Tau Distribution and testing residuals

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Introduction

Suppose that after a Least Squares adjustment of measurements we wish to test the residuals $\mathbf{v}$ to detect an outlier that may indicate a suspect measurement. A statistical test, first proposed by Thompson (1935) is based on the tau-statistic T where

$$\mathrm{T}_i = \frac{V_i}{S} = \frac{\bar{X} - X_i}{\sqrt{\dfrac{1}{n}\sum_{i=1}^{n}\left(\bar{X} - X_i\right)^2}} \tag{1}$$

T is a ratio of random variables V and S derived from a sample set of n random variables $\left\{X_1, X_2, ..., X_n\right\}$.

V_i is a residual defined as the difference between X_i (an arbitrary member of the sample set) and the mean $\bar{X}$ where $n\bar{X} = \sum_{i=1}^{n} X_i$ and S is a positive quantity that is a measure of the dispersion of the sample set.

T takes values τ of the τ-distribution (*tau*-distribution) having $\nu = n - 2$ degrees of freedom.

Pope (1976) expressed Thompson's tau-statistic in a more modern form as

$$\mathrm{T}_i = \frac{v_i}{\hat{\sigma}_{v_i}} = \frac{v_i}{\hat{\sigma}_0 \sqrt{q_{v_{ii}}}} \tag{2}$$

where v_i is an arbitrary residual from the vector of residuals $\mathbf{v}$ from a Least Squares adjustment of measurements, $\hat{\sigma}_{v_i}$ is the least squares estimate of the standard deviation of the residual v_i, $\hat{\sigma}_0$ is the square-root of the least squares estimate of the variance factor $\hat{\sigma}_0^2$ and $q_{v_{ii}}$ is the least squares estimate of the variance of the residual. $q_{v_{ii}}$ are the diagonal elements of the variance matrix $\mathbf{Q}_v$ and the variance factor $\hat{\sigma}_0^2 = \dfrac{\mathbf{v}^T \mathbf{W} \mathbf{v}}{n - u}$ where $\mathbf{W}$ is the weight matrix of the n observations and u is the number of unknowns in the adjustment.

T takes values τ of the τ-distribution (*tau*-distribution) having $\nu = n - u$ degrees of freedom.

The *tau*-distribution test is defined to be a test on a single $\mathrm{T}_{\max}$ where $\mathrm{T}_{\max}$ is defined to be the maximum value of $\left|\mathrm{T}_i\right|$ for $i = 1, 2, ..., n$. It should be noted that there may be one or more T_i to give an identical value $\mathrm{T}_{\max}$

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[1] The corrections relate to Example 1 in the section *Tests on data using the tau-distribution.* Dr Manoj Deo of Perth, Western Australia contacted the authors and pointed out several errors that have been corrected in this latest version. The authors are very grateful.

If α is the level of the one-sided statistical test, and τ_α defines the critical value such that the area to the right of τ_α under the probability density function $f_\tau(\tau)$ is α, then

$$\begin{aligned}
\alpha &= \Pr\left(T_{\max} > \tau_\alpha\right) \\
&= \Pr\left(\text{one or more of } T_i > \tau_\alpha\right) \\
&= 1 - \Pr\left(\text{no } T_i > \tau_\alpha\right)
\end{aligned}$$

Therefore

$$\begin{aligned}
1 - \alpha &= \Pr\left(\text{no } T_i > \tau_\alpha\right) \\
&= \Pr\left(\text{all } T_i \le \tau_\alpha\right)
\end{aligned}$$

For $\alpha = 0.05$, $1 - \alpha = 0.95$ and from Table 1 of the tau-distribution for $\nu = 13$ degrees of freedom $\tau_\alpha = 1.6495$.

Relation between the tau-distribution and the t–distribution

The tau-distribution and Student's t-distribution are closely related and Thompson (1935) showed that if X has a t-distribution with ν degrees of freedom, that is $X \sim t_\nu$, then $Y = X\sqrt{(\nu+1)/(\nu+X^2)}$ has a tau-distribution with $\nu+1$ degrees of freedom. Conversely, if $Y \sim \tau_\nu$ then $X = Y\sqrt{(\nu-1)/(\nu-Y^2)}$ has a t-distribution with $\nu-1$ degrees of freedom (Chu 1956). Thus we may compute values of the tau-distribution from the t-distribution using

$$\tau_\nu = \frac{t_{\nu-1}\sqrt{\nu}}{\sqrt{\nu - 1 + t_{\nu-1}^2}} \tag{3}$$

and from this expression it is easily seen that (supressing the subscripts)

$$\lim_{t \to \pm\infty} \tau = \lim_{t \to \pm\infty} \frac{t\sqrt{\nu}}{\sqrt{\nu - 1 + t^2}} = \sqrt{\nu}\lim_{t \to \pm\infty} \frac{t}{\sqrt{\nu - 1 + t^2}} = \pm\sqrt{\nu}$$

The random variable $X \sim t_{\nu-1}$ has a domain $(-\infty, +\infty)$ and from this last result the random variable $X \sim \tau_\nu$ has a domain $\left[-\sqrt{\nu}, +\sqrt{\nu}\right]$.

Derivation of the probability density function of the tau-distribution

Squaring both sides of (3) and rearranging gives

$$t_{\nu-1} = t(\tau) = \frac{\tau_\nu\sqrt{\nu-1}}{\sqrt{\nu - \tau_\nu^2}} \tag{4}$$

from which

$$dt_{\nu-1} = \frac{\nu\sqrt{\nu-1}}{\left(\nu - \tau_\nu^2\right)^{3/2}} d\tau_\nu \tag{5}$$

Now it is known that the tau-distribution is bounded by $\pm\sqrt{\nu}$ and that areas under probability density function curves are equal to unity, we may write

$$\int_{-\infty}^{+\infty} f_t\left(t_{\nu-1}\right) dt = \int_{-\sqrt{\nu}}^{+\sqrt{\nu}} f_\tau\left(\tau_\nu\right) d\tau = 1 \tag{6}$$

2

The probability density function $f_t(t)$ for the t–distribution with $\nu - 1$ degrees of freedom is (Hogg & Craig 1970, p. 136)

$$f_t\left(t_{\nu-1}\right) = \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{(\nu-1)\pi}}\left(1 + \frac{t_{\nu-1}^2}{\nu-1}\right)^{-\nu/2} \tag{7}$$

where it is noted that $f_t(t) \geq 0$ for all t, and $\int_{-\infty}^{\infty} f(t)\,dt = 1$. And for positive integer ν, the gamma function $\Gamma(\nu+1) = \nu!$, while more generally $\Gamma(x+1) = x\Gamma(x)$, for real $x > 0$, with special results $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$

Substituting (4) and (5) into (7) gives

$$
\begin{aligned}
f_t\left(t_{\nu-1}\right)dt &= \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{(\nu-1)\pi}}\left[1 + \frac{\tau^2(\nu-1)}{\nu-\tau^2}\frac{1}{\nu-1}\right]^{-\nu/2}\frac{\nu\sqrt{\nu-1}}{\left(\nu-\tau^2\right)^{3/2}}d\tau_\nu \\
&= \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{\pi}}\,\nu^{1-\frac{1}{2}\nu}\left(\nu-\tau^2\right)^{\frac{1}{2}(\nu-3)}d\tau_\nu \\
&= \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{\nu\pi}}\left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)}d\tau_\nu
\end{aligned}
$$

and bearing in mind (6), the probability density function for the tau-distribution with ν degrees of freedom is (Cramer 1946, eq. 18.2.7)

$$f_\tau\left(\tau_\nu\right) = \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{\nu\pi}}\left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)} \tag{8}$$

To check this result note that $f_\tau(\tau) \geq 0$ for all $\tau \in \left[-\sqrt{\nu}, \sqrt{\nu}\right]$ and $\nu \geq 1$, and

$$
\begin{aligned}
\int_{-\sqrt{\nu}}^{\sqrt{\nu}} f_\tau(\tau) &= \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{\nu\pi}}\int_{-\sqrt{\nu}}^{\sqrt{\nu}}\left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)}d\tau \\
&= 2\frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{\nu\pi}}\int_{0}^{\sqrt{\nu}}\left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)}d\tau
\end{aligned}
$$

Now with the substitution $x = \dfrac{\tau^2}{\nu}$ then $dx = \dfrac{2\sqrt{x}}{\sqrt{\nu}}d\tau$, $x = 0$ when $\tau = 0$, $x = 1$ when $\tau = \sqrt{\nu}$ and

$$
\begin{aligned}
\int_{-\sqrt{\nu}}^{\sqrt{\nu}} f_\tau(\tau) &= \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{\pi}}\int_{0}^{1}\left(1 - x\right)^{\frac{1}{2}(\nu-3)}x^{-\frac{1}{2}}dx \\
&= \frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)\sqrt{\pi}}\int_{0}^{1}\left(1 - x\right)^{\frac{1}{2}(\nu-1)-1}x^{\frac{1}{2}-1}dx \\
&= 1
\end{aligned}
$$

noting that $\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right)$ and using the definition of the *beta function* $\mathrm{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ for

$m > 0, n > 0$ that is intimately related to the gamma function by $\mathrm{B}(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (Nahin 2015, pp. 120-122).

Properties of the tau-distribution

When $\nu = 1$, the density function (8) reduces to $f_\tau(\tau) = 0$ for all $\tau \in (-\infty, \infty)$.

When $\nu = 2$, the density function becomes $f_\tau(\tau) = \dfrac{1}{\pi\sqrt{2-\tau^2}}$, which is a U-shaped curve between

$\left(-\sqrt{2}, \sqrt{2}\right)$ with a minimum of $\dfrac{\sqrt{2}}{2\pi}$ at $\tau = 0$.

When $\nu = 3$, the distribution is rectangular with density function $f_\tau(\tau) = \dfrac{1}{2\sqrt{3}}$, for $\tau \in \left(-\sqrt{3}, \sqrt{3}\right)$, and

finally when integer $\nu > 3$ the distribution is symmetric about $\tau = 0$.

When $\nu \to \infty$ then $\left(1 - \dfrac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)} \to e^{-\frac{1}{2}\tau^2}$ (see Appendix) and using Stirling's formula[2] we write

$$\frac{\Gamma\left(\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu-1)\right)} \sim e^{-\frac{1}{2}}\left(\tfrac{1}{2}\nu\right)^{\frac{1}{2}}\left(\frac{\nu}{\nu-1}\right)^{\frac{1}{2}(\nu-1)-\frac{1}{2}} = e^{-\frac{1}{2}}\left(\tfrac{1}{2}\nu\right)^{\frac{1}{2}}\left(1-\frac{1}{\nu}\right)^{1-\frac{1}{2}\nu} \sim \left(\tfrac{1}{2}\nu\right)^{\frac{1}{2}} \text{ as } \nu \to \infty$$

Thus $f_\tau(\tau) \to \dfrac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\tau^2}$ which is the probability density function for the standard Normal $N(0,1)$ distribution.

Figure 1.
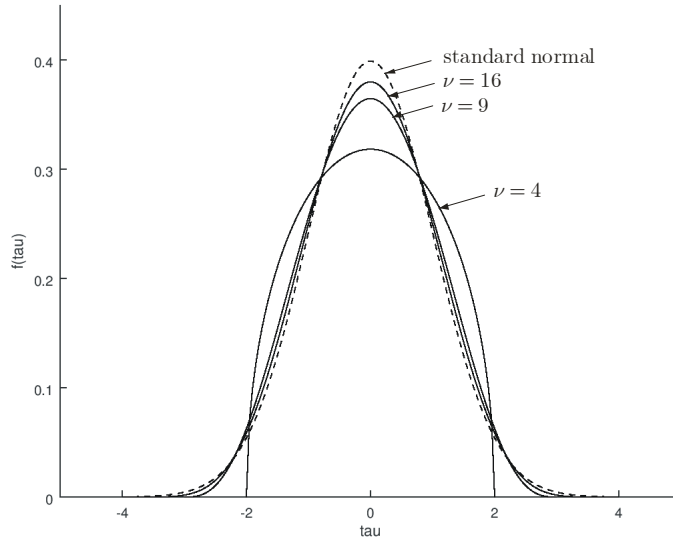tau-distribution curves for
$\nu = 4, 9$ and 16 degrees of freedom.

The dotted curve is a standard
normal distribution curve.

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[2] The Scottish mathematician James Stirling (1692–1770) proved that $C = \sqrt{2\pi}$ in the formula for integer

factorials: $n! \sim Cn^{n+\frac{1}{2}}e^{-n}$ as $n \to \infty$. This formula, now known as Stirling's formula, was first proposed by the French mathematician Abraham de Moivre (1667–1754) who only had an approximation for the constant

C. Stirling's formula can be extended to the gamma function as: $\Gamma(x) \sim Cx^{x-\frac{1}{2}}e^{-x}$ as $x \to \infty$ (Jameson)

For large values of ν the tau-distribution curve approaches the t-distribution and both curves approach the standard normal curve. For $\nu \geq 3$ the area under the tau-distribution curve equals 1.

The general shape of the tau-distribution curve can be verified by computer simulation. For example consider N = 50,000 samples each of size $n = 15$ drawn from a standard normal distribution $N(0,1)$. For each sample the mean $\bar{X}$ and S are calculated and the statistic T_1 computed from (1). These 50,000 values of T are then aggregated into a histogram. MATLAB code for this simulation is shown below.

```
% create an array of tau statistics
n = 15;
N = 50000;
tau = zeros(N,1);
for j=1:N
    x        = randn(n,1);                    % array of n standard normal variables
    tau(j,1) = (x(1)-mean(x))/std(x,1);   % tau statistic
end

% Clear Figure 1 and draw a histogram with 150 bins
figure(1);
clf(1);
box on;
hist(tau,150);
```
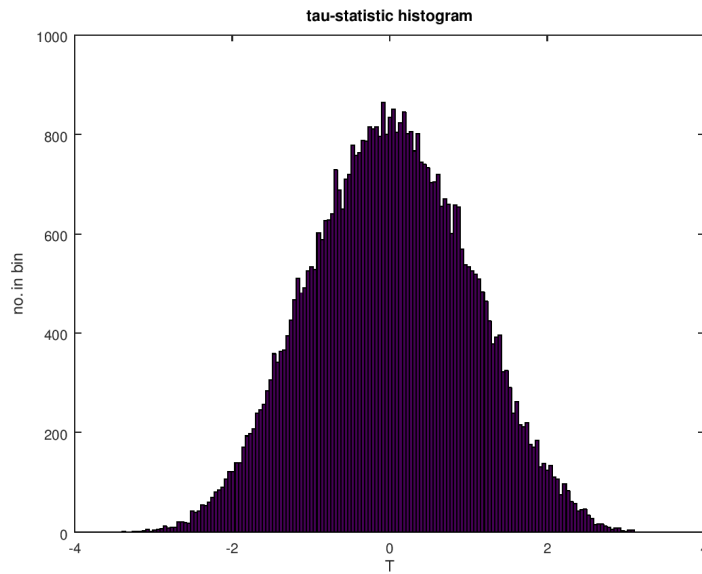

Figure 2. Histogram of T-statistic.
(50,000 values calculated from samples of size 15 drawn from standard normal population)

Values of areas under the tau-distribution (see Table 1) can be computed using the MATLAB function `tinv(1-alpha,nu-1)` for areas under the t-distribution curve and (3).

For example, the row $\nu = 13$ can be generated by the following

```
nu  = 13;                                    % degrees of freedom
t   = tinv([0.9 0.95 0.975 0.99 0.995],nu-1);  % t is a value of the t-distribution
tau = t.*sqrt(nu)./sqrt(nu-1+t.^2);          % tau is a value of the tau-distribution
```

and

```
tau = [1.3144   1.6495   1.9196   2.2068   2.3864]
```

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TABLE 1: TAU DISTRIBUTION

Probability for a given degree of freedom ν

(α is the area in the right-hand tail of the distribution between τ_α and $\sqrt{\nu}$)
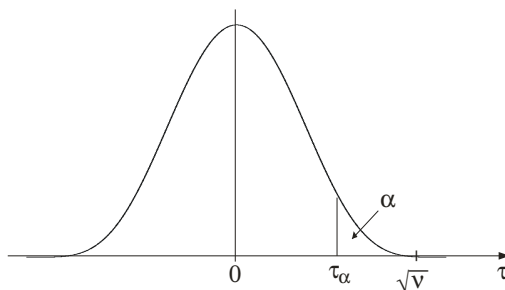
ν	α				
	0.10	0.05	0.025	0.01	0.005
3	1.3856	1.5588	1.6454	1.6974	1.7147
4	1.3741	1.6108	1.7567	1.8687	1.9175
5	1.3604	1.6308	1.8143	1.9726	2.0509
6	1.3493	1.6398	1.8481	2.0401	2.1421
7	1.3407	1.6443	1.8698	2.0868	2.2075
8	1.3339	1.6467	1.8848	2.1207	2.2562
9	1.3284	1.6481	1.8957	2.1464	2.2938
10	1.3239	1.6488	1.9039	2.1665	2.3236
11	1.3202	1.6492	1.9103	2.1826	2.3478
12	1.3171	1.6495	1.9154	2.1958	2.3678
13	1.3144	1.6495	1.9196	2.2068	2.3846
14	1.3122	1.6496	1.9231	2.2161	2.3989
15	1.3102	1.6495	1.9261	2.2240	2.4113
16	1.3084	1.6494	1.9286	2.2309	2.4220
17	1.3069	1.6494	1.9308	2.2370	2.4315
18	1.3055	1.6493	1.9327	2.2423	2.4398
19	1.3042	1.6491	1.9343	2.2470	2.4472
20	1.3031	1.6490	1.9358	2.2513	2.4539
21	1.3021	1.6489	1.9371	2.2551	2.4599
22	1.3012	1.6488	1.9383	2.2585	2.4654
23	1.3003	1.6487	1.9394	2.2616	2.4703
24	1.2996	1.6486	1.9403	2.2645	2.4749
25	1.2988	1.6485	1.9412	2.2671	2.4790
26	1.2982	1.6484	1.9420	2.2695	2.4829
27	1.2976	1.6483	1.9428	2.2717	2.4864
28	1.2970	1.6482	1.9435	2.2737	2.4897
29	1.2965	1.6482	1.9441	2.2756	2.4928
30	1.2960	1.6481	1.9447	2.2774	2.4956
40	1.2924	1.6474	1.9488	2.2900	2.5161
50	1.2902	1.6470	1.9512	2.2975	2.5282
60	1.2888	1.6467	1.9527	2.3024	2.5363
70	1.2878	1.6464	1.9538	2.3059	2.5420
80	1.2870	1.6463	1.9546	2.3085	2.5463
90	1.2864	1.6461	1.9552	2.3105	2.5496
100	1.2859	1.6460	1.9557	2.3121	2.5522
125	1.2850	1.6458	1.9566	2.3150	2.5570
150	1.2844	1.6456	1.9572	2.3169	2.5601
200	1.2837	1.6454	1.9579	2.3193	2.5641
250	1.2833	1.6453	1.9583	2.3207	2.5664

For $\nu = 9$ degrees of freedom with $\alpha = 0.05, 1 - \alpha = 0.95$ and $\tau_\alpha = 1.6481$.

(The area under the tau-distribution curve between $-\sqrt{\nu}$ and $\tau_\alpha = 1 - \alpha$)

Testing values of the tau-distribution with computer simulations

Values of the tau-distribution in Table 1 can be tested using computer simulations. For example, consider N = 50,000 samples, each of size $n = 15$, drawn from a standard normal distribution $N(0,1)$. For each sample the mean $\bar{X}$ and S are calculated and the statistic T_1 computed from (1). These 50,000 values of T are then stored in a vector *tau_vector*. For a significance level α a critical value τ_α is computed for the tau-distribution, e.g. $n = 15$, $\nu = n - 2 = 13$, $\alpha = 0.05$, $1 - \alpha = 0.95$, $\tau_\alpha = 1.6495$ and τ_α is used to determine how many of the 50,000 values of *tau_vector* are greater than τ_α. We should expect this number, expressed as a percentage of N to agree closely with the significance level α expressed as a percentage of the area under the distribution curve.

MATLAB code for this process is shown below.

```
% create an array of tau statistics
n = 15;
N = 50000;
tau_vector = zeros(N,1);
for j = 1:N
    x = randn(n,1);                              % array of standard normal variables
    tau_vector(j,1) = (x(1)-mean(x))/std(x,1);   % tau statistics
end
% find the maximum value of the tau statistics
max_tau = max(tau_vector);
% compute the value of tau from the probability distribution function for the
% particular significance level alpha
nu  = n-2;                          % degrees of freedom
t   = tinv(1-alpha,nu-1);           % t is a value of the t-distribution
tau = t*sqrt(nu)/sqrt(nu-1+t^2);    % tau is a value of the tau-distribution
% calculate the number of tau statistics greater than tau
count = 0;
for j = 1:N
  if(tau_vector(j,1) > tau)
    count = count+1;                % count is the number of values greater than tau
  end
end

% calculate the percentage of values that are greater than tau
percent = count/N*100;
```

The results from 20 such simulations are shown below

```
values from tau-simulation
N = 50000 simulations of sample of size n = 15 from standard normal distribution.
nu = n-2 = 13 degrees of freedom
alpha = 0.05, tau_alpha = 1.6495

simulation max_tau  count  percent
        1    3.2653  2503   5.0060
        2    3.1936  2515   5.0300
        3    3.2991  2557   5.1140
        4    3.3144  2494   4.9880
        5    3.1436  2566   5.1320
        6    3.2099  2516   5.0320
        7    3.2444  2345   4.6900
```

```
simulation max_tau   count   percent
        8   3.3125    2610    5.2200
        9   3.2709    2469    4.9380
       10   3.2512    2508    5.0160
       11   3.3616    2490    4.9800
       12   3.3679    2511    5.0220
       13   3.1612    2549    5.0980
       14   3.2125    2489    4.9780
       15   3.1985    2478    4.9560
       16   3.3376    2438    4.8760
       17   3.3434    2460    4.9200
       18   3.3227    2501    5.0020
       19   3.2301    2457    4.9140
       20   3.1782    2513    5.0260
```

The averages from the 20 simulations are: max_tau $= 3.2609$, count $= 2498.45$, percent $= 4.9969$

The tau-distribution curve for $\nu = 13$ is shown in Figure 3. The area under the curve between $\pm\sqrt{\nu} = \pm 3.6056$ is equal to 1 and the area under the curve between $\tau_\alpha = 1.6495$ and $\sqrt{\nu} = 3.6056$ is equal to $\alpha = 0.05$ which is 5% of the area.
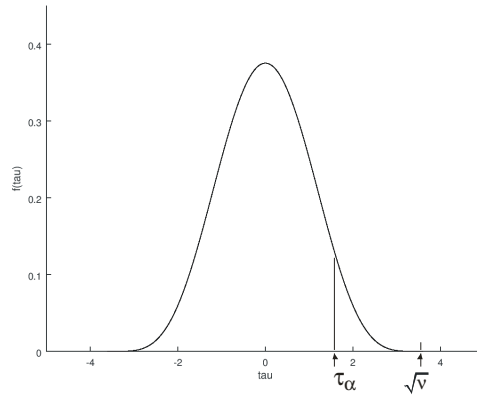

Figure 3. Tau-distribution curve for $\nu = 13$ degrees of freedom

The results of the simulations agree very well with the theoretical values

Tests on data using the tau-distribution

Three examples of testing residuals using the tau-distribution are shown below. The first is a simple test on a set of 15 residuals to decide if the largest could be regarded as an outlier and removed from the set. Nothing is known of the measurement or adjustment process and Thompson's original tau statistic (1) is used. The second and third examples (Level Network and 3D Resection respectively) involve testing of residuals that are the result of a least squares adjustment of measurements and Pope's expression of the tau statistic (2) is used.

Example 1. The data for this example is from *A Manual of Spherical and Practical Astronomy* by William Chauvenet[3] (1863) who gave the following residuals of $n = 15$ observations of the vertical semi-diameter of *Venus,* made by Lieut. Herndon, with the meridian circle at Washington in 1846:

$$-0''.30 \quad -0''.24 \quad -1''.40 \quad +0''.18$$
$$-0.44 \quad +0.06 \quad -0.22 \quad +0.39$$
$$+1.01 \quad +0.63 \quad -0.05 \quad +0.10$$
$$+0.48 \quad -0.13 \quad +0.20$$

The 7 largest residuals (in magnitude) are $-1''.40, +1''.01, +0''.63, +0''.48, -0''.44, +0''.39$ and $-0''.30$

Do some or all of these residuals fail the tau test for a significance level of $\alpha = 0.05$?

In this example we take Chauvenet's 15 residuals to be the set $\left\{ X_1 \quad X_2 \quad \ldots \quad X_{15} \right\}$ and using (1) with

$$\bar{X} = \frac{\sum X_i}{n} = \frac{0.27}{15} = 0.0180 \text{ and } S = \sqrt{\frac{1}{n}\sum\left(\bar{X} - X_i\right)^2} = \sqrt{\frac{4.2496}{15}} = 0.5323 \text{ then}$$

$$\mathrm{T}_{\max} = \frac{V_{\max}}{S} = \frac{\bar{X} - X_{\max}}{S} = \frac{0.0180 - \left(-1.40\right)}{0.5323} = 2.6639$$

The critical value τ_α from the tau-distribution for $\nu = n - 2 = 13$ degrees of freedom with significance value $\alpha = 0.05$ is found from Table 1 as $\tau_\alpha = 1.6495$. Now, since $\left|\mathrm{T}_{\max}\right| > \tau_\alpha$ the residual $-1''.40$ could be regarded as an outlier.

Removing the residual $-1''.40$ from the set leaves $n = 14$ residuals with the largest being $+1''.01$.

$$-0''.30 \quad -0''.24 \quad \cancel{-1''.40} \quad +0''.18$$
$$-0.44 \quad +0.06 \quad -0.22 \quad +0.39$$
$$+1.01 \quad +0.63 \quad -0.05 \quad +0.10$$
$$+0.48 \quad -0.13 \quad +0.20$$

We now test this residual.

Using (1) with $\bar{X} = \frac{\sum X_i}{n} = \frac{1.67}{14} = 0.1193$ and $S = \sqrt{\frac{1}{n}\sum\left(\bar{X} - X_i\right)^2} = \sqrt{\frac{2.0953}{14}} = 0.3869$ then

$$\mathrm{T}_{\max} = \frac{0.1193 - \left(1.01\right)}{0.3869} = -2.3021$$

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[3] William Chauvenet (1820–1870) was a professor of mathematics, astronomy, navigation, and surveying who was instrumental in the establishment of the U.S. Naval Academy at Annapolis, Maryland, and later the second chancellor of Washington University in St. Louis, USA. Volume 2 of his famous work on spherical and practical astronomy contained an appendix on the Method of Least Squares, within which was a section on Criterion for the Rejection of Doubtful Observations

The critical value τ_α from the tau-distribution for $\nu = n - 2 = 12$ degrees of freedom with significance value $\alpha = 0.05$ is found from Table 1 as $\tau_\alpha = 1.6495$. Now, since $\left|T_{max}\right| > \tau_\alpha$ the residual $+1''.01$ could be regarded as an outlier.

Removing the residual $+1''.01$ from the set leaves $n = 13$ residuals with the largest being $+0''.63$.

$$
\begin{array}{cccc}
-0''.30 & -0''.24 & \cancel{+1''.40} & +0''.18 \\
-0.44 & +0.06 & -0.22 & +0.39 \\
\cancel{+1.01} & +0.63 & -0.05 & +0.10 \\
+0.48 & -0.13 & +0.20 &
\end{array}
$$

We now test this residual.

Using (1) with $\bar{X} = \dfrac{\sum X_i}{n} = \dfrac{0.66}{13} = 0.0508$ and $S = \sqrt{\dfrac{1}{n}\sum\left(\bar{X} - X_i\right)^2} = \sqrt{\dfrac{1.2409}{13}} = 0.3090$ then

$$
T_{max} = \frac{0.0508 - \left(0.63\right)}{0.3090} = -1.8744
$$

The critical value τ_α from the tau-distribution for $\nu = n - 2 = 11$ degrees of freedom with significance value $\alpha = 0.05$ is found from Table 1 as $\tau_\alpha = 1.6492$. Now, since $\left|T_{max}\right| > \tau_\alpha$ the residual $+0''.63$ could be regarded as an outlier.

Removing the residual $+0''.63$ from the set leaves $n = 12$ residuals with the largest being $+0''.48$

$$
\begin{array}{cccc}
-0''.30 & -0''.24 & \cancel{+1''.40} & +0''.18 \\
-0.44 & +0.06 & -0.22 & +0.39 \\
\cancel{+1.01} & \cancel{+0.63} & -0.05 & +0.10 \\
+0.48 & -0.13 & +0.20 &
\end{array}
$$

We now test this residual, and following the procedure above, we find that this also fails and is removed from the set leaving $-0''.44$ as the largest residual in a set of 11. This is tested and fails and is removed from the set leaving $+0''.39$ as the largest residual in a set of 10. This also fails the test and is removed and now we have the set $n = 9$ residuals with the largest being $-0''.30$

$$
\begin{array}{cccc}
-0''.30 & -0''.24 & \cancel{+1''.40} & +0''.18 \\
\cancel{-0.44} & +0.06 & -0.22 & \cancel{+0.39} \\
\cancel{+1.01} & \cancel{+0.63} & -0.05 & +0.10 \\
\cancel{+0.48} & -0.13 & +0.20 &
\end{array}
$$

We now test this residual.

Using (1) with $\bar{X} = \dfrac{\sum X_i}{n} = \dfrac{-0.40}{9} = -0.0444$ and $S = \sqrt{\dfrac{1}{n}\sum\left(\bar{X} - X_i\right)^2} = \sqrt{\dfrac{0.2836}{9}} = 0.1775$ then

$$
T_{max} = \frac{-0.0444 - \left(-0.30\right)}{0.1775} = 1.4400
$$

The critical value τ_α from the tau-distribution for $\nu = n - 2 = 7$ degrees of freedom with significance value $\alpha = 0.05$ is found from Table 1 as $\tau_\alpha = 1.6443$. Now, since $\left|T_{max}\right| < \tau_\alpha$ the residual $-0''.30$ is accepted.

We now have a set of 9 residuals with a mean value $\bar{X} = -0''.0444$ and we have rejected 6 of the 7 largest (in magnitude) residuals in the original set of 15 residuals.

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Note 1: Chauvenet had a completely different method of determining outliers that is not used today, and his method rejected only the two residuals $-1''.40$ and $+1''.01$

Note 2: Tetreault (1965) uses Chauvenet's data and Thompson's tau-test with a similar result to ours above but does not go further than rejecting the two largest (magnitude) residuals. And she does not use critical values of the tau-distribution; instead, by re-arrangement of (3), she uses critical values of Student's t-distribution.

Example 2. The data for this example is from *Notes on Least Squares* (Deakin 2005).

Consider the small levelling network of height differences shown in Figure 4. A and B are stations of known elevation of 102.440 m and 104.565 m respectively. X, Y and Z are stations of unknown elevation. The arrows on the diagram indicate the direction of rise. The Table below left shows height differences in metres for each line of the network and the distance (in kilometres) of the line.

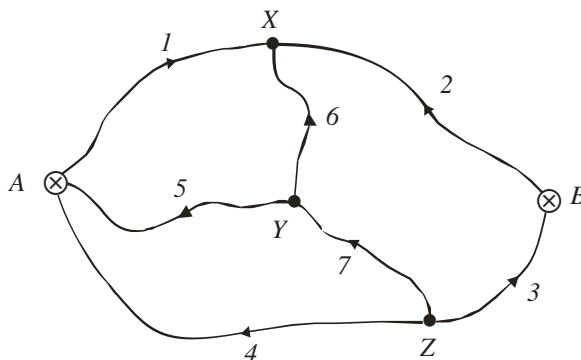| Line	Height Diff	Dist (km)
1	6.345	1.7
2	4.235	2.5
3	3.060	1.0
4	0.920	3.8
5	3.895	1.7
6	2.410	1.2
7	4.820	1.5

Figure 4. Level Network diagram

Least Squares is to be used to determine the best estimates of the elevations of X, Y and Z.

Observation equations of the form $P + \Delta H_{PQ} + v_{PQ} = Q$ can be written for each observed height difference ΔH. P, Q are relevant stations of the set $\{A, B, X, Y, Z\}$ and residuals v_{PQ} are small unknown corrections to the measured height differences. There will be $n = 7$ observation equations involving $u = 3$ unknown elevations X, Y and Z.

$$
\begin{aligned}
A + \Delta H_1 + v_1 &= X & Y + \Delta H_5 + v_5 &= A \\
B + \Delta H_2 + v_2 &= X & Y + \Delta H_6 + v_6 &= X \\
Z + \Delta H_3 + v_3 &= B & Z + \Delta H_7 + v_7 &= Y \\
Z + \Delta H_4 + v_4 &= A
\end{aligned}
$$

The n observation equations can be re-arranged into a standard least squares form

$$\mathbf{v} + \mathbf{Bx} = \mathbf{f} \tag{9}$$

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}
+
\begin{bmatrix}
-1 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & -1 & 0 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
=
\begin{bmatrix}
-A - \Delta H_1 \\
-B - \Delta H_2 \\
B - \Delta H_3 \\
A - \Delta H_4 \\
-A - \Delta H_5 \\
-\Delta H_6 \\
-\Delta H_7
\end{bmatrix}
=
\begin{bmatrix}
-108.785 \\
-108.800 \\
101.505 \\
101.520 \\
-106.335 \\
-2.410 \\
-4.820
\end{bmatrix}
$$

where $\mathbf{v}$ is the $n,1$ vector of residuals, $\mathbf{B}$ is an n,u coefficient matrix of the $u,1$ vector of unknowns $\mathbf{x}$ and $\mathbf{f}$ is an $n,1$ vector of numeric terms.

Applying the least squares principle to the set of observation equations (9) gives rise to a set of normal equations of the form $\left(\mathbf{B}^T\mathbf{W}\mathbf{B}\right)\mathbf{x} = \mathbf{B}^T\mathbf{W}\mathbf{f}$ from which the vector $\mathbf{x}$ is given by

$$\mathbf{x} = \left(\mathbf{B}^T\mathbf{W}\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{W}\mathbf{f} = \mathbf{N}^{-1}\mathbf{t} \tag{10}$$

where $\mathbf{N} = \mathbf{B}^T\mathbf{W}\mathbf{B}$ is the u,u symmetric coefficient matrix of the normal equations ($\mathbf{N}^{-1}$ denote the inverse and $\mathbf{B}^T$ denotes the transpose), $\mathbf{t} = \mathbf{B}^T\mathbf{W}\mathbf{f}$ is a $u,1$ vector of numeric terms and $\mathbf{W}$ is an n,n diagonal weight matrix whose elements are reciprocals of the distance (in km) of the levelling run and

$$\begin{aligned}\mathbf{W} &= \operatorname{diag}\left[\frac{1}{1.7}\ \ \frac{1}{2.5}\ \ \frac{1}{1.0}\ \ \frac{1}{3.8}\ \ \frac{1}{1.7}\ \ \frac{1}{1.2}\ \ \frac{1}{1.5}\right]\\ &= \operatorname{diag}\left[0.588235\ \ 0.4\ \ 1\ \ 0.263158\ \ 0.588235\ \ 0.833333\ \ 0.666667\right]\end{aligned}$$

[In general, weight matrices $\mathbf{W}$ and cofactor matrices $\mathbf{Q}$ containing estimates of variances (diagonal terms) and covariances (off-diagonal terms) are related by $\mathbf{Q} = \mathbf{W}^{-1}$ and both are symmetric matrices. For independent variables covariances will be zero and it is generally assumed that the weight matrix $\mathbf{W}$ (and the covariance matrix $\mathbf{Q}$) of the observations are diagonal matrices.]

For the solution we have

$$\mathbf{N} = \begin{bmatrix} 1.821568 & -0.833333 & 0.000000 \\ -0.833333 & 2.088235 & -0.666667 \\ 0.000000 & -0.666667 & 1.929825 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 109.519477 \\ 63.754971 \\ 125.007465 \end{bmatrix} \quad \text{and}$$

$$\mathbf{N}^{-1} = \begin{bmatrix} 6.9071\text{e-}001 & 3.0980\text{e-}001 & 1.0702\text{e-}001 \\ 3.0980\text{e-}001 & 6.7719\text{e-}001 & 2.3394\text{e-}001 \\ 1.0702\text{e-}001 & 2.3394\text{e-}001 & 5.9900\text{e-}001 \end{bmatrix}$$

The unknowns $\mathbf{x}$ (elevations of X, Y and Z) and residuals $\mathbf{v}$ are

$$\mathbf{x} = \begin{bmatrix} 108.775518 \\ 106.347074 \\ 101.514671 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} -0.009482 \\ -0.024482 \\ -0.009671 \\ 0.005329 \\ 0.012073 \\ 0.018445 \\ 0.012403 \end{bmatrix}$$

The estimate of the variance factor $\hat{\sigma}_0^2$ is given by (Mikhail 1976, Deakin 2005)

$$\hat{\sigma}_0^2 = \frac{\mathbf{v}^T\mathbf{W}\mathbf{v}}{n-u} = \frac{\mathbf{f}^T\mathbf{W}\mathbf{f} - \mathbf{x}^T\mathbf{t}}{n-u} \tag{11}$$

and for this example $\hat{\sigma}_0^2 = 2.163576\text{e-}004$

Applying the matrix rules for propagation of variances to the sequence of operations for the least squares solution (Mikhail 1976, pp. 160-161; Deakin, 2005, Ch. 5) gives the variance matrix for the residuals $\mathbf{Q}_{vv}$ as

$$\mathbf{Q}_{vv} = \mathbf{Q} - \mathbf{B}\mathbf{N}^{-1}\mathbf{B}^T \tag{12}$$

$$\mathbf{Q}_{vv} = \begin{bmatrix} 1.0093 & -0.6907 & 0.1070 & 0.1070 & -0.3098 & -0.3809 & -0.2028 \\ & 1.8093 & 0.1070 & 0.1070 & -0.3098 & -0.3809 & -0.2028 \\ & & 4.0100 & -0.5990 & 0.2339 & -0.1269 & -0.3651 \\ & & & 3.2010 & 0.2339 & -0.1269 & -0.3651 \\ & & & & 1.0228 & 0.3674 & -0.4433 \\ & & & & & 0.4517 & 0.2405 \\ & & & & & & 0.6917 \end{bmatrix}$$

With this information we may now test the residuals using (2)

For $v_1 = -0.009482$, $\hat{\sigma}_0 = \sqrt{\hat{\sigma}_0^2} = 0.014709$, $q_{v_{11}} = 1.0093$ and

$$T_1 = \frac{v_1}{\hat{\sigma}_0 \sqrt{q_{v_{11}}}} = \frac{-0.009482}{0.014709\sqrt{1.0093}} = -0.6417$$

For $v_2 = -0.024482$, $\hat{\sigma}_0 = \sqrt{\hat{\sigma}_0^2} = 0.014709$, $q_{v_{22}} = 1.8093$ and

$$T_2 = \frac{v_2}{\hat{\sigma}_0 \sqrt{q_{v_{11}}}} = \frac{-0.024482}{0.014709\sqrt{1.8093}} = -1.2374$$

and similarly for the other tau statistics giving the set of tau-statistics

$$T_i = \left\{ -0.6417 \quad -1.2374 \quad -1.0383 \quad 0.2025 \quad 0.8116 \quad 1.8658 \quad 1.0139 \right\}$$

The critical value τ_α from the tau-distribution for $\nu = n - u = 7 - 3 = 4$ degrees of freedom with significance value $\alpha = 0.05$ is found from Table 1 as $\tau_\alpha = 1.6108$. Now, ignoring the negative sign, $\left| T_6 \right| = 1.8658 > \tau_\alpha$ and the test fails for the residual $v_6 = 0.018445$ which could be regarded as an outlier.
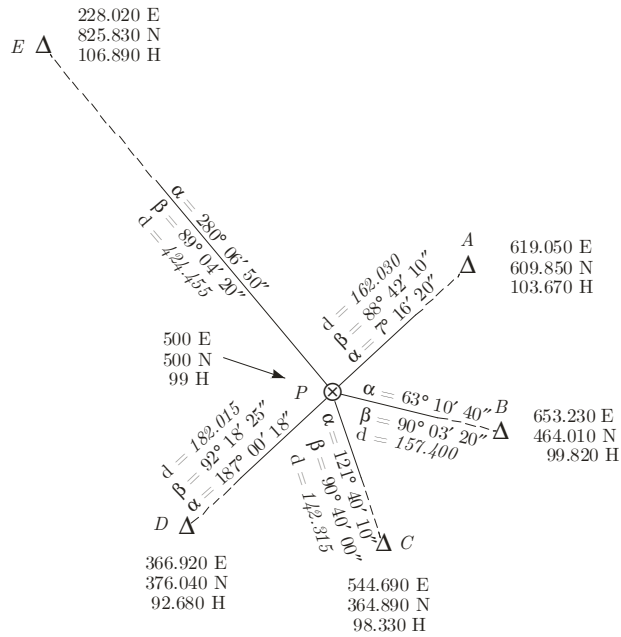
Example 3. 3D Resection.

Figure 5. Plan of 3D Resection and observations

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The diagram above shows Total Station observations from P (unknown) to five (known) stations A, B, C, D and E. The observations are: α = direction, β = zenith angle and d = slope distance. (Directions and zenith angles are Total Station horizontal and vertical circle readings respectively). Observations are made to fixed targets (prisms) at the known stations and it can be assumed that instrument height at P and target heights at the fixed stations are identical. Coordinates (E,N) and elevations (H) of stations are also shown on the diagram and the coordinates and elevation of P are approximate.

A least squares solution for the coordinates and elevation of P can be obtained from a set of $n = 15$ observation equations involving $u = 4$ unknowns (orientation constant z; E, N coordinates and elevation H at P). The observation equations are linear approximations of non-linear functions of the coordinates and elevations and the process of solution is an iterative improvement of approximate coordinates and elevation of P. This process is known as Variation of Coordinates and the solution shown below (a single iteration) will be sufficient for mm accuracy.

The observation equations for directions α, zenith angles β and slope distances d respectively are

$$
\begin{aligned}
v_i + a_i \Delta N + b_i \Delta E \quad\quad\quad +\Delta z &= \phi_i^0 - \left(\alpha_i + z^0\right) \\
v_i + l_i \Delta N + m_i \Delta E + n_i \Delta H \quad &= \beta_i^0 - \beta_i \\
v_i + e_i \Delta N + f_i \Delta E + g_i \Delta H \quad &= d_i^0 - d_i
\end{aligned}
\tag{13}
$$

where v_i are residuals (in angular or length units as appropriate); $\Delta E, \Delta N, \Delta H$ are small (unknown) corrections to approximate coordinates $E_P^0 = 500$ m, $N_P^0 = 500$ m and approximate elevation $H_P^0 = 99$ m and Δz is a small (unknown) correction to the approximate orientation constant $z^0 = 47° 18' 06''$ that is the bearing of the line P-A computed using the approximate coordinates of P.

a_i, b_i are direction coefficients; l_i, m_i, n_i are zenith angle coefficients; e_i, f_i, g_i are slope distance coefficients and ϕ_i^0 are bearings computed using approximate coordinates of P.

Numerical values for the direction coefficients are computed from

$$
\begin{aligned}
a_i &= \frac{-\sin \phi_i^0}{s_i^0} \times \rho''; & b_i &= \frac{\cos \phi_i^0}{s_i^0} \times \rho'' \\
l_i &= \frac{\left(H_i - H_P^0\right)\cos \phi_i^0}{d_i^2} \times \rho''; & m_i &= \frac{\left(H_i - H_P^0\right)\sin \phi_i^0}{d_i^2} \times \rho''; & n_i &= -\frac{s_i^0}{d_i^2} \times \rho'' \\
e_i &= \frac{N_i - N_P^0}{d_i}; & f_i &= \frac{E_i - E_P^0}{d_i}; & g_i &= \frac{H_i - H_P^0}{d_i}
\end{aligned}
\tag{14}
$$

where s_i^0 are distances computed using approximate coordinates of P and $\rho'' = \dfrac{180}{\pi} \times 3600$ is the number of seconds of arc in one radian.

The n observation equations arranged in a standard least squares form are

$$
\mathbf{v} + \mathbf{Bx} = \mathbf{f}
\tag{15}
$$

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \hline v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ \hline v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \\ v_{15} \end{bmatrix} +
\begin{bmatrix}
-9.3582 & 8.6351 & 0 & 1 \\
-12.7573 & -2.9964 & 0 & 1 \\
-4.5517 & -13.7609 & 0 & 1 \\
8.2989 & -7.7302 & 0 & 1 \\
3.1143 & 3.7309 & 0 & 1 \\
\hline
0.1955 & 0.2119 & -12.7269 & 0 \\
0.0034 & -0.0146 & -13.1045 & 0 \\
0.1615 & -0.0534 & -14.4922 & 0 \\
0.3106 & 0.3335 & -11.3231 & 0 \\
0.0605 & -0.0505 & -4.8586 & 0 \\
\hline
0.6780 & 0.7347 & 0.0227 & 0 \\
-0.2287 & 0.9735 & -0.0011 & 0 \\
-0.9493 & 0.3140 & -0.0117 & 0 \\
-0.6810 & -0.7311 & -0.0402 & 0 \\
0.7676 & -0.6407 & 0.0162 & 0
\end{bmatrix}
\begin{bmatrix} \Delta N \\ \Delta E \\ \Delta H \\ \Delta z \end{bmatrix} =
\begin{bmatrix} 0 \\ 38 \\ -5 \\ -9 \\ 14 \\ \hline -2 \\ 36 \\ 20 \\ -8 \\ -8 \\ \hline -0.1 \\ 0.0 \\ 0.4 \\ 0.1 \\ 2.8 \end{bmatrix}
$$

In these equations the corrections $\Delta E, \Delta N, \Delta H$ are in cm's and Δz is in seconds of arc. In the coefficient matrix **B** the units of the direction coefficients (1st block) and zenith angle coefficients (2nd block) are sec/cm and the units of the slope distances coefficients (3rd block) are dimensionless. In the vector of residuals **v** and the numeric terms **f** the units in the 1st, 2nd and 3rd blocks are sec's, sec's and cm's respectively.

Applying the least squares principle to the set of observation equations (15) gives rise to a set of normal equations of the form $\left(\mathbf{B}^T \mathbf{W} \mathbf{B} \right) \mathbf{x} = \mathbf{B}^T \mathbf{W} \mathbf{f}$ from which the vector **x** is given by (10) [see Example 2]

For this example the observations are considered to be independent and estimates of the standard deviations of the observations are taken to be: directions $s_\alpha = 15''$, zenith angles $s_\beta = 15''$ and slope distances $s_d = 0.005$ m + 5 ppm . The squares of these estimates are variances and these will be the diagonal elements of the cofactor matrix **Q.** Weight matrices **W** and cofactor matrices **Q** containing estimates of variances (diagonal terms) and covariances (off-diagonal terms) are related by $\mathbf{W} = \mathbf{Q}^{-1}$ and both are symmetric matrices, hence

$$
\mathbf{W} = \mathrm{diag}\left[\frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \ \frac{1}{\left(15\right)^2} \right.
$$

$$
\left. \frac{1}{\left(0.58\right)^2} \ \frac{1}{\left(0.58\right)^2} \ \frac{1}{\left(0.57\right)^2} \ \frac{1}{\left(0.59\right)^2} \ \frac{1}{\left(0.71\right)^2} \right]
$$

[The units of the first ten elements are $1/\sec^2$ and the units of the last five elements are $1/\mathrm{cm}^2$]

For the solution we have

$$
\mathbf{N} = \begin{bmatrix}
8.351313 & 0.212383 & 0.145401 & -0.067796 \\
0.212383 & 8.616139 & 0.075538 & -0.053873 \\
0.145401 & 0.075538 & 3.098428 & 0.000000 \\
-0.067796 & -0.053873 & 0.000000 & 0.022222
\end{bmatrix}, \quad
\mathbf{t} = \begin{bmatrix}
0.506063 \\ -3.278521 \\ -2.639157 \\ 0.168889
\end{bmatrix} \quad \text{and}
$$

$$
\mathbf{N}^{-1} = \begin{bmatrix}
1.2289\mathrm{e}\text{-}01 & -6.4433\mathrm{e}\text{-}04 & -5.7511\mathrm{e}\text{-}03 & 3.7335\mathrm{e}\text{-}01 \\
-6.4433\mathrm{e}\text{-}04 & 1.1788\mathrm{e}\text{-}01 & -2.8435\mathrm{e}\text{-}03 & 2.8380\mathrm{e}\text{-}01 \\
-5.7511\mathrm{e}\text{-}03 & -2.8435\mathrm{e}\text{-}03 & 3.2308\mathrm{e}\text{-}01 & -2.4439\mathrm{e}\text{-}02 \\
3.7335\mathrm{e}\text{-}01 & 2.8380\mathrm{e}\text{-}01 & -2.4439\mathrm{e}\text{-}02 & 4.6827\mathrm{e}\text{-}01
\end{bmatrix}
$$

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The unknowns $\mathbf{x}$ (coordinate and elevation corrections $\Delta N, \Delta E, \Delta H$ in cm's and orientation correction Δz in seconds of arc) and residuals $\mathbf{v}$ (in three columns: directions, seconds of arc; zenith angles, seconds of arc; slope distances, cm's) are

$$
\mathbf{x} = \begin{bmatrix} 0.142534 \\ -0.331351 \\ -0.850383 \\ 7.231549 \end{bmatrix} \quad
\mathbf{v} = \begin{bmatrix} -3.036431 & -12.780397 & 0.066109 \\ 31.593945 & 24.850828 & 0.354233 \\ -16.142467 & 7.635359 & 0.629403 \\ -19.975840 & -17.562742 & -0.079370 \\ 7.560794 & -12.157030 & 2.492070 \end{bmatrix}
$$

The adjusted coordinates and elevation of P are: 499.997 E, 500.001 N, 98.991 H. The orientation constant $z = z^0 + \Delta z = 47° 18' 06'' + 7.2'' = 47° 18' 13''$

The estimate of the variance factor is $\hat{\sigma}_0^2 = 2.487612$ (the variance factor is a dimensionless quantity).

The variance matrix for the residuals $\mathbf{Q}_{vv}$ is given by (12) and the diagonal elements of $\mathbf{Q}_{vv}$ are

$$
\mathbf{Q}_{vv}^{\text{diag}} = \begin{bmatrix} 160.60 & 168.39 & 164.60 & 160.77 & 170.91 \\ 172.62 & 169.52 & 157.12 & 183.49 & 217.37 \\ 0.21703 & 0.21797 & 0.20221 & 0.22870 & 0.38267 \end{bmatrix}
$$

[The order of the diagonal elements is: 1st row, directions, units of $\sec^2$; 2nd row, zenith angles, units of $\sec^2$; 3rd row, slope distances, units of cm^2]

With this information we may now test the residuals using (2)

For $v_1 = -3.036431$, $\hat{\sigma}_0 = \sqrt{\hat{\sigma}_0^2} = 1.577217$, $q_{v_{11}} = 160.60$ and

$$
T_1 = \frac{v_1}{\hat{\sigma}_0 \sqrt{q_{v_{11}}}} = \frac{-3.036431}{1.577217\sqrt{160.60}} = -0.1519
$$

For $v_2 = 31.593945$, $\hat{\sigma}_0 = \sqrt{\hat{\sigma}_0^2} = 1.577217$, $q_{v_{22}} = 168.39$ and

$$
T_2 = \frac{v_2}{\hat{\sigma}_0 \sqrt{q_{v_{11}}}} = \frac{31.593945}{1.577217\sqrt{168.39}} = 1.5437
$$

and similarly for the other tau statistics giving the set of tau-statistics

$$
T_i = \left\{ \begin{matrix} -0.1519 & 1.5437 & -0.7977 & -0.9989 & 0.3667 \\ -0.6167 & 1.2101 & 0.3862 & -0.8220 & -0.5228 \\ 0.0900 & 0.4811 & 0.8874 & -0.1052 & 2.5538 \end{matrix} \right\}
$$

[The order of the elements in the set is: directions 1st row, zenith angles 2nd row, slope distances 3rd row]

The critical value τ_α from the tau-distribution for $\nu = n - u = 15 - 4 = 11$ degrees of freedom with significance value $\alpha = 0.05$ is found from Table 1 as $\tau_\alpha = 1.6492$. Now, $T_{15} = 2.5538 > \tau_\alpha$ and the test fails for the residual $v_{15} = 2.492070$ which could be regarded as an outlier. This residual is the correction to the observed slope distance P-E.

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Appendix

To show that for fixed τ as $\nu \to \infty$ then $\left(1 - \dfrac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)} \to e^{-\frac{1}{2}\tau^2}$

Proof: Fix τ. Since $\left(1 - \dfrac{\tau^2}{\nu}\right)^{-\frac{3}{2}} \to 1$ as $\nu \to \infty$, that is $\left(1 - \dfrac{\tau^2}{\nu}\right)^{-\frac{3}{2}} \sim 1$, then

$$\left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)} = \left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}\nu}\left(1 - \frac{\tau^2}{\nu}\right)^{-\frac{3}{2}} \sim \left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}\nu} = \left(1 - \frac{\tau^2}{2m}\right)^{m}$$

where integer $\nu = 2m$.

Let $\frac{1}{2}\tau^2 = s$, then the binomial theorem gives

$$\left(1 - \frac{s}{m}\right)^{m} = 1 + m\left(-\frac{s}{m}\right) + \frac{m(m-1)}{2!}\left(-\frac{s}{m}\right)^2 + \frac{m(m-1)(m-2)}{3!}\left(-\frac{s}{m}\right)^3 + \cdots + \left(-\frac{s}{m}\right)^{m}$$

$$= 1 - s + \frac{1}{2!}\left(1 - \frac{1}{m}\right)s^2 - \frac{1}{3!}\left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)s^3 + \cdots$$

$$\to 1 - s + \frac{1}{2!}s^2 - \frac{1}{3!}s^3 + \cdots$$

$$= e^{-s}$$

as $m \to \infty$. So $\left(1 - \dfrac{s}{m}\right)^{m} \to e^{-s}$ as $m \to \infty$, therefore

$$\left(1 - \frac{\tau^2}{\nu}\right)^{\frac{1}{2}(\nu-3)} \to e^{-\frac{1}{2}\tau^2} \text{ as } \nu \to \infty$$

Notes.   The $\sim$ symbol is used here in the following sense: $f(\nu) \sim g(\nu)$ as $\nu \to \infty$ means

$$\frac{f(\nu)}{g(\nu)} \to 1 \text{ as } \nu \to \infty$$

The series for the exponential function $e^x = 1 + x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \cdots$