\\ y^o\end{bmatrix} \quad (19)$$

Rearranging in a matrix form yields:

$$\begin{bmatrix}dx\\ dy\end{bmatrix}=\begin{bmatrix}1 & 0 & y^o & x^o\\ 0 & 1 & -x^o & y^o\end{bmatrix}\begin{bmatrix}t_x\\ t_y\\ \alpha\\ k\end{bmatrix} \quad (20)$$

The above equations can be written for every point in the network as:

$$\begin{bmatrix}d_{x1}\\ d_{y1}\\ d_{x2}\\ d_{y2}\\ .\\ .\\ d_{xn}\\ d_{yn}\end{bmatrix}=\begin{bmatrix}1 & 0 & y_1^o & x_1^o\\ 0 & 1 & -x_1^o & y_1^o\\ 1 & 0 & y_2^o & x_2^o\\ 0 & 1 & -x_2^o & y_2^o\\ . & . & . & .\\ . & . & . & .\\ 1 & 0 & y_n^o & x_n^o\\ 0 & 1 & -x_n^o & y_n^o\end{bmatrix}\begin{bmatrix}t_x\\ t_y\\ \alpha\\ k\end{bmatrix} \quad (21)$$

By considering this to be over determined system of equations, then:

$$f\approx B\Delta \quad (22)$$

Then it could be solved in the least squares sense by the usual normal equations:

$$\Delta=\left(B^tB\right)^{-1}B^tf \quad (23)$$

Now suppose that we would like to enforce the condition between the point coordinates before and after the iterative correction where will be no net shift, rotation, or scale change.
In other words,

$$\Delta=\begin{bmatrix}t_x\\ t_y\\ \alpha\\ k\end{bmatrix}=0 \quad (24)$$

This can be done by setting

$$B^t f = 0 \qquad (25)$$

or, it can be written as,

$$\begin{bmatrix} 1 & 0 & . & . & 1 & 0 \\ 0 & 1 & . & . & 0 & 1 \\ y_1^o & -x_1^o & . & . & y_n^o & -x_n^o \\ x_1^o & y_1^o & . & . & x_n^o & y_n^o \end{bmatrix} = \begin{bmatrix} d_{x1} \\ d_{y1} \\ . \\ . \\ d_{xn} \\ d_{yn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (26)$$

The geometric interpretation of the inner constraint solution is that when advancing from one iteration to the next, there will be no net shift, rotation, or scale change between the approximate and refined coordinate positions. Thus, rather than arbitrarily fixing two points (four coordinate components) out of many, one fixes four geometric relationships. All points then equal play roles in connecting the network to the coordinate system. This can have dramatic effects upon the a posteriori confidence ellipse of the network points. If a point is fixed, then its confidence ellipse vanishes. For a three point network, after fixing two points, all of the error is cast into the uncertainty of the third point. With the inner constraint or free net solution, however, each point has a finite confidence ellipse that reflects its strength of determination in the network.

First Order Design problem (FOD)

Determination of the optimum geometric design for geodetic network is one of the classical design problems in geodesy, known as the first-order design problem, **Berné, J. L.,** and **Baselga, S., 2004.** When trying to define the best geometric configuration of a new geodetic network, the range of situation can vary from the case where the possible location of the station is so constrained by exterior conditions - visibility, natural features, private properties, etc. – that there is almost no choice for the most adequate location because there is no margin of movement, to the case where any possible location within an area is acceptable. As the margin of choice grows, there is an increasing need for reliable criteria to determine the most appropriate network design. Criteria for appropriateness should rely on the minimum in determination at the defined points, considering both the type and the number of observations to be done. Some mathematical methods are required to search for the best design.

One of the most rigorous approaches is to determine the position of all the stations to be located by means of minimizing the hyper volume of error hyper ellipsoid inherent to the solution. The system of equations for the parametric least squares method in its parametric form, and for any kind of observations (angles, distance, etc.) can be used and as the aim is to minimize the hyper volume of the hyper ellipsoid define by $\sum_{xx}$ this involves minimizing its determined

$$\text{Min. det.} (\textstyle\sum_{xx}) \quad \text{or} \quad \text{Max. det.} (\textstyle\sum_{xx})^{-1} \qquad (27)$$

The question of finding optima for the determinant function (and even to calculate the determinant function itself) is far from being a trivial problem. Moreover, if the desired solution has to be not a local but the global optimum, the solution is almost unattainable, at least from the classical mathematical point of view.

Measures and Criteria for Precision

The precision measures and criteria of a geodetic network are based on the covariance matrix of estimated coordinates. One measure of precision takes the form of scalar function of the elements of the covariance matrix of the coordinates varieties. The purpose is to fill the need for an overall representation of the precision of a network**, Kuang, S. L., 1991.**
A scalar function (objective function) may be one of the following, **Berné, J. L.,** and **Baselga, S., 2004.**

1. A – Optimality ●
   This refers to minimizing the trace in the covariance matrix. As a result, the average variances of the parameter estimates are minimized.

   $$f = trace\ (\textstyle\sum_{xx}) \rightarrow \min \qquad (28)$$

2. D – Optimality ●
   This seeks to minimize the covariance matrix determinant. It has the statistical significance of minimizing the volume of the error hyper ellipsoid.

   $$f = det\ (\textstyle\sum_{xx}) \rightarrow \min \qquad (29)$$

3. E – Optimality ●

This aims to minimize the largest eigenvalue of the covariance matrix for the parameter estimates.

$$f = \lambda_{max} \rightarrow \min \qquad (30)$$

4. S – Optimality
This related to network stiffness and supposes an eigenvalue and eigenvector approach to the problem of optimization in terms of network stiffness, i.e. the maximum flattening of the eigenvalue spectrum.

$$f = \lambda_{max} - \lambda_{min} \rightarrow \min \qquad (31)$$

5. Criterion Matrices
These consist of defining a desired matrix for the result – the criterion matrix – and then finding the solution that is closest to this ideal assumption. They sometimes appear together with some of the other methods.

Rosenbrock Method for Unconstrained Optimization Problems

Rosenbrock Method falls within the numerical category of optimization methods. The method, due to, **Rosenbrock, H. H., 1960**, uses a direct search technique making use of the Orthogonalization principle of Gram-Schmidt in order to give an acceleration in both the direction and the step length. The method is fully explained in many references, **Rosenbrock, H. H., 1960, Kuester, J. L.,** and **Mize, J. H., 1973, Bazaraa, M. S.,** and **Shetty, C. M., 1979, AL-Fahdawi, D. KH., 2000, Al-Hity, A. A, 2006.** It can be reviewed by the following algorithm:

1. Choose initial values for the design variables and set of step sizes ($\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$) (where n is the number of independent variables). Also choose a set of search directions ($S_1^j, S_2^j, S_3^j, \ldots, S_n^j$) which are usually selected as being parallel to the basic axes when j=1, and evaluate the function.
2. Sequentially search parallel to each of the n directions in turn, adopting the new point if the move is successful (objective function is less than or equal to the previous one) and retaining the last point if the move is unsuccessful. If a move is successful the step length is multiplied by a factor (α), (α >1) then this direction is next searched in its turn. If the move is unsuccessful then the step length to be used for the next search is (β) times the previous length ($0 < \beta < 1$) and the direction of move is reversed. However Rosenbrock recommended the use of ($\alpha = 3$) and ($\beta = 0.5$) as suitable values.
3. The search of step (2) continues until at success is followed by a failure in every direction, **Himmelblau, D. M., 1972.**
4. Compute the new set of direction ($S_1^{j+1}, S_2^{j+1}, S_3^{j+1}, S_n^{j+1}$) for use in next $(j+1)^{th}$ stage of method by using the Gram-Schmidt Orthogonalization procedure.
   a. Compute a set of independent direction ($P_1, P_2, P_3, \ldots, P_n$) as

$$P_{(n,n)} = \begin{bmatrix} P_1 & P_2 & P_3 \ldots\ldots\ldots P_n \end{bmatrix} \qquad (32)$$

$$= \begin{bmatrix} S_1^j & S_2^j & S_3^j \ldots\ldots & S_n^j \end{bmatrix} \begin{bmatrix} a_1 & 0 & . & 0 \\ a_2 & a_2 & . & 0 \\ . & . & . & . \\ a_n & a_n & . & a_n \end{bmatrix}$$

where:
a_i = The algebraic sum of the all the successful step length in the corresponding direction.
S_i^j = The direction and i= 1,2,3,…,n
P_1 = The vector joining the starting point and the final obtained after the sequence of the searches in the $(j)^{th}$ stage.
$P_2, P_{3,\ldots,} P_n$ = The algebraic sum of the successful step length in the all directions except the first one, and so on these linearly independent vector $P_2, P_{3,\ldots,} P_n$ can be used to generate a new set of orthogonal directions.

b. Set $D_1^j = P_1^j \qquad (33)$
with:

$$S_1^j = \frac{D_1^j}{\sqrt{\left(D_1^j\right)^t . D_1^j}} \qquad (34)$$

c. Compute $D_i^j = P_i^j - \sum_{m=1}^{i-1}\left[\left(\left(P_{m+1}^j\right)^t . S_m^j\right). S_m^j\right] \qquad (35)$

with:

$$S_i^j = \frac{D_1^j}{\sqrt{\left(D_i^j\right)^t . D_i^j}}$$

5. Take the best point obtained in the present stage, and repeat the same procedure of searching from step (2).

i.e. $new\,[x^j_{L,i}] = old\,[x^j_{L,i}] + \lambda_i\,[S^j_{L,i}] \qquad (36)$
where:
L = variable index, L=1,2,….,n
i = direction index, i=1,2,......,n

j = stage index.

6. The procedure terminates when the convergence criterion between initial and final value of the objective function is satisfied.

Case Study Geodetic Network

A trilateration network part of Mosul Dam Network was selected as a case study for the purpose of application of the proposed optimization network technique. The network consists of 19 points, six of the observations pillars are on the dam body, and 58 designed distance with standard deviation equal to 5*mm* as shown in **Fig. 5, Table 1** lists the initial coordinates of the network points. This initial coordinates were based on the epoch 36. **Table 2.** lists the designed distances of the network. The designed distances were derived from the observed distances of Mosul Dam Network in epoch 36 from **AL-Kanani, Y. H., 2009.** The designed distances difference slightly from the observed distance for study purpose. Additional designed distances were added to make a difference in the original network and to strengthen the network to get the more realistic control points. These distances were from P44 to P54, P55 to P3, P42 to P52, P42 to P61, P01 to P71, P61 to P71, and from P71 to P73.

ZOD Application to the Case Study

During the design stage of the geodetic networks, a selection of some control points must be made. The ZOD used to select the best points which can be detected to be the control points for the network of the case study. A program using MATLAB language was prepared to solve ZOD as shown in **Fig. 6.** which shows a flow chart describing the steps of free network program, this program has a capability of:

1. Compute the free network adjustment by calculating the general inverse or the so-called false inverse through Helmert Method.
2. Compute the post-analysis for the adjustment including variance-covariance matrix and compute the elements of ellipse of error.
3. Plot of the points and the ellipse of errors for each point in the network.

The results of application of the ZOD, semi-major, semi-minor axis and area of the ellipse of errors, for each point presented in **Table 3**. **Fig. 7** presents the ellipses of error for each point in the network exaggerated by a factor of 30000. The ellipse of error for the network points varies between 24.50mm^2 and 146.24 mm^2. Points P66, P55, and P73 have the smallest ellipse of error compared to the other points. Because of, Point P66 is lie on the dam body, therefore; points P55 and P73 considered as the best points and selected as control points.

FOD with High Precision

First Order Design problem is one of the classical design problems that face surveyors. By using FOD one can determinate the optimum geometric design for a geodetic network through optimizing station positions. The variables in FOD problem are the observations "design matrix" or (B-Matrix geometry of the network). To solve FOD problem, one of the optimization methods was selected that is "Rosenbrock Method". A program was prepared using MATLAB language to solve FOD with precision using Rosenbrock Method based on the equations from (32) to (36), and using least square techniques. The two objective functions to solve FOD with high precision are:

1. A-Optimality:

$$\text{Objective function} = \text{Min. trace}\ (Q_{xx}) \qquad (37)$$

Constraint: $E_L \le E_i \le E_U$
$N_L \le N_i \le N_U$

2. D-Optimality:

$$\text{Objective function} = \text{Min. det.}\ (Q_{xx}) \qquad (38)$$

Constraint: $E_L \le E_i \le E_U$
$N_L \le N_i \le N_U$

where:

$\sum_{xx} = \sigma_o^2\ Q_{xx}$,
$Q_{xx} = (B^t w B)^{-1}$,
$\sigma_o^2 = 1$ in design stage,
E_i and N_i = coordinates of points,
i = number of point,
E_L and E_U= lower and upper limit in X-direction for each point, and
N_L and N_U= lower and upper limit in Y-direction for each point.

A program was prepared using MATLAB language to solve A-Optimality and D-Optimality. FOD applying to the network of case study after applying ZOD and selecting the points P55 and P73 as control points, then the FOD was applied to the network of case study in three cases. The case which satisfies the best condition of precision will be identified as the optimal design of the micro geodetic network. These cases are:

- The 1st case: considering the dam points are fixed, and the other points are moving.
- The 2nd case: fixing the dam points, and using the resulting points from the ZOD results as control points, these points are P55 and P73.
- The 3rd case: fixing the dam points, and using the points which are used by the general directorate for survey as control points, these points are P42 and P44.

In FOD, different values of range of movement of 50,100, and 300*m* were tried. It was found that when the range of movement of 50 and 100*m* , the movements in the points of the network were not clear . 300*m* as a range of the movement in each direction for each point was selected and assuming no obstacles to exist. The upper and lower limits of movement for each point can write as:

$$E_i^o - 150\ m \le E_i \le E_i^o + 150\ m$$
$$N_i^o - 150\ m \le N_i \le N_i^o + 150\ m$$

The limits of movement for each point of the network in E and N directions computed and presented in **Table 7.** The final coordinates of network points obtained according to the FOD will be compared to the lower and upper limits, obtained below. If the final coordinates within the limits of movement, the solution continue to the next step of calculation of FOD until exceeding the limits of movement of all points. FOD with high precision, high reliability, and high precision and high reliability were applied to the three cases, mentioned above.

FOD with High Precision Application to The Case Study

In FOD with high precision computations, The initial coordinates of points were used to calculate the initial value of the objective function. Then the prepared program of FOD with high precision applied to each case to compute the final coordinates of the points that satisfy the range of movement. The final value of the objective function of each case was computed. To make sure that the errors in the final positions of points are less than the errors in the initial positions and the errors not distributed between points of the network, area of the ellipse of errors has been added to the computations. The results of applying FOD with high precision to the three cases were as follows:

1. The 1st case: the initial value of the objective function was $905348144335858*10^{-17}$, which is equal the trace of Q_{xx} matrix with [26×26] and the final value of the objective function was $672977706671926*10^{-17}$. This indicates that there was an improvement in the value of the objective function. **Table 4.** lists the initial and final easting and northing coordinates for each point with their standard deviations and area of ellipse of error for each point. It can be noticed that the improvement in the standard deviation in most of points and this lead to improvement the area of the ellipse of error and all that indicates the improvement in precision, see **Fig. 8.**

2. The 2nd case: the initial value of the objective function, A-Optimality, was $767947323337789*10^{-17}$, for a trace of Q_{xx} matrix with [22×22] and the final value of the objective function was $558262174843364*10^{-17}$. **Table 5** presents the results of application of the FOD with high precision. The value of the objective function and the results of application of the FOD indicated that there was an enhancement in the value of the objective function, in the standard deviation in most of points, and in the area of the ellipse of error. All of these enhancements of sure will be reflected on the improvement of precision, see **Fig. 9.**

3. The 3rd case: the initial value of the objective function, A-Optimality, was $68147035762434*10^{-17}$, which is equal the trace of Q_{xx} matrix with [22×22] and the final value of the objective function was $496224444075659*10^{-17}$. The results of applying FOD with high precision to the 3rd case are list in **Table 6.** that lists the final coordinates of network points, the standard deviation, and area of ellipse of error for each point. There was an improvement in the value of the objective function, in the standard deviation in most of points, and in the area of the ellipse of error. With these enhancements the network becomes more precision than the initial network, see **Fig. 10.**

The objective function values of the 2nd case and the 3rd case was better than the objective function values of the 1st case. The results of FOD with high precision of the 3rd case were better than that of the 2nd case. Therefore, the 3rd case network

was considered as optimal design due to its high precision.
Different values of the constants α and β of the Rosenbrock Method than the used values of 3 and 0.5 respectively, in the above analysis were investigated. These values were α=1, 1.5, and 2, and β=0.1 and 0.25. Results showed that these values gave almost similar results of that when α=3 and β=0.5 were used. The precision of all cases were analyzed by using D-Optimality as objective function, which means the determinates of Q_{xx} matrix. The results of applying FOD with high precision to all cases using D-Optimality indicate that there was an improvement in the precision. The obtained improvement is exactly as that obtained by applying FOD with high precision using A-Optimality.

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List Of Symbols

Symbol	Symbol
B	Coefficient matrix.
C	Constraints matrix.
COMD	Combined Design.
E_L and E_U	Lower and upper limit in X-direction for each point
FOD	First Order Design.
g	Constant terms in constraint equation.
GDS	General Directorate for Survey.
n	No. of observations.
N_L and N_U	Lower and upper limit in Y-direction for each point
Q_{xx}	Covariance matrix
S_i^j	The direction and $i = 1,2,3,........,n$
SOD	Second Order Design.
TOD	Third Order Design.
v_s	Residual of observation.
w	The weight matrix.
ZOD	Zero Order Design
$\sum_{xx}$	Variance-covariance matrix of adjusted unknowns.
α_p, α_r and α_c	Weight coefficients for precision, reliability and cost.
θ	The angle of ellipse orientation.
ρ''	206265
σ_o^2	Variance of unit weight.

Table 1. The initial coordinates of the network points.

Point	Easting, *m*	Northing, *m*	Notes
P61	70384.0000	49113.0000	on Dam body
P62	70795.0000	48989.0000	on Dam body
P63	71178.0000	48873.0000	on Dam body
P64	71571.0000	48754.0000	on Dam body
P65	71963.0000	48636.0000	on Dam body
P66	72272.0000	48402.0000	on Dam body
P51	69547.0000	49265.0000	
P52	69856.0000	48865.0000	
P53	70582.0000	48516.0000	
P54	71138.0000	47976.0000	
P55	71754.0000	47914.0000	
P71	69904.0000	49524.0000	
P72	70620.0000	49999.0000	
P73	72344.0000	48898.0000	
P01	68672.0000	49162.0000	
P32	71705.0000	50937.0000	
P03	73417.0000	47589.0000	
P42	70191.3921	48310.7710	
P44	71851.7091	47153.8040	

Table 2. The designed distances.

Station Occupied	Station Sighted	Distance, *m*	Station Occupied	Station Sighted	Distance, *m*
P42	P53	441.8518	P53	P62	518.6125
P42	P66	2083.2556	P53	P63	694.6583
P42	P55	1612.8824	P53	P66	1693.9374
P42	P54	1004.4418	P53	P55	1317.6971
P44	P55	766.7708	P53	P54	774.9212
P44	P66	1317.3135	P54	P55	619.3717
P44	P03	1625.3395	P54	P63	897.8043
P03	P44	1625.3431	P54	P64	890.3940
P03	P66	1404.1610	P54	P65	1056.1644
P03	P73	1692.5980	P54	P66	1211.3027
P72	P51	1299.9509	P55	P53	1317.6951
P72	P32	1434.6604	P55	P64	860.3537
P72	P61	916.6018	P55	P65	751.2943
P72	P63	1256.2973	P55	P66	711.2337
P72	P52	1367.2003	P61	P62	429.9238
P73	P32	2137.0400	P62	P63	400.1617
P73	P63	1166.4082	P63	P64	409.8950
P73	P64	786.8815	P64	P65	410.0221
P73	P65	462.5056	P65	P66	388.0540
P73	P66	501.2474	P32	P63	2130.4762
P52	P62	946.6672	P32	P66	2598.0246
P52	P61	582.6254	P71	P72	859.0721
P52	P72	1367.1868	P44	P54	1088.1253
P52	P51	505.5532	P55	P03	1694.8752
P52	P01	1221.3664	P42	P52	647.2145
P51	P01	881.4129	P42	P61	825.4356
P51	P71	441.1126	P01	P71	1284.3542
P51	P61	849.7613	P61	P71	631.4236
P51	P53	1277.4954	P71	P73	2519.5667

Table 3. The semi-major, semi-minor axis and area of ellipse of errors for each point in the network.

Point	Semi-major axis, *m*	Semi-minor axis, *m*	Area of ellipse of error, mm^2	Point	Semi-major axis, *m*	Semi-minor axis, *m*	Area of ellipse of error, mm^2
P61	0.0043	0.0034	45.93	P55	0.0034	0.0028	29.91
P62	0.0064	0.0035	70.37	P71	0.0057	0.0030	53.72
P63	0.0036	0.0030	33.93	P72	0.0055	0.0030	51.84
P64	0.0046	0.0035	50.58	P73	0.0035	0.0030	32.99
P65	0.0041	0.0036	46.37	P1	0.0133	0.0035	146.24
P66	0.0030	0.0026	24.50	P32	0.0058	0.0031	56.49
P51	0.0044	0.0029	40.09	P3	0.0068	0.0032	68.36
P52	0.0038	0.0034	40.59	P42	0.0044	0.0030	41.47
P53	0.0046	0.0026	37.57	P44	0.0056	0.0033	58.06
P54	0.0039	0.0029	35.53				

Table 4. The results of FOD with high precision in 1st case.

Point	Easting, *m*		Northing, *m*		σ_E, *m*		σ_N, *m*		Area of ellipse of error, mm^2	
	Initial	Final	Initial	Final	Initial	Final	Initial	Final	Initial	Final
P51	69547.0000	69631.8139	49265.0000	49258.8553	0.0036	0.0036	0.0070	0.0058	73.26	65.94
P52	69856.0000	69748.9006	48865.0000	48763.9506	0.0033	0.0033	0.0049	0.0048	49.44	50.71
P53	70582.0000	70485.2286	48516.0000	48422.7215	0.0027	0.0027	0.0037	0.0036	32.41	32.76
P54	71138.0000	71047.7465	47976.0000	47888.4522	0.0031	0.0031	0.0032	0.0032	29.86	30.47
P55	71754.0000	71668.9221	47914.0000	47831.2088	0.0032	0.0032	0.0028	0.0027	28.98	28.95
P71	69904.0000	69823.3478	49524.0000	49649.1831	0.0042	0.0040	0.0066	0.0060	81.60	78.61
P72	70620.0000	70645.6995	49999.0000	50049.1453	0.0062	0.0060	0.0034	0.0033	65.01	62.17
P73	72344.0000	72491.2871	48898.0000	49042.3992	0.0027	0.0029	0.0035	0.0032	30.56	33.10
P01	68672.0000	68813.7930	49162.0000	49208.4240	0.0041	0.0042	0.0186	0.0129	239.93	171.91
P32	71705.0000	71842.0263	50937.0000	50869.5914	0.0077	0.0073	0.0030	0.0031	73.96	71.95
P03	73417.0000	73350.6781	47589.0000	47719.5048	0.0053	0.0044	0.0081	0.0072	95.12	83.26
P42	70191.3921	70319.8802	48310.7710	48247.4835	0.0032	0.0032	0.0042	0.0039	41.69	41.27
P44	71851.7091	71841.3424	47153.8040	47276.6001	0.0068	0.0057	0.0035	0.0037	76.29	67.45

Table 5. The results of FOD with high precision in 2nd case.

Point	Easting, *m*		Northing, *m*		σ_E, *m*		σ_N, *m*		Area of ellipse of error, mm^2	
	Initial	Final	Initial	Final	Initial	Final	Initial	Final	Initial	Final
P51	69547.0000	69632.2526	49265.0000	49258.8593	0.0036	0.0035	0.0066	0.0054	66.97	60.10
P52	69856.0000	69753.4218	48865.0000	48768.4718	0.0033	0.0033	0.0047	0.0047	46.28	47.01
P53	70582.0000	70489.7498	48516.0000	48427.2428	0.0024	0.0023	0.0034	0.0033	24.96	25.08
P54	71138.0000	71052.2678	47976.0000	47892.9734	0.0026	0.0025	0.0030	0.0030	23.71	24.07
P71	69904.0000	69904.0369	49524.0000	49648.8212	0.0039	0.0042	0.0066	0.0056	73.87	67.92
P72	70620.0000	70543.8264	49999.0000	49998.9320	0.0061	0.0056	0.0034	0.0033	62.33	58.10
P01	68672.0000	68817.0506	49162.0000	49303.9264	0.0041	0.0041	0.0181	0.0125	229.83	161.93
P32	71705.0000	71844.1505	50937.0000	50868.6863	0.0072	0.0069	0.0029	0.0029	64.91	63.73
P03	73417.0000	73349.9212	47589.0000	47720.7375	0.0045	0.0039	0.0070	0.0066	71.32	65.48
P42	70319.8802	70320.8827	48247.4835	48247.0789	0.0028	0.0028	0.0041	0.0038	35.32	34.81
P44	71841.3424	71977.0182	47276.6001	47277.1503	0.0059	0.0051	0.0033	0.0034	60.65	55.65

Table 6. The results of FOD with high precision in 3rd case.

Point	Easting, *m*		Northing, *m*		σ_E, *m*		σ_N, *m*		Area of ellipse of error, mm^2	
	Initial	Final	Initial	Final	Initial	Final	Initial	Final	Initial	Final
P51	69547.0000	69632.2357	49265.0000	49258.8599	0.0036	0.0036	0.0063	0.0053	65.01	58.55
P52	69856.0000	69754.1618	48865.0000	48769.2118	0.0031	0.0031	0.0043	0.0042	40.00	41.33
P53	70582.0000	70490.4898	48516.0000	48427.9827	0.0024	0.0023	0.0034	0.0036	26.21	26.88
P54	71138.0000	71052.0077	47976.0000	47892.7134	0.0025	0.0025	0.0028	0.0027	21.65	21.61
P55	71754.0000	71674.1833	47914.0000	47926.5815	0.0026	0.0023	0.0026	0.0027	21.27	21.07
P71	69904.0000	69828.2941	49524.0000	49671.9188	0.0041	0.0042	0.0065	0.0058	78.22	74.23
P72	70620.0000	70595.8732	49999.0000	49998.9351	0.0060	0.0056	0.0034	0.0033	61.31	57.25
P73	72344.0000	72482.8647	48898.0000	49034.1665	0.0027	0.0027	0.0035	0.0035	29.88	31.43
P01	68672.0000	68805.7023	49162.0000	49293.4201	0.0041	0.0044	0.0176	0.0121	226.34	157.64
P32	71705.0000	71834.2848	50937.0000	50873.3644	0.0075	0.0071	0.0031	0.0031	71.94	69.54
P03	73417.0000	73354.3575	47589.0000	47712.3250	0.0027	0.0027	0.0055	0.0051	45.82	43.88

Table 7. The limits of movement for points of the network.

Point	Limits of E_i^o, *m*		Limits of N_i^o, *m*	
	Lower	**Upper**	**Lower**	**Upper**
P51	69397	69697	49115	49415
P52	69706	70006	48715	49015
P53	70432	70732	48366	48666
P54	70988	71288	47826	48126
P55	71604	71904	47764	48064
P71	69754	70054	49374	49674
P72	70470	70770	49849	50149
P73	72194	72494	48748	49048
P1	68522	68822	49012	49312
P32	71555	71855	50787	51087
P3	73267	73567	47439	47739
P42	70041.3921	70341.3921	48160.7710	48460.7710
P44	71701.7091	72001.7091	47003.8040	47303.8040

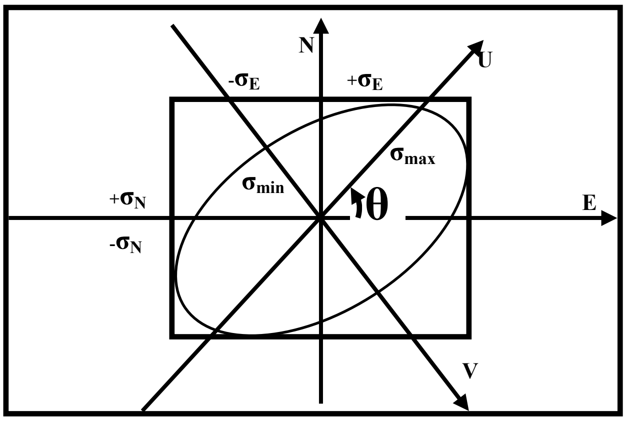

Figure 1. ellipse of error

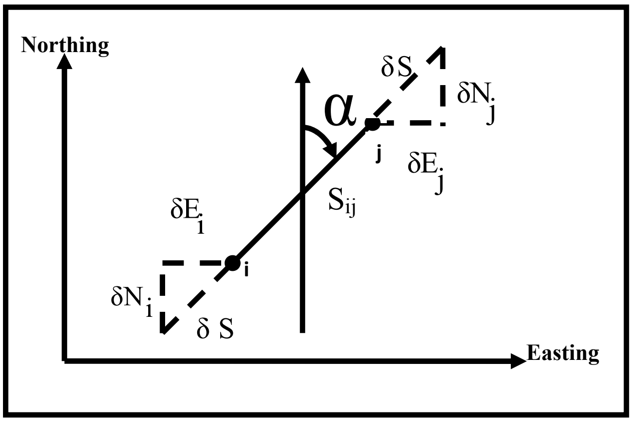

Figure 2. Distance observation.

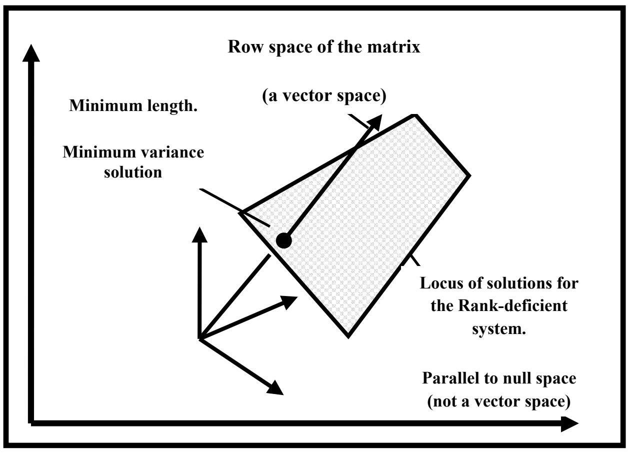

Figure 3. Intersection of solution space and row space for a rank-deficient system, ***after Mikhail, et al., 2000***.

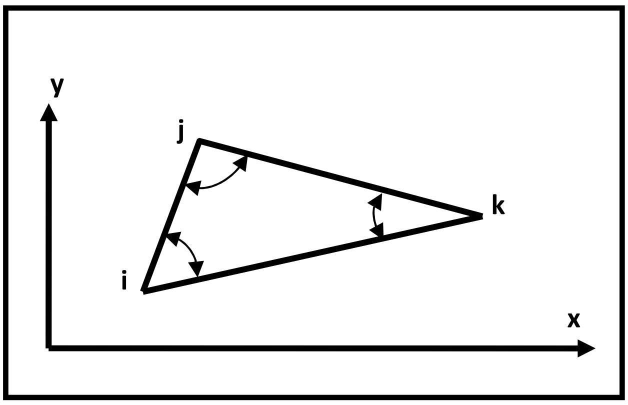

Figure 4. Triangle network with only angle observation

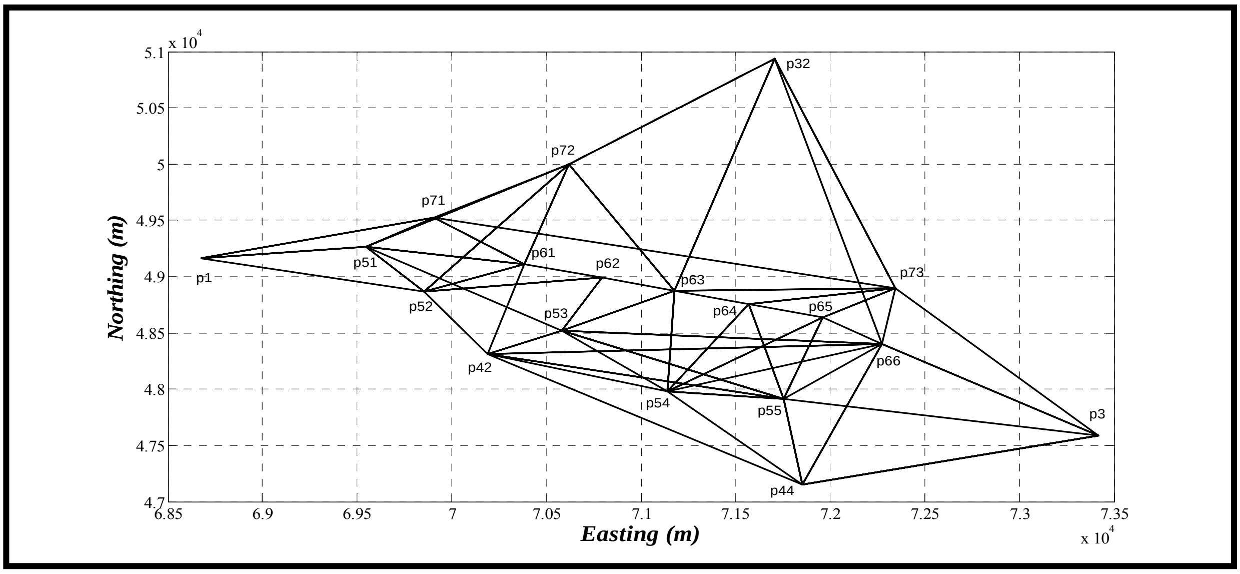

Figure 5. The selected network as a case study

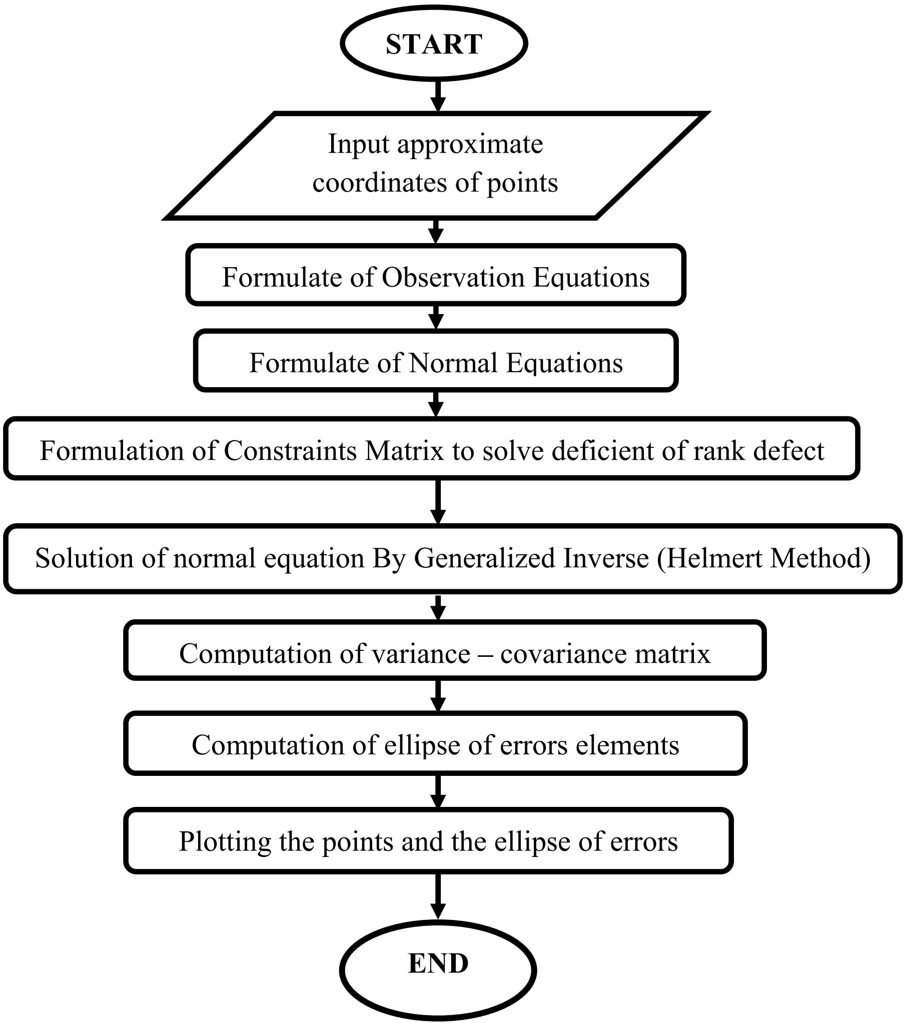

Figure 6. Flow chart of ZOD program.

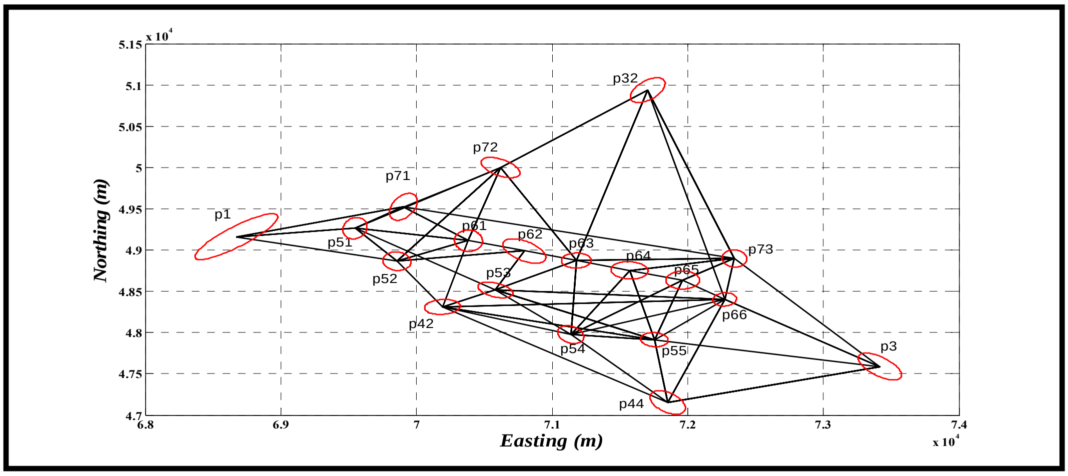

Figure 7. The ellipses of error for each point in the network.

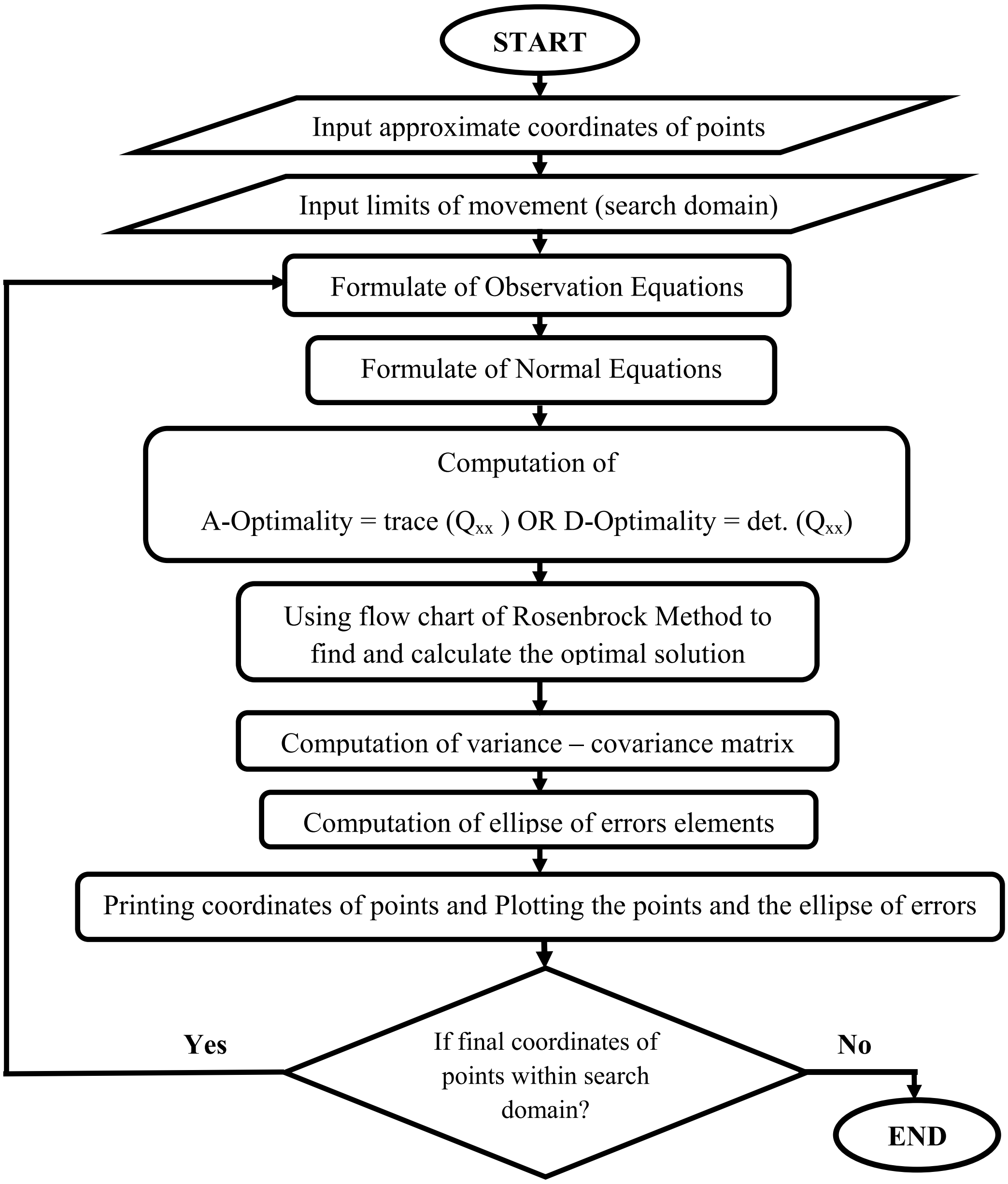

Figure 8. Flow chart of FOD with high precision program.

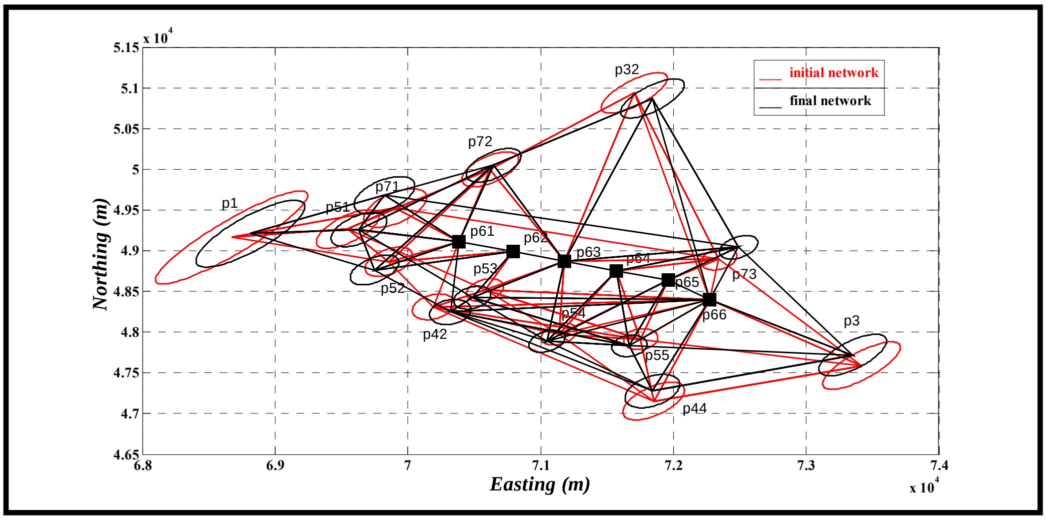

Figure 9. The initial and final network of the 1st case.

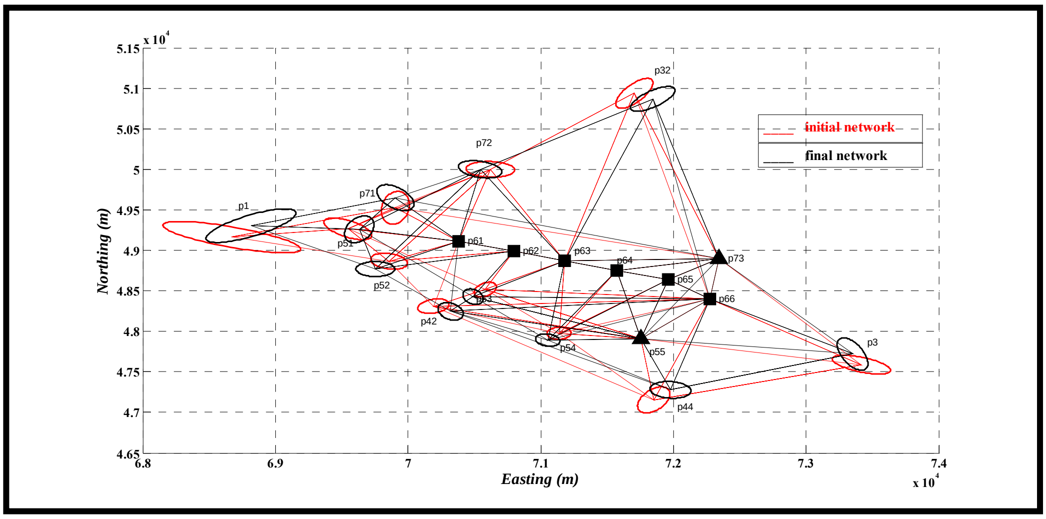

Figure 10. The initial and final network of the 2nd case.

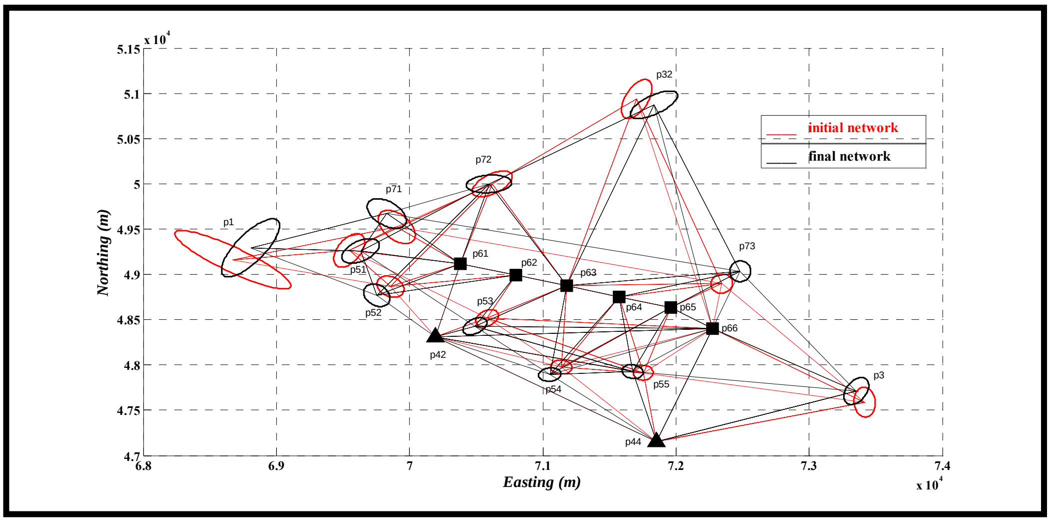

Figure 11. The initial and final network of the 3rd case.