GEODETIC DEFORMATION ANALYSIS

Short Lecture Notes for Graduate Students

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CONTENTS

PREFACE

This short lecture notes aims to give some fundamental subjects in geodetic conventional deformation analysis. It has been written for graduate students who take *Error Theory and Parameter Estimation* and *Adjustment Computation* courses in their Geodesy departments. For reading the notes, it is highly recommended to have knowledge on geodetic network adjustment solutions, especially trace-min, partial trace-min and S-transformation.

The updated version with numerical examples and literature review will appear soon. Your comments on the lecture notes will be appreciated.

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İstanbul, July-2014

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1. INTRODUCTION

Because of acting forces, a physical body may displace from $\mathbf{x}_1$ initial position to $\mathbf{x}_2$ present position in time in 3D space. The displacement vector

$$\mathbf{d}=\mathbf{x}_2-\mathbf{x}_1 \quad (1.1)$$

consists of two parts

- relative part,
- non-relative part.

The relative part, the so-called rigid body displacement, represents **translation** and **rotation**. They are relative because they change depending on where we are observing the body from. For instance, a body may not move or rotate for an observer moving and rotating along the same direction simultaneously with the body. The non-relative part, on the other hand, does not depend on the observer position. This part represents shape change, i.e. **deformation**.

Engineering buildings, such as dams, bridges, tunnels etc., or the Earth's crust are such bodies affected by some physical forces all the time. Monitoring their responses to the forces is an essential task not only for understanding the body mechanism but also for taking some precautions before any possible damage. The responses, which are monitored as afore-mentioned displacements, are very small compared to the body size. Geodetic methods and instruments may overcome this problem sufficiently having provided milimeter accuracy in positioning of the stations distributed in even large areas, and therefore, today they are indispensable in crustal and constructional deformation studies. Although geodetic deformation analysis is now around 30 years old and there are plenty of approaches, techniques, methods etc., the surveying principle is the same. For monitoring the bodies (the "objects" as commonly pronounced in geodesy), we establish a deformation network. There are two types of deformation network,

- absolute deformation networks,
- relative deformation networks.

An absolute deformation network consists of two parts; 1) reference points and 2) object points. The reference points are established in a stable region, the so-called reference block, whereas the object points are located at some specific places of the object such that they are able to characterize the investigated dynamical property of the object itself. Both point groups are predefined in absolute networks. Or, if the reference points and object points are defined after deformation analysis or depending on a prior information beforehand, we call such deformation networks absolute ones. On the other hand, if a deformation network may not be partitioned into two parts beforehand, this type of network is called relative deformation network.

Before realization of a deformation network, a practitioner knows naturally (intuitively or depending on pre-analysis of the studied area) which part is reference block and which part is object. However, after realization we should test whether the reference block has undergone any deformation or not. In other words, the reference points are not exact in a deformation network. Therefore, in theory, all deformation networks are (should be!) described as relative ones unless verifying the reference points by some statistical tests based on the corresponding deformation measurements.

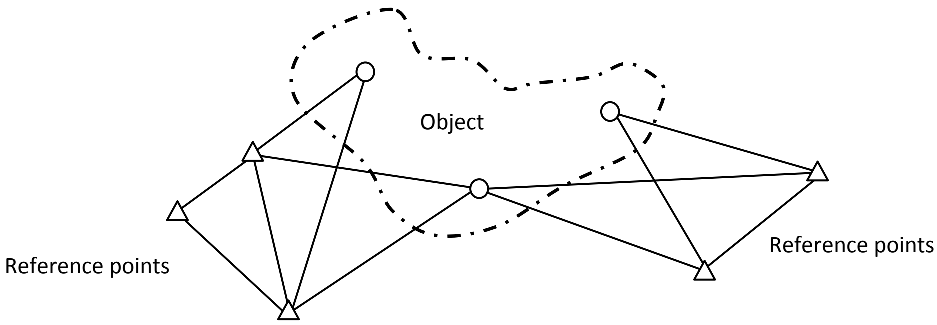

Fig. 1.1. Configuration of an absolute deformation network

In that course we mainly concentrate on conventional (geometrical) deformation analysis for absolute and reference deformation networks, i.e., **global test and localization procedures**; **sensitivity analysis** to derive the capacity of our deformation networks; **kinematic models** to derive velocity and/or acceleration of the bodies which move in time as well as **strain analysis** to interprete the deformation of an object.

2. GLOBAL TEST

Global test is realized for two aims; to learn (or verify)

- *whether whole network has undergone any deformation or not*,
- *whether a part of the network (for instance, reference points) has any deformation or not.*

By global test it is not possible to answer if the corresponding points have translated or rotated as a block (remember these kind of displacements are relative changes), therefore the aim is sometimes expressed as follows;

- *To learn whether the corresponding part has some points whose coordinates have significant changes.*

2.1 Testing Whole Network-Method I

We here desire to test if a whole network has any deformation or not. Let our c-dimensional deformation network with u=cp coordinate unknowns of p points be measured in two periods. Applying trace-minimum solution to each period observations, suppose that we get

$$\mathbf{x}_i = \mathbf{Q}_{x_i x_i} \mathbf{A}_i^T \mathbf{P}_i \mathbf{y}_i \;,\; \mathbf{Q}_{x_i x_i} = (\mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i)^+ = (\mathbf{A}_i^T \mathbf{P}_i \mathbf{A}_i + \mathbf{G}\mathbf{G}^T)^{-1} - \mathbf{G}\mathbf{G}^T \quad \text{(i=1,2)} \qquad (2.1a)$$

as well as

$$\mathbf{v}_i = \mathbf{A}_i \mathbf{x}_i - \mathbf{y}_i \;,\; s_{0,i}^2 = (\mathbf{v}_i^T \mathbf{P}_i \mathbf{v}_i)/f_i \quad \text{(i=1,2)} \qquad (2.1b)$$

where

$\mathbf{A}_i$	:	$n_i \times u$ design matrix with rank$\mathbf{A}_i$=u–r,
$\mathbf{Q}_{x_i x_i}$	:	$u \times u$ cofactor matrix of the unknowns,
$\mathbf{P}_i$	:	$n_i \times n_i$ weight matrix of the observations
$\mathbf{y}_i$	:	$n_i \times 1$ (diminished) observation vector,
$(\ldots)^+$	:	denotes pseudo-inverse,
$\mathbf{v}_i$	:	$n_i \times 1$ residual vector,
$s_{0,i}^2$	:	a-posteriori variance of unit weight,
n_i	:	number of observations,
r	:	number of datum parameters,
f_i	:	degrees of freedom ($f_i=n_i-u+r$) and
$\mathbf{G}^T$	:	$r \times u$ coefficient matrix of constraint equations (datum matrix) to define the datum of the network.

Using the solutions given in Eq. (2.1a), the displacement vector **d** and its cofactor matrix $\mathbf{Q}_{dd}$ are obtained as

$$\mathbf{d}=\mathbf{x}_2-\mathbf{x}_1 \quad , \quad \mathbf{Q}_{dd}=\mathbf{Q}_{x_1x_1}+\mathbf{Q}_{x_2x_2} \tag{2.2}$$

The test procedure depends on discriminating the following null hypothesis (H_0) against its alternative (H_1);

$$H_0\text{: } E(\mathbf{d})=\mathbf{0} \quad \text{vs.} \quad H_1\text{: } E(\mathbf{d})\neq\mathbf{0} \tag{2.3}$$

For this, we have two possible ways;

- F-(Fisher) test
- χ^2-(Chi-square) test.

The test statistics (T) and the threshold values (κ) of these two tests are given as follows:

$$\begin{cases} F \;\rightarrow\quad T = \dfrac{\mathbf{d}^T \mathbf{Q}_{dd}^{+} \mathbf{d}}{h s_d^2} \sim F(h,f) \quad , \quad \kappa = F_{h,f,1-\alpha} \\ \chi^2 \rightarrow\quad T = \dfrac{\mathbf{d}^T \mathbf{Q}_{dd}^{+} \mathbf{d}}{\sigma_d^2} \sim \chi^2(h) \quad , \quad \kappa = \chi^2_{h,1-\alpha} \end{cases} \tag{2.4}$$

where

h	:	rank $\mathbf{Q}_{dd}$ (see Note 2.1),
f	:	total degrees of freedom, i.e., $f=f_1+f_2$,
s_d^2	:	pooled variance factor (see Note 2.2),
α	:	total significance level (Type I error) and
σ_d^2	:	a-priori variance factor (see Note 2.3).

We simply compare the corresponding test statistic T with its threshold value κ in Eq. (2.4). There is two possible outcomes and results;

i) If $T<\kappa$, H_0 is accepted with α significance level. It means that there does not exist any deformation in the network. In other words, the vector **d** of the monitored displacements is only the result of random errors in two periods.

ii) If $T\geq\kappa$, H_1 is accepted with $1-\alpha$ confidence level. Then we conclude that the network has undergone deformation between two periods. Or, we may say that at least one point in the network has significant coordinate change.

Note 2.1: If both periods are identical, then h=rank $\mathbf{Q}_{dd}$ =u−r holds.

Note 2.2: Pooled variance factor is obtained as follows

$$s_d^2 = \frac{f_1 s_{0,1}^2 + f_2 s_{0,2}^2}{f_1 + f_2} = \frac{\mathbf{v}_1^T \mathbf{P}_1 \mathbf{v}_1 + \mathbf{v}_2^T \mathbf{P}_2 \mathbf{v}_2}{f} = \frac{\Omega}{f}$$

However, to consider the pooled variance factor in Eq. (2.4), the ratio "$s_{0,1}^2 / s_{0,2}^2$" should be smaller than $F_{f_1, f_2, 1-\alpha}$ threshold value (Variance test)*. Otherwise, we may not put the pooled variance into Eq. (2.4); in other words, it means that the periods are not proper for any comparison.

*) This is valid if $s_{0,1}^2$ is numerically bigger than $s_{0,2}^2$. Otherwise, the ratio "$s_{0,2}^2 / s_{0,1}^2$" is compared with $F_{f_2, f_1, 1-\alpha}$.

Note 2.3: In statistical point of view, χ^2-test is more powerful than F-test. In other words, the probability of correctly accepting the alternative hypothesis (the power of the test) in χ^2-test is bigger. However, it requires a precise knowledge on the a-priori variance σ_d^2. Since it can be derived from long time experience on the data of the surveying methods applied in the studied area, this requirement may not be ensured always or in a short time. Therefore, commonly F-test is chosen because it needs only the variances from the current measurements of the periods. Herafter, in the test procedures we will consider only F-distributed test statistics.

Note 2.4: For each period, adjustment procedure should be realized with common approximate coordinates in the network. However this may not be ensured everytime because in some cases (for example in monitoring of landslides which may cause big displacements) iterative adjustment is required; so, the approximate coordinates inevitably changes in each iteration. For such cases, instead of using diminished coordinate unknown vectors ($\mathbf{x}_i$), adjusted coordinates of the periods should be considered to estimate the displacement vector in Eq. (2.2).

2.2 Testing Whole Network-Method II

The previous test statistic values are deduced from the theory of “generalized linear hypothesis”. In deformation analysis, there is a second method which substitutes the hypotheses **implicitly** into a corresponding Gauss-Markoff model. It is called therefore implicit hypothesis method.

We may gather the separate adjustment models of the periods in a unique Gauss-Markoff model as

$$E\left\{\begin{pmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \end{pmatrix}\right\} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix}\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \quad , \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix} \tag{2.5}$$

Now we will consider the null hypothesis H_0: E(**d**)=**0** or H_0: E($\mathbf{x_1}$)=E($\mathbf{x_2}$) in model (2.5). For this we write the following model, which implies that “there is no any difference between two periods”;

$$E\left\{\begin{pmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \end{pmatrix}\right\} = \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix}\mathbf{x}_H \quad , \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix} \tag{2.6}$$

Solving model (25) we get the following quadratic form, which is equivalent to the sum of the weighted sum of the squared residuals of the periods,

$$\Omega = \mathbf{v}_1^T \mathbf{P}_1 \mathbf{v}_1 + \mathbf{v}_2^T \mathbf{P}_2 \mathbf{v}_2 = f_1 s_{0,1}^2 + f_2 s_{0,2}^2 \tag{2.7}$$

On the other hand, the solution of model (2.6), which has f_H=f+u–r degrees of freedom, results in

$$\Omega_H = \mathbf{v}_H^T \mathbf{P} \mathbf{v}_H \tag{2.8}$$

If there is no any difference between two periods, the difference between two quadratic forms of the models, i.e., R=Ω_H–Ω, should go to zero. For this the test statistic from "testing of linear hypotheses" is set as follows

$$T = \frac{(\Omega_H - \Omega)/(f_H - f)}{\Omega/f} = \frac{R}{hs_d^2} \sim F(h,f) \tag{2.9}$$

because of f_H–f=u–r=h and (Ω/f)= s_d^2 (Note 2.2). The test statistic is identical with the one of F-test in Eq. (2.4). So, the test procedure is similar.

Note 2.5: In each free adjustment method (trace-min, partial trace-min and minimum-constrained), we get a unique residual vector. Therefore, solution of model (2.6) may be realized by any free adjustment method. Since the minimum-constrained solution is a normal adjustment procedure, it is easy to use this solution in the implicit hypothesis method. For this, arbitrary r columns of the design matrix in model (2.6) are deleted before adjustment.

2.3 Testing Whole Network for Non-Identical Case

A deformation network may be augmented or renovated with newer points in different periods. In that case we should handle with different configurations in the periods to be compared. To make them identical there are two possible ways;

i) The corresponding periods are separately adjusted using the observations connecting the identical points in both periods.

ii) The periods are re-adjusted such that only identical points in the periods define the datum of the network.

The latter is more advantageous because we do not modify the network design. Let us consider that our c-dimensional deformation network with p points is augmented with k newer points in the second period and we have already their trace-minimum solutions;

	Coordinates		Cofactors		
	1st Period	2nd Period	1st Period	2nd Period	
Identical points	$\mathbf{x}_1$ (cp×1)	$\mathbf{x}'_2$ (cp×1)	$\mathbf{Q}_{x_1x_1}$	$\mathbf{Q}_{x'_2x'_2}$	$\mathbf{Q}_{x'_2x'_n}$
New points	–	$\mathbf{x}'_n$ (ck×1)		$\mathbf{Q}_{x'_nx'_2}$	$\mathbf{Q}_{x'_nx'_n}$
		$\underbrace{}_{\tilde{\mathbf{x}}_2}$		$\underbrace{}_{\mathbf{Q}_{\tilde{x}_2\tilde{x}_2}}$	

In the second period, new points should be extracted from the datum definition. We may use S-transformation for doing this. Let c(k+p)×r coefficient matrix of the constrained equations for the second period be had the following form

$$\mathbf{G}_2^{\mathrm{T}} = (\mathbf{G}^{\mathrm{T}} \;\; \mathbf{G}_{\mathrm{n}}^{\mathrm{T}}) \tag{2.10}$$

where $\mathbf{G}_{\mathrm{n}}^{\mathrm{T}}$ is the ck×r datum sub-matrix for the new points. Taking $\mathbf{G}_{\mathrm{n}}^{\mathrm{T}}=\mathbf{0}$ above (in other words, new points are extracted from the datum definition) we set a new coefficient matrix

$$\mathbf{B}_2^{\mathrm{T}} = (\mathbf{G}^{\mathrm{T}} \;\; \mathbf{0}) \tag{2.11}$$

With Eqs. (2.10) and (2.11), the S-transformation matrix is computed as

$$\mathbf{S}_2 = \mathbf{I} - \mathbf{G}_2(\mathbf{B}_2^{\mathrm{T}}\mathbf{G}_2)^{-1}\mathbf{B}_2^{\mathrm{T}} \tag{2.12}$$

Using the matrix $\mathbf{S}_2$ we define a new datum for the second period

$$\mathbf{S}_2\tilde{\mathbf{x}}_2 = \begin{pmatrix} \mathbf{x}_2 \\ \mathbf{x}_{\mathrm{n}} \end{pmatrix} \quad \text{and} \quad \mathbf{S}_2\mathbf{Q}_{\tilde{x}_2\tilde{x}_2}\mathbf{S}_2^{\mathrm{T}} = \begin{pmatrix} \mathbf{Q}_{x_2x_2} & \mathbf{Q}_{x_2x_n} \\ \mathbf{Q}_{x_nx_2} & \mathbf{Q}_{x_nx_n} \end{pmatrix} \tag{2.13}$$

The sub-vector $\mathbf{x}_2$ and the sub-matrix $\mathbf{Q}_{x_2x_2}$ in Eq. (2.13) are now compatible with $\mathbf{x}_1$ and $\mathbf{Q}_{x_1x_1}$. Hence, both periods are identical and ready for comparison as usual.

2.4 Testing a Part of a Network

Our reference points should be stable because we define the cloud of these points as our “observer” to monitor the object. We should therefore verify whether the reference points have undergone any deformation or whether they include any point whose coordinates have changed significantly. This procedure is one of the most important stage in monitoring of the object.

Suppose that $\Re$ and $\otimes$ represent $p_{\Re}$ reference points and $p_{\otimes}=p-p_{\Re}$ object points, respectively. The displacement vector **d** and its cofactor matrix $\mathbf{Q}_{dd}$ in Eq. (2.2) may be written explicitly for these point groups as follows

$$\mathbf{d} = \begin{pmatrix} \tilde{\mathbf{d}}_{\Re} \\ \tilde{\mathbf{d}}_{\otimes} \end{pmatrix} \quad , \quad \mathbf{Q}_{dd} = \begin{pmatrix} \tilde{\mathbf{Q}}_{\Re\Re} & \tilde{\mathbf{Q}}_{\Re\otimes} \\ \tilde{\mathbf{Q}}_{\otimes\Re} & \tilde{\mathbf{Q}}_{\otimes\otimes} \end{pmatrix} \tag{2.14}$$

To test the reference points first we should define the network datum according to them. This may be realized by the following S-transformation

$$\overline{\mathbf{d}} = \mathbf{S}_{\Re}\mathbf{d} = \begin{pmatrix} \mathbf{d}_{\Re} \\ \mathbf{d}_{\otimes} \end{pmatrix} \quad , \quad \overline{\mathbf{Q}}_{dd} = \mathbf{S}_{\Re}\,\mathbf{Q}_{dd}\,\mathbf{S}_{\Re}^{T} = \begin{pmatrix} \mathbf{Q}_{\Re\Re} & \mathbf{Q}_{\Re\otimes} \\ \mathbf{Q}_{\otimes\Re} & \mathbf{Q}_{\otimes\otimes} \end{pmatrix} \tag{2.15}$$

where the transformation matrix $\mathbf{S}_{\Re}$ is set as follows;

$$\mathbf{S}_{\Re} = \mathbf{I} - \mathbf{G}(\mathbf{B}^{T}\mathbf{G})^{-1}\mathbf{B}^{T} \quad \text{with} \quad \mathbf{G}^{T} = (\mathbf{G}_{\Re}^{T}\ \ \mathbf{G}_{\otimes}^{T}) \ \text{ and } \ \mathbf{B}^{T} = (\mathbf{G}_{\Re}^{T}\ \ \mathbf{0}) \tag{2.16}$$

In the test, the sub-vector $\mathbf{d}_{\Re}$ and the sub-matrix $\mathbf{Q}_{\Re\Re}$ in Eq. (2.15) are used: The test statistic having F-distribution, similar to the one in Eq. (2.4), is given as follows

$$T_{\Re} = \frac{\mathbf{d}_{\Re}^{T}\mathbf{Q}_{\Re\Re}^{+}\mathbf{d}_{\Re}}{h_{\Re}s_{d}^{2}} \sim F(h_{\Re}, f) \tag{2.17}$$

where $h_{\Re}$=h–c(p–$p_{\Re}$)=rank$\mathbf{Q}_{\Re\Re}$. The test statistic is compared with the threshold value $F_{h_{\Re},f,1-\alpha}$;

i) If $T_{\Re} < F_{h_{\Re},f,1-\alpha}$, then our reference points are accepted as stable with α error.

ii) If $T_{\Re} \geq F_{h_{\Re},f,1-\alpha}$, there is at least one point whose coordinates has changed significantly. In that case, we should find the responsible point(s) and extract it (them) from the reference definition. One possible way for point detection is adding each reference point one by one to the point cloud ⊗ in Eq. (2.14) in each time and repeat the above test procedure for the remaining reference points until the test statistic becomes smaller than the corresponding threshold value. Other possible way is applying localization procedure, which is given in Section 3.2.3.

Note 2.6: The degrees of freedom $h_{\mathfrak{R}}$ must be bigger than 0, i.e., $h_{\mathfrak{R}} \geq 1$. Since there may be a significantly changed point among them, we should take into account $h_{\mathfrak{R}} \geq 2$. This natural limitation gives information about the minimum number of reference points ($p_{\mathfrak{R}}$) for a network: From the inequality, we get

$$h_{\mathfrak{R}} = h - c(p - p_{\mathfrak{R}}) = cp - r - cp + cp_{\mathfrak{R}} = cp_{\mathfrak{R}} - r \geq 2.$$

Then we see that $p_{\mathfrak{R}}$ should be equal to $(2+r)/c$ in a worst case. This means that our absolute deformation networks (1, 2 or 3D) should be designed so that it consists of approximately at least 3 reference points.

3. TESTING OBJECT POINTS

3.1 Testing Object Points in Absolute Deformation Networks

As we mentioned previously, an absolute deformation network has two parts; reference points and object points. The reference points should be verified by the global test given in Section 2.2 such that they can be defined as "observer" to monitor the object.

Let our reference points $\Re$ be already verified as stable. Then, each object point may be tested to learn whether its observed displacement relative to the reference points is significant or not by using the following test statistic

$$T_{\otimes_i} = \frac{\mathbf{d}_{\otimes_i}^T \mathbf{Q}_{\otimes_i \otimes_i}^{-1} \mathbf{d}_{\otimes_i}}{c s_d^2} \sim F(c,f) \quad \forall i=1,\ldots,p_\otimes \tag{3.1}$$

where

$\mathbf{d}_{\otimes_i}$	:	$c \times 1$ ith object point's displacement vector, which is the corresponding sub-vector of the object displacement vector $\mathbf{d}_\otimes$ in Eq. (2.15),
$\mathbf{Q}_{\otimes_i \otimes_i}$	:	$c \times c$ cofactor matrix of the ith object point displacements, which is the corresponding sub-matrix of the object cofactor matrix $\mathbf{Q}_{\otimes\otimes}$ in Eq. (2.15) and
c	:	dimension of the corresponding network.

Each test statistic in Eq. (3.1) is compared with the threshold value $F_{c,f,1-\alpha}$;

i) If $T_{\otimes_i} < F_{c,f,1-\alpha}$, the corresponding displacement is not significant.

ii) If $T_{\otimes_i} \geq F_{c,f,1-\alpha}$, it is accepted that the corresponding displacement is significant with $1-\alpha$ confidence level.

Note 3.1: The above given test procedure is equivalent to "relative confidence interval/ellipse/ellipsoid" method applied in deformation analysis studies. For example, let us consider 2D cases: First, the displacement vector ($\mathbf{d}_{\otimes_i}$) of a corresponding point is plotted on a map and, then, the confidence ellipse obtained from $\mathbf{Q}_{\otimes_i \otimes_i}$ (this ellipse is called "relative" in deformation analysis) is centered on the end point of the displacement vector (see Fig 3.1). If the displacement vector on the plot remains outside of the ellipse, than this displacement of the corresponding point is said to be significant with the preassumed probability of confidence level.

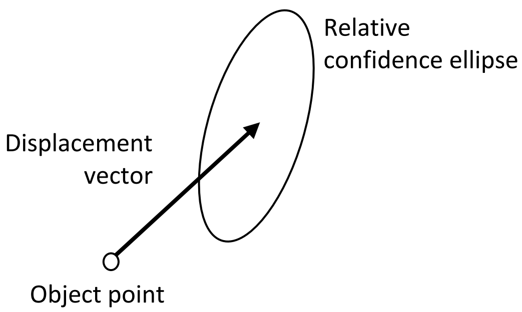

Fig 3.1. Displacement vector and relative confidence ellipse

Note 3.2: If our reference points are verified as undeformed (stable) by the global test in Section 2.4, this does not mean that we also verified that they has not moved in time as a block. It means that the observed displacements of the object points with the above testing procedure carry the relative effect of the reference points if they has undergone a rigid body displacement. This fact may not be so important for some studies, for instance in tectonic/velocity studies, which are mostly based on some relative functions. However, for some cases, for example in damage monitoring of engineering buildings, it may be a problematic issue because an analysist may interprete mistakenly that the object is under a damaging force system. To be clear and not to cause a wrong or over interpretation, two different/independent relative blocks may be chosen in the studied area, if it is possible. Depending on these two relative-tested blocks two different results are obtained for the object points. Then we may check our object's displacements by comparing two results. Of course, there will be some statistical unbalancies between two results, but it may be ignorable effect compared to early or wrong emergency alarm for a possible damage.

For 1D networks, i.e., levelling and gravity monitoring networks, square root of the test statistic $T_{\otimes_i}$ in Eq. (3.1) becomes t (Student)-distributed, because of the probability distribution function properties,

$$\sqrt{T_{\otimes_i}} = \sqrt{\frac{d_{\otimes_i}^2 Q_{\otimes_i \otimes_i}^{-1}}{s_d^2}} = \frac{|\, d_{\otimes_i} \,|}{s_d \sqrt{Q_{\otimes_i \otimes_i}}} \sim t(f) \qquad (3.2)$$

where

$d_{\otimes_i}$ and $Q_{\otimes_i \otimes_i}$: the ith point's displacement and its cofactor, respectively

The threshold value is then taken as $\sqrt{F_{c=1,f,1-\alpha}} = t_{f,1-\alpha/2}$ which denotes α-percentage point of the t-distribution.

3.2 Localization

Localization is a procedure to identify the points having significant coordinate changes. Mostly it is adapted in relative deformation networks to separate reference points and object points, however it may be used also in absolute deformation networks to detect the disturbing point(s) in the reference point set.

There are three localization methods commonly applied in deformation analysis;

i) Gauss-elimination method

ii) Implicit hypothesis method

iii) S-transformation method

We discuss only first two of them for relative deformation networks. Afterwards, the localization procedure in absolute deformation networks by Gauss-elimination method is explained.

3.2.1 Localization with Gauss-elimination method

If the global test shows significantly changed points in our relative deformation network, the next step is identifying or localizing these points. For each point in our c-dimensional network we compute the following effect values

$$R_i=\delta_i^T \tilde{\mathbf{P}}_{ii}^{-1} \delta_i \quad \text{with} \quad \delta_i = \tilde{\mathbf{d}}_i + \tilde{\mathbf{P}}_{ii}^{-1} \tilde{\mathbf{P}}_{iA} \tilde{\mathbf{d}}_A \quad , \quad (i=1,2,\ldots,p) \qquad (3.3)$$

where

δ_i : c×1 reduced displacement vector of the ith point,

$\tilde{\mathbf{d}}_i$: c×1 displacement vector of the ith point

$\tilde{\mathbf{P}}_{ii}$: c×c weight matrix belonging to the ith point (ith block diagonal of $\mathbf{P} = \mathbf{Q}_{dd}^{+}$),

$\tilde{\mathbf{d}}_A$: (cp–c)×1 vector of displacements of the remaining points denoted A not including the point i and,

$\tilde{\mathbf{P}}_{iA}$: c×(cp–c) weight matrix between point i and the remaining points (see Note 3.3).

Note 3.3: Displacement vectors and their weight matrices used in Eq.(3.3) may be represented as

$$\mathbf{d} = \begin{pmatrix} \tilde{\mathbf{d}}_A \\ \tilde{\mathbf{d}}_i \end{pmatrix}, \ \mathbf{P}_{dd} = \mathbf{Q}_{dd}^{+} = \begin{pmatrix} \tilde{\mathbf{P}}_{AA} & \tilde{\mathbf{P}}_{iA} \\ \tilde{\mathbf{P}}_{iA} & \tilde{\mathbf{P}}_{ii} \end{pmatrix} \quad (3.4)$$

The point resulting in maximum effect value by Eq. (3.3) is accepted as significantly changed point. Let this point be the jth point. Then first object point is being defined;

$$\otimes = \{ j \} \quad (3.5)$$

Now we should define the datum of our network depending on the remaining p–1 points. Let us denote these remaining points with B. Using the transformation matrix $\mathbf{S}_B$ defining the datum according to the points B, we obtain

$$\mathbf{S}_B \mathbf{d} = \begin{pmatrix} \tilde{\mathbf{d}}_B \\ \tilde{\mathbf{d}}_\otimes \end{pmatrix} \quad , \quad \mathbf{S}_B \, \mathbf{Q}_{dd} \, \mathbf{S}_B^T = \begin{pmatrix} \tilde{\mathbf{Q}}_{BB} & \tilde{\mathbf{Q}}_{B\otimes} \\ \tilde{\mathbf{Q}}_{\otimes B} & \tilde{\mathbf{Q}}_{\otimes\otimes} \end{pmatrix}, \quad (3.6)$$

Our next aim is to investigate the remaining points B. If the test statistic

$$T_B = \frac{\tilde{\mathbf{d}}_B^T \tilde{\mathbf{Q}}_{BB}^+ \tilde{\mathbf{d}}_B}{h_{\mathbf{B}} s_d^2} \sim F(h_B, f) \quad , \quad (h_B = \text{rank}\, \tilde{\mathbf{Q}}_{BB} = \text{rank}\, \mathbf{Q}_{dd} - c) \tag{3.7}$$

is smaller than the threshold value $F_{h_B, f, 1-\alpha}$, the localization procedure is ended. The object points and the reference points are being separated already as $\otimes=\{ j \}$ and $\Re=B$, respectively. Otherwise, it is decided that the group of points B has significantly changed point(s) and a new localization procedure is required. In that case, we consider $\tilde{\mathbf{d}}_B$ and $\tilde{\mathbf{Q}}_{BB}$ as **d** and $\mathbf{Q}_{dd}$ in Eq. (3.4),

$$\tilde{\mathbf{d}}_B \rightarrow \mathbf{d} \quad , \quad \tilde{\mathbf{Q}}_{BB} \rightarrow \mathbf{Q}_{dd}, \tag{3.8}$$

and we search the point giving the maximum effect using Eq.(3.3) among the remaining p–1 points. The procedure is repeated until the global test shows no-more significantly changed points. In each localization step the object point set $\otimes$ in Eq. (3.5) is augmented with the newer identified points.

For 1D networks, the effect R_i in Eq. (3.3) may be computed directly: Let the displacement vector and its weight matrix given by Eq. (3.4) be written as follows

$$\mathbf{d} = \begin{pmatrix} \tilde{d}_1 \\ \vdots \\ \tilde{d}_n \end{pmatrix} \quad , \quad \mathbf{P}_{dd} = \mathbf{Q}_{dd}^+ = \begin{pmatrix} \tilde{P}_{11} & \cdots & \tilde{P}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{P}_{n1} & \cdots & \tilde{P}_{nn} \end{pmatrix} \tag{3.9}$$

The multiplication $\mathbf{dP}_{dd}$ results in

$$\delta = \mathbf{dP}_{dd} = \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}, \tag{3.10}$$

Dividing the elements of the vector in (3.10) by the weights in (3.9) gives the corresponding effect

$$R_i=\delta_i/\tilde{P}_{ii} \quad , \quad (i=1,2,....,p) \qquad (3.11)$$

Hence, for 1D networks, the computation burden of Eq. (3.6) is drastically being reduced.

3.2.2 Localization with implicit hypothesis method

Two periods' Gauss-Markov models are gained into a single Gauss-Markov model by Eq. (2.5) as

$$E\left\{\begin{pmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \end{pmatrix}\right\} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix}\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \quad , \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix}$$

Our hypothesis now assumes that the group of points A, which does not contain the suspected point i, has not deformed. This hypothesis may be implicity incorporated into the above Gauss-Markov model as follows

$$E\left\{\begin{pmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \end{pmatrix}\right\} = \begin{pmatrix} \mathbf{A}_{A1} & \mathbf{A}_{i1} & \mathbf{0} \\ \mathbf{A}_{A2} & \mathbf{0} & \mathbf{A}_{i2} \end{pmatrix}\begin{pmatrix} \mathbf{x}_A \\ \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{pmatrix} \quad , \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix} \qquad (3.12)$$

Suppose that the solution of model (3.12) yields the weighted sum of the squared residuals $(\Omega_H)_i = (\mathbf{v}_H^T\mathbf{P}\mathbf{v}_H)_i$. Each point in the network is attained as i in model (3.12) in turn, and we get p effects for the points in the network as follows

$$R_i=(\Omega_H)_i-\Omega \quad , \quad (i=1,2,...,p) \qquad (3.13)$$

If R_j belonging to the jth point is the minimum effect value among the others, this point is accepted as the point with significant coordinate change. It is our first object point:

Then we may define the object point group $\otimes$ as

$$\otimes=\{ j \} \tag{3.14}$$

Now we should learn whether the remaining p–1 points, denoted B, still consist of any significantly changed points. For this we use the minimum deficiency R_j to set the following test statistic

$$T_B = \frac{(R_j/h_B)}{(\Omega/f)} = \frac{R_j}{h_B s_d^2} \sim F(h_B, f) \tag{3.15}$$

where $h_B=f_H-f-c=h-c$. If the test statistic in Eq. (3.15) is bigger than the corresponding threshold value, i.e.

$$T_B \geq F_{h_B,f,1-\alpha}, \tag{3.16}$$

we should identify the responsible point(s) among the group of points B. For the ith point of B, we set our Gauss-Markov model as follows;

$$E\left\{\begin{pmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \end{pmatrix}\right\} = \begin{pmatrix} \underbrace{\mathbf{A}_{D1} \quad \mathbf{A}_{i1} \quad \mathbf{0}}_{B} & \mathbf{A}_{\otimes 1} & \mathbf{0} \\ \mathbf{A}_{D2} \quad \mathbf{0} \quad \mathbf{A}_{i2} & \mathbf{0} & \mathbf{A}_{\otimes 2} \end{pmatrix} \begin{pmatrix} \mathbf{x}_D \\ \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \\ \mathbf{x}_{\otimes 1} \\ \mathbf{x}_{\otimes 2} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix} \tag{3.17}$$

where

D	:	denotes the points except the ith point in B ($D \cup \{i\}=B$)
$\otimes$	:	shows the object point identified in the previous localization step (we do not change of the place of $\otimes$ in Eq. (3.17) during the all localization steps anymore).

Now in the second localization step, each point in B is attained as i in Eq. (3.17) by turn and the effects of all p–1 points are computed as realized by Eq. (3.13). The point giving minimum effect is taken as the new object point and we update our object point definition $\otimes$ in Eq. (3.14). If the global test, which is set according to the effect of the point identified in that second localization step (h_B becomes h–2c), shows more suspected points among the remaining points, similarly we continue localizing these points. The procedure is repeated until the corresponding global test shows no-more significantly changed point in the network.

3.2.3 Localization in absolute deformation networks

If the global test in Section 2.4 results in that our reference points $\Re$ is not stable, we should identify the changed point(s) among them. One way for doing this is applying localization procedure to the reference points: For this we consider $\mathbf{d}_{\Re}$ and $\mathbf{Q}_{\Re\Re}$ belonging to the $p_{\Re}$ reference points in Eq. (2.15) as **d** and $\mathbf{Q}_{dd}$ in (3.4),

$$\mathbf{d}_{\Re} \rightarrow \mathbf{d} \quad , \quad \mathbf{Q}_{\Re\Re} \rightarrow \mathbf{Q}_{dd} \tag{3.18}$$

and we start localization procedure with Eq. (3.3) to identify the responsible point(s) among $p_{\Re}$ reference points. The localization procedure is similarly realized until the corresponding global test shows that remaining reference points have no any points disturbing the stability of our reference. The identified points are removed from the reference point set and added to the object points.

4. SENSITIVITY ANALYSIS

Sensitivity analysis is used to optimize a deformation network such that it becomes sensitive to the expected displacement, movement or deformation or to derive the minimum detectable displacement, movement or deformation parameter for measuring the quality of our design.

In theory it can be adapted to all kind of changes to be monitored in a studied area, however we express it just for displacements here.

4.1 Global Sensitivity Analysis

4.1.1 Optimization Criteria

The hypotheses given by Eq. (2.3) is originally set as

$$H_0: E(\mathbf{d})=\mathbf{0} \quad \text{vs.} \quad H_1: E(\mathbf{d})\neq\mathbf{0}=\Delta \qquad (4.1)$$

where

Δ : $u\times1$ vector of expected displacements.

Because we are now at the design stage, we know that the alternative hypothesis H_1 in Eq. (4.1) is true. In that case, second test statistic in Eq. (2.4) follows a non-central χ^2-distribution;

$$T=\frac{\mathbf{d}^T\mathbf{Q}_{dd}^{+}\mathbf{d}}{\sigma_d^2}\sim\chi'^2\,(h,\lambda) \qquad (4.2)$$

where λ is the non-centrality parameter computed from the vector of expected displacements Δ as follows;

$$\lambda = \frac{\Delta^{T}\mathbf{Q}_{dd}^{+}\Delta}{\sigma_{d}^{2}} \tag{4.3}$$

Once we obtain the non-centrality parameter in Eq. (4.3), the power of the global test, i.e., the probability of correcty accepting the true alternative hypothesis, may be computable from the distribution function of the non-central χ^2-distribution, $F(\chi'^2=\chi^2_{h,1-\alpha};h, \lambda)$, as

$$\gamma=1-F(\chi'^2=\chi^2_{h,1-\alpha};h, \lambda) \tag{4.4}$$

The power of the test γ mathematically increases with increasing non-centrality parameter λ and with decreasing degrees of freedom h. It means that the power of our test will be better for more precise network and less points (remember that h is related with number of points in an identical network, i.e. $h=u-r=cp-r$).

Instead of computing the power of the test by Eq. (4.4), the non-centrality parameter is compared with a non-centrality parameter giving a desired power of the test γ_0 and significance level α_0. This parameter is called lower bound of the non-centrality parameter λ_0 (see Table 4.1). If the following inequality is fulfilled,

$$\lambda \geq \lambda_{0,h} \tag{4.5}$$

the network is defined as "sensitive to the expected displacements". Otherwise, the network is re-designed such that it becomes sensitive.

Table 4.1 Some lower bound of the non-centrality parameters ($\lambda_{0,h}$) for γ_0=80 and 90%, α_0= 5% and 1≤h≤100

h	γ_0=80%	γ_0=90%
1	7.85	10.51
2	9.64	12.65
3	10.90	14.17
4	11.94	15.41
5	12.83	16.47
10	16.24	20.53
20	20.96	26.13
30	24.55	30.38
40	27.56	33.94
50	30.20	37.07
100	40.56	49.29

4.1.2 Minimum detectable displacement

Forecasting the directions of the displacements is easier than setting the vector Δ of expected displacements itself. Let the vector Δ be the product of a scale factor (b) and a given direction vector (**g**),

$$\Delta = b\mathbf{g} \tag{4.6}$$

Substituting Eq. (4.6) into Eq. (4.3) and using Eq. (4.5) we obtain

$$b_{min} = \sigma_d \sqrt{\frac{\lambda_{0,h}}{\mathbf{g}^T \mathbf{Q}_{dd}^{+} \mathbf{g}}} \tag{4.7}$$

Then we define the minimum detectable displacement vector as follows;

$$\Delta_{min} = b_{min}\mathbf{g} \tag{4.8}$$

In some cases, even directions of the displacements are not be available. To obtain the minimum displacement vector in such a case, we use the eigen-vector Λ_{max} belonging to the maximum eigen value λ_{max} of $\mathbf{Q}_{dd}$. This results in the minimum value of the scale factor

$$b_{min}=\sigma_d\sqrt{\frac{\lambda_{0,h}}{\Lambda_{max}^T\mathbf{Q}_{dd}^{+}\Lambda_{max}}}=\sigma_d\sqrt{\lambda_{0,h}\lambda_{max}} \quad (4.9)$$

Then, if we consider Eq. (4.9) in Eq. (4.8), we obtain the displacement vector which is just detectable on the directions of the eigen-vector with a specified power of the test γ_0 and significance level α_0.

Note 4.1: The direction vector **g** for leveling monitoring networks consists of "1" for the uplifted points, "-1" for the subsided points and "0" for stable points. In 2D networks (or in a horizontal plane) it includes "cosθ" and "sinθ" where θ is the forecasted azimuth of the displacement vector of the corresponding point.

4.2 Sensitivity Analysis in Absolute Deformation Networks

Let our object points $\otimes$ be defined related to the reference points $\Re$. The hypotheses to test each object point are set follows;

$$H_0: E(\mathbf{d}_{\otimes_i})=\mathbf{0} \quad , \quad H_1: E(\mathbf{d}_{\otimes_i})\neq\mathbf{0}=\Delta_{\otimes_i} \quad (4.10)$$

where $\Delta_{\otimes_i}$ is the expected displacement vector of the ith object point.

Let us assume that our test statistic set for discriminating the hypotheses in Eq. (4.10) is χ^2-distributed. Since we know that the alternative hypothesis is true now, the test statistic has a non-central χ^2-distribution;

$$T_{\otimes_i} = \frac{\mathbf{d}_{\otimes_i}^{T} \mathbf{Q}_{\otimes_i \otimes_i}^{-1} \mathbf{d}_{\otimes_i}}{\sigma_d^2} \sim \chi'^2(c, \lambda_i) \tag{4.11}$$

where λ_i is the non-centrality parameter,

$$\lambda_i = \frac{\Delta_{\otimes_i}^{T} \mathbf{Q}_{\otimes_i \otimes_i}^{-1} \Delta_{\otimes_i}}{\sigma_d^2} . \tag{4.12}$$

To learn whether our expected displacements for the ith point is detectable or not, the non-centrality parameter λ_i is compared with its boundary value $\lambda_{0,c}$; If

$$\lambda_i \geq \lambda_{0,c} \tag{4.13}$$

holds, the corresponding displacement is said to be detectable with the corresponding power of the test.

To derive the minimum detectable displacement, we may follow the same methodology given in the previous section. Instead of giving the formula for a specific direction, we consider the eigenvector of the maximum eigen-value $\lambda_{max,i}$ of $\mathbf{Q}_{\otimes_i \otimes_i}$. Similar to Eq. (4.9), we obtain

$$(b_{min})_i = \sigma_d \sqrt{\lambda_{0,c} \lambda_{max,i}} = (\sigma_d \sqrt{\lambda_{max,i}}) \sqrt{\lambda_{0,c}} \tag{4.14}$$

With this scale factor, the minimum detectable displacement vector of the ith point becomes

$$(\Delta_{\otimes_i})_{min} = (b_{min})_i \Lambda_{max,i} \tag{4.15}$$

where $\Lambda_{max,i}$ is the eigen-vector belonging to the maximum eigenvalue $\lambda_{max,i}$.

Note 4.2: For 2D networks, first term of the right hand of Eq. (4.14), i.e., $\sigma_d \sqrt{\lambda_{max,i}}$, is in fact the semi-major axis of the relative error ellipse of the ith point. Then, $(b_{min})_i$ may be considered as the semi-major axis of the relative error ellipse for the power of the test (we may call this ellipse relative power ellipse!). Furthermore, (4.15) shows that minimum detectable displacement is on the direction overlapped with the direction of the relative error ellipse of the ith point. Then in 2D examples, $(b_{min})_i$ denotes the minimum detectable displacement magnitude of the corresponding point. The boundary value is obtained as $\lambda_{0,c=2}$=9.64 for 80% power of the test from Table 4.1. It means that a displacement whose magnitude is 3.1 times of the semi-major axis of the relative confidence ellipse may be just detectable with 80% power of the test in an object point of an absolute deformation network.

For 1D networks, $\lambda_{max,i}$ becomes equal to the displacement's cofactor value of the corresponding point. From Table 4.1 we read $\lambda_{0,c=1}$=7.85 for 80% power of the test. Then it is clear that a displacement (uplift or subsidence) may be just detectable with 80% power, if its magnitude is 2.8 times of the displacement's standard deviation.

With the above-mentioned simple statistics in two previous notes, one is able to speak about the capacity of the designed network. The cofactor matrices of displacements with respect to the reference points may easily be derived and the pooled variance may be guessed depending on the experiences before any realization. Then the minimum detectable displacement of an object point is about "3 times" of the semi-major axis of its relative confidence ellipse and displacement's standard deviation in 2D and 1D networks, respectively. If our expectation does not match with this magnitude, then we may re-design our network such that its object points have smaller ellipses or smaller standard deviations depending on the network type. For example, let us assume that we expect 5 mm vertical movement in a region, and our leveling network's object points relative to a reference point set have around 2 mm standard deviation in a period. Then the displacement's standard deviation is expected as $2\sqrt{2}$ =2,8 mm. With 80% power (or more), the minimum detectable displacement is around 3×2,8=8.4 mm. It means that we may not able to detect 5 mm movement with 80% power of the test using the corresponding design. For that reason, we should plan additional observations or should measure the network with more precise levels to improve the precision of our points. Such an optimization is called "trial and error" method; but we may also set some analytical target functions to obtain a global solution to this optimization problem. This is called "analytical optimization of deformation networks"; however, nowadays, because of our improved computing capabilities by normal PCs, trial and error methods may be more preferable. They are more realistic because we may produce the problem depending on some experiments which we may face with in reality. In analytical tools, sometimes, if we do not consider the constraints realistically (it is a little bit hard to consider all conditions mathematically!), the solution may go far away from the global solution and stop at a local one, which may mislead the practitioner, or, which may cause an another problem waiting for a different solution.

5. INTERPRETATION MODELS

After defining the reference points in our network, we may investigate the object points' movements and strain elements of some object blocks to interprete the motion and the deformation of the object, respectively. For this we use kinematic model and strain model. In this section we briefly explain these models used in deformation analysis.

5.1 Kinematic Model

An object may change its position in time continuously related to a reference frame. This motion is expressed by the following well-known equation

$$\beta(t) = \beta(t_0) + (t - t_0)\dot{x} + \frac{1}{2}(t - t_0)^2 \ddot{x} \quad (5.1)$$

where

t	:	current time,
t_0	:	initial time,
$\beta(t)$	:	current (present) position,
$\beta(t_0)$	:	initial position,
$\dot{x}$	:	velocity and
$\ddot{x}$	:	acceleration.

If the parameters $\beta(t_0)$, $\dot{x}$ and $\ddot{x}$ are available, one may predict the object's position in any period from Eq. (5.1). Reversely, if we have coordinates of the point in different periods, we may estimate the parameters to create a model for the object's movement. This is called kinematic model in deformation analysis. To satisfy redundancy in kinematic modelling, there should be at least 4 periods. Hereafter we assume therefore that we have m≥4 periods.

5.1.1 Single point model for an object point's movement

5.1.1.1 Model I

Now suppose that we work with a 1D network, or we would like to model only one component of a point among its other components (as realized mostly for North, East and Up components of a GNSS station). For this we set the following Gauss-Markov model from Eq. (5.1) having taken the initial period as t_1 and partitioning the initial (unknown) position $\beta(t_0)$ into two parts as $\beta(t_0)=\beta(t_1)+\delta\beta$,

$$E\left\{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}\right\}=\begin{pmatrix} 1 & (t_1-t_1) & 0.5(t_1-t_1)^2 \\ 1 & (t_2-t_1) & 0.5(t_2-t_1)^2 \\ \vdots & \vdots & \vdots \\ 1 & (t_m-t_1) & 0.5(t_m-t_1)^2 \end{pmatrix}\begin{pmatrix} \delta\beta \\ \dot{x} \\ \ddot{x} \end{pmatrix}, \mathbf{P}=\sigma_0^2\begin{pmatrix} Q_{11}^{-1} & 0 & \cdots & 0 \\ 0 & Q_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{mm}^{-1} \end{pmatrix} \quad (5.2)$$

where

y_i	:	$c\times1$ (diminished) coordinate component (observation), ($y_i=\beta(t_i)-\beta(t_0)$),
$\beta(t_i)$	:	coordinate component in the ith period,
$\delta\beta$	:	unknown shift parameter,
$\dot{x}$	:	unknown velocity,
$\ddot{x}$	:	unknown acceleration,
σ_0^2	:	a-priori variance of unit weight and
Q_{ii}	:	cofactor value of the corresponding coordinate component.

5.1.1.2 Model II

For c-dimensional networks, considering model (5.2) may cause over-optimistic results because in that case we neglect the correlations between the coordinate components of **a point**. They are in fact highly correlated therefore they may be considered in a single Gauss-Markov model

$$E\left\{\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\\ \mathbf{y}_m\end{bmatrix}\right\}=\begin{pmatrix}\mathbf{I} & (t_1-t_1)\mathbf{I} & 0.5(t_1-t_1)^2\mathbf{I}\\ \mathbf{I} & (t_2-t_1)\mathbf{I} & 0.5(t_2-t_1)^2\mathbf{I}\\ \vdots & \vdots & \vdots\\ \mathbf{I} & (t_m-t_1)\mathbf{I} & 0.5(t_m-t_1)^2\mathbf{I}\end{pmatrix}\begin{pmatrix}\delta\beta\\ \dot{\mathbf{x}}\\ \ddot{\mathbf{x}}\end{pmatrix},\ \mathbf{P}=\sigma_0^2\begin{pmatrix}\mathbf{Q}_{11}^{-1} & \mathbf{0} & \cdots & \mathbf{0}\\ \mathbf{0} & \mathbf{Q}_{22}^{-1} & \cdots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{mm}^{-1}\end{pmatrix} \quad (5.3)$$

where

$\mathbf{y}_i$	:	c×1 (diminished) coordinate (observation) vector ($\mathbf{y}_i=\beta(t_i)-\beta(t_1)$),
$\beta(t_i)$	:	c×1 coordinate vector,
$\mathbf{I}$	:	c×c identity matrix,
$\delta\beta$	:	c×1 unknown shift parameter vector,
$\dot{\mathbf{x}}$	:	c×1 vector of unknown velocities,
$\ddot{\mathbf{x}}$	:	c×1 vector of unknown accelerations along the corresponding axes,
σ_0^2	:	a-priori variance of unit weight and
$\mathbf{Q}_{ii}$	:	c×c cofactor matrix belonging to **the corresponding point** in the ith period.

5.1.1.3 Model III

In some cases previous two models may not yield satisfactory estimates because of some unmodelled physical effects on the coordinates. They are mostly observed as periodic changes in time series of a point's coordinate component as shown in Fig 5.1.

If we have a prior information about the periodical part of the changes, instead of Eq. (5.19) we may consider

$$\beta(t) = \beta(t_0) + (t - t_0)\dot{x} + \text{periodic part} \tag{5.4}$$

where the periodic part is a linear function of the effect of the corresponding physical sources. (The point's movement may include also an acceleration part; however, for simplicity we drop it in Eq. (5.4))

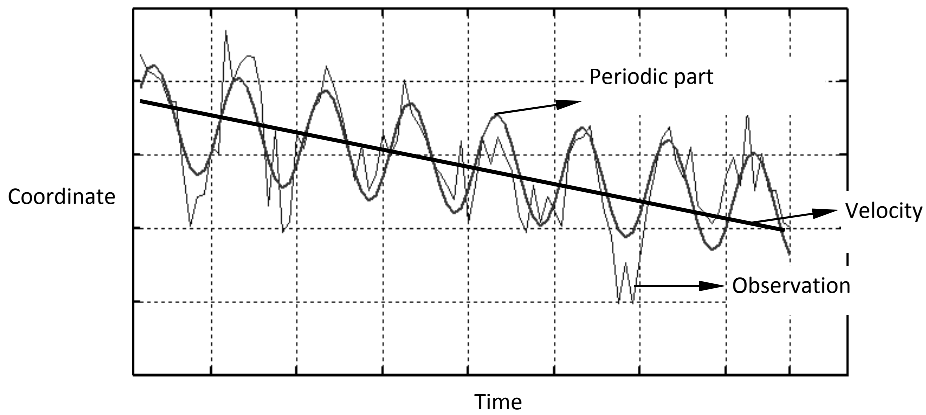

Fig 5.1 Coordinate changes in time domain

The periodicity may happen hourly, daily, annually or semi-annually depending on the sources. They are commonly modelled using "$a_1\cos(2\pi t/T_p)+a_2\sin(2\pi t/T_p)$" function, where T_p is the known period, a_1 and a_2 are the unknown amplitutes. Let us consider that our periodic part may be modelled with this function: Then instead of model (5.2) we may set the following Gauss-Markov model

$$E\left\{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}\right\} = \begin{pmatrix} 1 & \Delta t_1 & \cos(2\pi\Delta t_1/T_p) & \sin(2\pi\Delta t_1/T_p) \\ 1 & \Delta t_2 & \cos(2\pi\Delta t_2/T_p) & \sin(2\pi\Delta t_2/T_p) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \Delta t_m & \cos(2\pi\Delta t_m/T_p) & \sin(2\pi\Delta t_m/T_p) \end{pmatrix} \begin{pmatrix} \delta\beta \\ \dot{x} \\ a_1 \\ a_2 \end{pmatrix}, \ldots$$

$$\ldots \qquad \mathbf{P} = \sigma_0^2 \begin{pmatrix} Q_{11}^{-1} & 0 & \cdots & 0 \\ 0 & Q_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_{mm}^{-1} \end{pmatrix} \tag{5.5}$$

where $y_i = \beta(t_i) - \beta(t_0)$ and $\Delta t_i = t_i - t_1$.

Note 5.1: In most problems, the period T_p is not available or not exact. In that case, $w=2\pi/T_p$ angular frequency may be obtained by applying Fast-Fourier transform to the **observations** beforehand.

5.1.2 Single model for whole object points' movement

Kinematic model (5.1) may be established for all object points under a single model. For $p_{\otimes}=p-p_{\Re}$ object points in the network, which is measured m periods, the Gauss-Markov model for such a modelling is written as follows

$$E\left\{\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_m \end{bmatrix}\right\} = \begin{pmatrix} \mathbf{I} & (t_1-t_1)\mathbf{I} & 0.5(t_1-t_1)^2\mathbf{I} \\ \mathbf{I} & (t_2-t_1)\mathbf{I} & 0.5(t_2-t_1)^2\mathbf{I} \\ \vdots & \vdots & \vdots \\ \mathbf{I} & (t_m-t_1)\mathbf{I} & 0.5(t_m-t_1)^2\mathbf{I} \end{pmatrix} \begin{pmatrix} \delta\beta \\ \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix}, \ \mathbf{P} = \sigma_0^2 \begin{pmatrix} \mathbf{Q}_{x_1x_1}^{-1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{x_2x_2}^{-1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{Q}_{x_mx_m}^{-1} \end{pmatrix} \tag{5.6}$$

where

$\mathbf{y}_i$	:	$cp_{\otimes}\times 1$ (diminished) coordinate (observation) vector for the ith period ($\mathbf{y}_i = \beta_i - \beta_1$),
β_i	:	$cp_{\otimes}\times 1$ coordinate vector of the ith period;
$\mathbf{I}$	:	$cp_{\otimes}\times cp_{\otimes}$ identity matrix,
$\delta\beta$	:	$cp_{\otimes}\times 1$ unknown shift parameter vector,
$\dot{\mathbf{x}}$	:	$cp_{\otimes}\times 1$ vector of unknown velocities,
$\ddot{\mathbf{x}}$	:	$cp_{\otimes}\times 1$ vector of unknown accelerations,
σ_0^2	:	a-priori variance of unit weight and
$\mathbf{Q}_{x_i x_i}$	:	$cp_{\otimes}\times cp_{\otimes}$ cofactor matrix belonging to the object points in the ith period.

5.1.3 Testing model parameters

Before publishing the kinematic model of an object point, we should test its parameters (mostly velocity and acceleration) to learn whether they are significant or not. Testing procedure is therefore called significancy test.

Let us consider that the velocity estimate $\dot{x}$ with standard deviation $s_{\dot{x}}$ is desired to be tested; for this we set the following hypotheses

$$H_0: E(\dot{x})=0 \quad , \quad H_1: E(\dot{x})\neq 0 \tag{5.7}$$

Then the test statistic follows t-distribution

$$T_{\dot{x}} = \frac{|\dot{x}|}{s_{\dot{x}}} \sim t(f) \tag{5.8}$$

where f is the degrees of freedom of the corresponding model. If $T_{\dot{x}} < t_{f,1-\alpha/2}$, the estimated value $\dot{x}$ is not significant. Otherwise, it is accepted that it has a significant-physical meaning.

Note 5.2: For each estimated parameter the same testing procedure given above is realized. Insignificant parameters may be extracted from the corresponding model, and the estimation procedure is repeated having established the corresponding model with the remaining-significant parameters. This will increase the redundancy, i.e. degrees of freedom, and will result in more precise estimation. In some applications, for example in GNSS studies with long-time series, the redundancy is already big, therefore, the testing procedure may be unnecessary: The estimated values and their standard deviations are declared, for example, as "**velocity±its standard deviation**". This is called sometimes "**velocity with 1-sigma error**": If we have big redundancy, the accepted one-dimensional t-distribution gets close to normal-distribution and this interval shows a confidence interval with about 40% probability. If we declare "**velocity with 2-sigma error**", from the normal-distribution function, we understand that the interval shows a confidence inverval with a probability more than 95%.

5.1.4 Model test

There may exist different kinematic models for the time-dependent observations. To verify which model fits better to the observations, we may apply model test: For example, let us take the model (5.2) and call it model 1. Its alternative one may be the model without an acceleration parameter, i.e. velocity model; let us call it model 2. From each model we estimate the unknowns and obtain the weighted squares of the residuals; i.e. we obtain Ω_1 and Ω_2 quadratic forms independently. The following test statistic follows F-distribution, with $f_2–f_1$ and f_2 degrees of freedom,

$$T_M = \frac{(\Omega_2 - \Omega_1)/(f_2 - f_1)}{\Omega_1 / f_1} \sim F(f_2–f_1,\ f_2) \tag{5.9}$$

where

f_1 : degrees of freedom of model 1 and

f_2 : degrees of freedom of model 2.

We compare T_M with the threshold value $F_{f_2-f_1,f_1,1-\alpha}$:

i) If $T_M < F_{f_2-f_1,f_1,1-\alpha}$, model 1 is not necessary. In other words, instead of acceleration and velocity parameters, it is better to consider only velocity parameter.

ii) $T_M \geq F_{f_2-f_1,f_1,1-\alpha}$, model 1 fits better to the observations: Model 2 does not ensure the essential information to model the time-dependent observations.

Testing model 1 against model 2 with the above-mentioned model test practically may be done with the previous significancy test: If the acceleration's significancy test fails, it means that, model 2 (the model with only velocity) should be considered to model the observations. But, at this point, we should remind that, statistically and theoretically, "model test" is more correct because the previous testing procedure neglects the correlations between the estimated parameters.

The given model test procedure may be applied for comparing different types of models, not only for comparing the velocity model and velocity+acceleration model: We should just care about that model 1 is to be attained as an augmented model with additional parameters which are not included in model 2.

5.2 Strain Analysis

5.2.1 Definition

Strain is defined as the ratio of increase or decrease in length to its original length. It is a normalized measure for deformation. For instance, let us consider a wire with L_1=100 m length has extended to L_2=100.02 m; then the engineering strain, the so-called nominal strain, is computed as follows

$$\varepsilon=\frac{L_2-L_1}{L_1}=\frac{100.02-100}{100}=2\times10^{-5}\ \text{strain=20 μstrain=20 ppm.}$$

This strain may be denoted as

$$\varepsilon=\frac{\text{Displacement}}{\text{Original Length}}=\frac{dL}{L_1} \tag{5.10}$$

On the other hand, scale factor λ, the so-called stretch ratio, is related with the engineering strain ε by

$$\lambda=1+\varepsilon \tag{5.11}$$

which is the one commonly used in geodesy to explain the deformations of the coordinate axes, for example in similarity and Affine transformations.

In two dimensional, instead of a single strain measure, there exists a strain tensor,

$$\mathbf{E}=\begin{pmatrix}\varepsilon_{xx} & \varepsilon_{xy}\\ \varepsilon_{yx} & \varepsilon_{yy}\end{pmatrix}=\begin{pmatrix}\partial dx/\partial x & \partial dx/\partial y\\ \partial dy/\partial y & \partial dy/\partial y\end{pmatrix} \tag{5.11}$$

where

dx and dy : displacements of a particle in the object in x and y directions.

Since we assume the object is continuum, i.e. the object is full of homogeneous particles, the tensor elements in Eq. (5.11) represent the deformation of the whole object. There are some other quantities obtained from the elements of this strain tensor, such as,

Dilation (mean strain): $$\varepsilon_{mean}=\frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy}) \tag{5.12a}$$

Pure shear: $$\varepsilon_{pure}=\frac{1}{2}(\varepsilon_{xx}-\varepsilon_{yy}) \tag{5.12b}$$

Simple shear: $$\varepsilon_{simple}=\frac{1}{2}(\varepsilon_{xy}+\varepsilon_{yx}) \tag{5.12c}$$

Total shear: $$\varepsilon_{shear}=\sqrt{\varepsilon_{pure}^2+\varepsilon_{simple}^2}=\frac{1}{2}\left(\sqrt{(\varepsilon_{xx}-\varepsilon_{yy})^2+(\varepsilon_{xy}+\varepsilon_{yx})^2}\right) \tag{5.12d}$$

Differential rotation: $$\psi=\frac{1}{2}(\varepsilon_{yx}-\varepsilon_{xy}) \tag{5.12e}$$

In earth sciences, instead of the strain tensor **E**, symmetrical strain tensor $\mathbf{E}_s$, which is derived from Eq. (5.11), is used;

$$\mathbf{E}_s=\begin{pmatrix}\varepsilon_{xx} & (\varepsilon_{xy}+\varepsilon_{yx})/2 \\ (\varepsilon_{xy}+\varepsilon_{yx})/2 & \varepsilon_{yy}\end{pmatrix}=\begin{pmatrix}\varepsilon_{xx} & \varepsilon_{\mathbf{simple}} \\ \varepsilon_{\mathbf{simple}} & \varepsilon_{yy}\end{pmatrix} \tag{5.13}$$

To show the object deformation in 2D, principal strain components, i.e., the eigenvalues of $\mathbf{E}_s$ are derived from (5.13)

$$\varepsilon_{max}=\frac{1}{2}(\varepsilon_{xx}+\varepsilon_{yy}+\sqrt{(\varepsilon_{xx}-\varepsilon_{yy})^2+4\varepsilon_{simple}^2})=\varepsilon_{mean}+\varepsilon_{shear} \tag{5.14a}$$

$$\varepsilon_{min} = \frac{1}{2}(\varepsilon_{xx} + \varepsilon_{yy} - \sqrt{(\varepsilon_{xx} - \varepsilon_{yy})^2 + 4\varepsilon_{simple}^2}) = \varepsilon_{mean} - \varepsilon_{shear} \tag{5.14b}$$

with the direction of the maximum principal axis, clockwise from x-axis (see Note 5.3),

$$\vartheta = \frac{1}{2}\text{atan}\left(\frac{2\varepsilon_{simple}}{\varepsilon_{xx} - \varepsilon_{yy}}\right) = \frac{1}{2}\text{atan}\left(\frac{\varepsilon_{simple}}{\varepsilon_{pure}}\right) \tag{5.14c}$$

ε_{max} shows the greatest change while ε_{min} is the smallest change of length per unit length. They are plotted on the centroid of the object as shown in Fig 5.1. The negative sign of any component shows **contraction** whereas positive sign denotes **extension** through the corresponding direction.

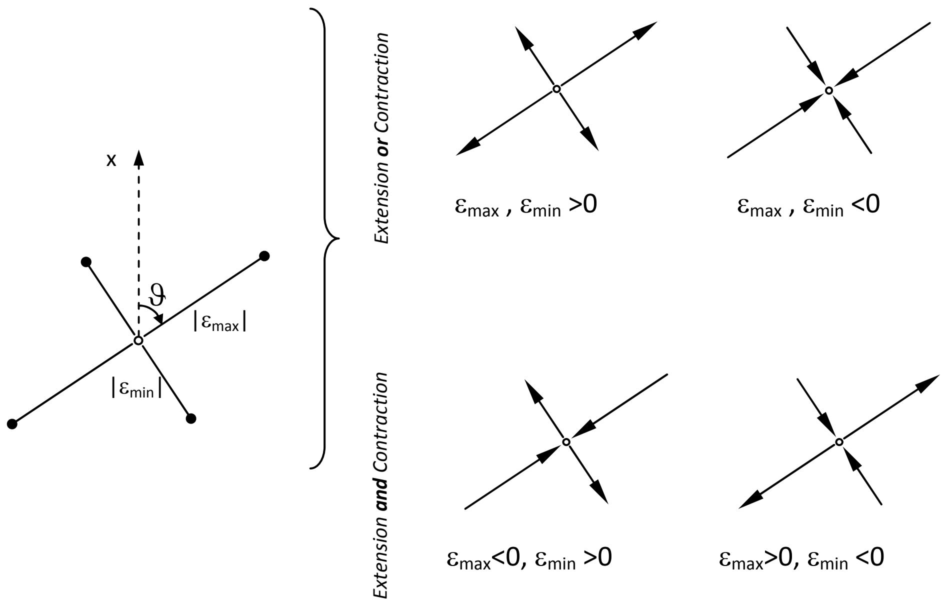

Fig. 5.1. Principal strain components

Note 5.3: The principle of computation of the principal strain angle ϑ is similar to the one of computation of azimuth: First, "$2\vartheta = \text{atan}(\varepsilon_{simple}/\varepsilon_{pure}) = a$" is obtained; **i)** if $\varepsilon_{simple}>0$ and $\varepsilon_{pure}>0$, $\vartheta=a/2$, **ii)** if $\varepsilon_{simple}>0$ and $\varepsilon_{pure}<0$, $\vartheta=200+a/2$, **iii)** if $\varepsilon_{simple}<0$ and $\varepsilon_{pure}<0$, $\vartheta=200+a/2$ and **iv)** if $\varepsilon_{simple}<0$ and $\varepsilon_{pure}>0$, $\vartheta=400+a/2$. Without considering the regions of the angle, direct computation may be realized by "$\vartheta = \text{atan}(\varepsilon_{simple}/(\varepsilon_{pure}-\varepsilon_{shear}))+90^{o}$". However this direct computation is sensitive to the numerical errors in ε_{simple}, ε_{pure} as well as ε_{shear}. Therefore it should be considered in double precision computing tools. Moreover, if $\varepsilon_{simple}=0$, then "0/0" vague happens in that formula: For this, the reader should notice the following two conditions; **i)** if $\varepsilon_{simple}=0$ and $\varepsilon_{pure}>0$, $\vartheta=0^{o}$ and **ii)** if $\varepsilon_{simple}=0$ and $\varepsilon_{pure}<0$, $\vartheta=90^{o}$ while using that direct computation formula.

5.2.2 Strain modelling in geodetic deformation analysis

For the ith object point having dx_i and dy_i displacements, we may write

$$dx_i=t_x+\varepsilon_{xx}x_i+\varepsilon_{xy}y_i \quad \text{and} \quad dy_i=t_y+\varepsilon_{yx}x_i+\varepsilon_{yy}y_i \tag{5.15}$$

which are the fundamental equations for modelling strain of the corresponding object. In addition to 4 strain parameters (ε_{xx}, ε_{xy}, ε_{yx}, ε_{yy}) we have two translation parameters t_x and t_y in Eq. (5.15), therefore, to obtain these 6 parameters, mathematically we need at least 3 points.

For modelling strain of the studied object, the structural properties of the object should be known priorily. From such a prior information the object is divided into the different blocks as demonstrated in Fig 5.2. For each block we consider different strain model (see Note 5.4).

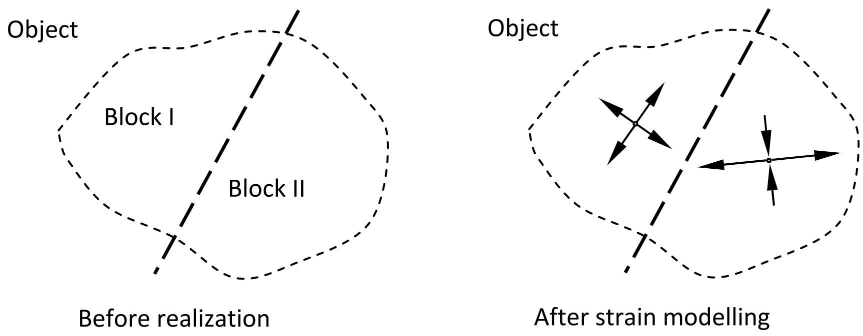

Fig 5.2. Object blocks and their principal strain components

Note 5.4: Object blocks should be considered at the design stage so that each block has its own object points characterizing the deformation to be monitored. An attempt for deciding object blocks considering only the observed displacements may yield wrong interpretations.

Now suppose that our object consists of one block; then all object points are included in a single strain model. For this, our Gauss-Markov model is set as follows

$$E\left\{\begin{pmatrix} dx_1 \\ dy_1 \\ \vdots \\ dx_{p_\otimes} \\ dy_{p_\otimes} \end{pmatrix}\right\} = \begin{pmatrix} 1 & 0 & x_1 & y_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & x_1 & y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_{p_\otimes} & y_{p_\otimes} & 0 & 0 \\ 0 & 1 & 0 & 0 & x_{p_\otimes} & y_{p_\otimes} \end{pmatrix} \begin{pmatrix} t_x \\ t_y \\ \varepsilon_{xx} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \\ \varepsilon_{yy} \end{pmatrix} \rightarrow E\{\mathbf{d}_\otimes\}=\mathbf{Mx} \qquad (5.16a)$$

with the $2p_\otimes \times 2p_\otimes$ matrix of weights of displacements,

$$\mathbf{P}=\sigma_0^2\begin{pmatrix} Q_{dx_1dx_1} & Q_{dx_1dy_1} & \cdots & Q_{dx_1dx_{p\otimes}} & Q_{dx_1dy_{p\otimes}} \\ & Q_{dy_1dy_1} & \cdots & Q_{dy_1dx_{p\otimes}} & Q_{dy_1dy_m} \\ & & \ddots & \vdots & \vdots \\ & & & Q_{dx_{p\otimes}dx_{p\otimes}} & Q_{dx_{p\otimes}dy_{p\otimes}} \\ & & & & Q_{dy_{p\otimes}dy_{p\otimes}} \end{pmatrix}^{-1} \rightarrow \mathbf{P}=\sigma_0^2\,\mathbf{Q}_{\otimes\otimes} \qquad (5.16b)$$

where

$\mathbf{d}_\otimes$ and $\mathbf{Q}_{\otimes\otimes}$	:	vector of displacements and its cofactor matrix belonging to the object points ⊗, respectively, from Eq. (2.15),
M	:	$2p_\otimes \times 6$ coefficient matrix and
x	:	6×1 unknown parameter vector.

Solving model (5.16) by least-squares method, the parameter vector **x** is estimated and so we get strain parameters ε_{xx}, ε_{xy}, ε_{yx} and ε_{yy} for the object. By using them the principal strain parameters in Eq. (5.14) are obtained and they are plotted on the centroid of the object under-consideration as shown in Fig 5.3.

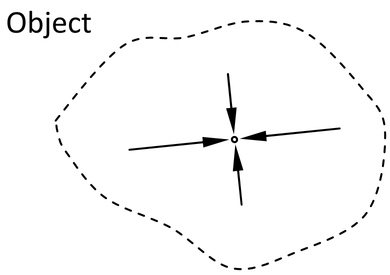

Fig 5.3 Principal strain parameters for an object

For more blocks, their independent strain models may be considered in a single Gauss-Markov model: For instance, for two object blocks (Block I and Block II) we establish the following Gauss-Markov model;

$$E\left\{\begin{pmatrix}\mathbf{d}_{\otimes_{I}}\\ \mathbf{d}_{\otimes_{II}}\end{pmatrix}\right\}=\begin{pmatrix}\mathbf{M}_{I} & 0\\ 0 & \mathbf{M}_{II}\end{pmatrix}\begin{pmatrix}\mathbf{x}_{I}\\ \mathbf{x}_{II}\end{pmatrix} \quad , \quad \mathbf{P}=\sigma_0^2\begin{pmatrix}\mathbf{Q}_{\otimes_{I}\otimes_{I}} & \mathbf{Q}_{\otimes_{I}\otimes_{II}}\\ \mathbf{Q}_{\otimes_{II}\otimes_{I}} & \mathbf{Q}_{\otimes_{II}\otimes_{II}}\end{pmatrix}^{-1} \tag{5.17}$$

From the solution of model (5.17), we obtain $\mathbf{x}_{I}$ and $\mathbf{x}_{II}$ parameter vectors including the blocks' strain parameters.

APPENDIX A: GLOSSARY

Absolute deformation network: Mutlak deformasyon ağı

Acceleration: İvme

Block: Blok

Confidence level: Güven düzeyi

Contraction: Küçülme

Constraint equations: Koşul denklemleri

Current period: Mevcut periyot

Current time: Mevcut zaman

Deformation: Deformasyon

Degrees of freedom: Serbestlik derecesi

Diminished observation: Küçültülmüş ölçü

Direction vector: Yön vektörü

Displacement: Yerdeğişim

Extension: Genişleme

Gauss-elimination method: Gauss-eliminasyon yöntemi

Global test: Global test

Identical: Eşdeğer

Identity matrix: Birim matris

Implicit hypothesis method: Kapalı hipotez yöntemi

Initial period: Başlangıç periyodu

Initial time: Başlangıç zamanı

Kinematic model: Kinematik model

Localization: Yerelleştirme

Lower bound of the non-centrality parameter: Dış merkezlik parametresinin sınır değeri

Minimum detectable displacement: Belirlenebilir en küçük yerdeğişim

Minimum constrained: Zorlamasız

Monitoring: İzleme

Model test: Model testi

Non-central: Merkezsel olmayan

Non-centrality parameter: Dış merkezlik parametresi

Non-identical: Eşdeğer olmayan

Object: Nesne

Object block: Obje bloğu

Object point: Obje noktası

Partial trace minimum: Kısmi iz minimum

Period: Periyot

Pooled variance factor: Birleştirilmiş varyans çarpanı

Power of the test: Test gücü

Principal strain parameters: Asal gerinim parametreleri

Reference block: Referans bloğu

Reference point: Dayanak noktası

Relative confidence ellipse: Bağıl güven elipsi

Relative deformation network: Bağıl deformasyon ağı

Rotation: Dönüklük

Quadratic form: Karesel biçim

Sensitivity: Duyarlılık

Shift parameter: Sıfır eki

Significance level: Yanılma olasılığı

Significancy test: Anlamlılık testi

Significant: Anlamlı

Stable: Durağan

Strain: Gerinim

Strain tensor: Gerinim tensörü

Subsidence: Çökme

Test statistic: Test büyüklüğü

Threshold value: Karşılaştırma değeri

Trace minimum: Tüm iz minimum

Translation: Öteleme

Undeformed: Deforme olmamış

Uplift: Yükselme

Velocity: Hız

APPENDIX B: THRESHOLD VALUES (F and t-distributions)

Table B1. Threshold values for F-distribution (*) for α=5% ($F_{a,b,1-\alpha}$)

a \ b	1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90	100
1	161.45	18.51	10.13	7.71	6.61	5.99	5.59	5.32	5.12	4.96	4.35	4.17	4.08	4.03	4.00	3.98	3.96	3.95	3.94
2	199.50	19.00	9.55	6.94	5.79	5.14	4.74	4.46	4.26	4.10	3.49	3.32	3.23	3.18	3.15	3.13	3.11	3.10	3.09
3	215.71	19.16	9.28	6.59	5.41	4.76	4.35	4.07	3.86	3.71	3.10	2.92	2.84	2.79	2.76	2.74	2.72	2.71	2.70
4	224.58	19.25	9.12	6.39	5.19	4.53	4.12	3.84	3.63	3.48	2.87	2.69	2.61	2.56	2.53	2.50	2.49	2.47	2.46
5	230.16	19.30	9.01	6.26	5.05	4.39	3.97	3.69	3.48	3.33	2.71	2.53	2.45	2.40	2.37	2.35	2.33	2.32	2.31
6	233.99	19.33	8.94	6.16	4.95	4.28	3.87	3.58	3.37	3.22	2.60	2.42	2.34	2.29	2.25	2.23	2.21	2.20	2.19
7	236.77	19.35	8.89	6.09	4.88	4.21	3.79	3.50	3.29	3.14	2.51	2.33	2.25	2.20	2.17	2.14	2.13	2.11	2.10
8	238.88	19.37	8.85	6.04	4.82	4.15	3.73	3.44	3.23	3.07	2.45	2.27	2.18	2.13	2.10	2.07	2.06	2.04	2.03
9	240.54	19.38	8.81	6.00	4.77	4.10	3.68	3.39	3.18	3.02	2.39	2.21	2.12	2.07	2.04	2.02	2.00	1.99	1.97
10	241.88	19.40	8.79	5.96	4.74	4.06	3.64	3.35	3.14	2.98	2.35	2.16	2.08	2.03	1.99	1.97	1.95	1.94	1.93
20	248.01	19.45	8.66	5.80	4.56	3.87	3.44	3.15	2.94	2.77	2.12	1.93	1.84	1.78	1.75	1.72	1.70	1.69	1.68
30	250.10	19.46	8.62	5.75	4.50	3.81	3.38	3.08	2.86	2.70	2.04	1.84	1.74	1.69	1.65	1.62	1.60	1.59	1.57
40	251.14	19.47	8.59	5.72	4.46	3.77	3.34	3.04	2.83	2.66	1.99	1.79	1.69	1.63	1.59	1.57	1.54	1.53	1.52
50	251.77	19.48	8.58	5.70	4.44	3.75	3.32	3.02	2.80	2.64	1.97	1.76	1.66	1.60	1.56	1.53	1.51	1.49	1.48
60	252.20	19.48	8.57	5.69	4.43	3.74	3.30	3.01	2.79	2.62	1.95	1.74	1.64	1.58	1.53	1.50	1.48	1.46	1.45
70	252.50	19.48	8.57	5.68	4.42	3.73	3.29	2.99	2.78	2.61	1.93	1.72	1.62	1.56	1.52	1.49	1.46	1.44	1.43
80	252.72	19.48	8.56	5.67	4.41	3.72	3.29	2.99	2.77	2.60	1.92	1.71	1.61	1.54	1.50	1.47	1.45	1.43	1.41
90	252.90	19.48	8.56	5.67	4.41	3.72	3.28	2.98	2.76	2.59	1.91	1.70	1.60	1.53	1.49	1.46	1.44	1.42	1.40
100	253.04	19.49	8.55	5.66	4.41	3.71	3.27	2.97	2.76	2.59	1.91	1.70	1.59	1.52	1.48	1.45	1.43	1.41	1.39

***)** Square root of F value for a=1 and b in the first row yields $t_{b,1-\alpha/2}$