YARRA BEND LEVEL NET ADJUSTMENT EXERCISE

An example of a LEVEL NET is shown below. The network, in the vicinity of the RMIT Surveying Annexe at Yarra Bend Park consists of four points; three are Bench Marks (BM's - star pickets set in concrete) and the fourth is a Temporary Bench Mark (TBM) placed near the Surveying Annexe for some future purpose. The BM's have known Reduced Levels (RL's) on the Australian Height Datum (AHD) but one of the BM's is thought to have moved and its RL may be unreliable. The purpose of the exercise is to use the method of least squares to:

    (i)   determine which BM may have an unreliable RL and

   (ii)   compute the best estimate of the RL of the TBM.

Three least squares solutions will be demonstrated

    1:    Least squares adjustment of observations only

    2:    Least squares adjustment of indirect observations

    3:    Free Net adjustment (least squares adjustment of indirect observation with inner constraint)

The diagram below shows the level network of height differences observed between BM's 707, 726, 727 and TBM X (shown as points *A, B, C* and *X*). The arrows on the diagram indicate the direction of rise. The table shows the height difference (in metres) for each line of the network and the distance (in metres) of each level run. The observations were made with a Leica digital level and calibrated invar staves.

ADJUSTMENT OF OBSERVATIONS ONLY

Line	Height Diff	Dist (m)
1	0.89650	99
2	0.10980	130
3	0.29355	215
4	1.18899	264
5	0.18418	365
6	1.00653	113
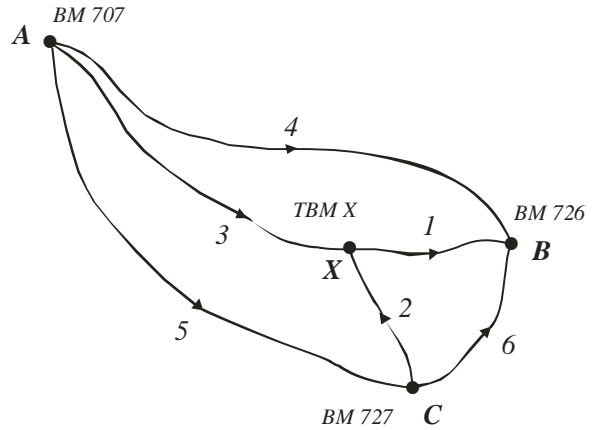
There are $n = 6$ levelled height differences (observations) and only $n_0 = 3$ observations are required to determine the heights of all points, hence there are $r = n - n_0$ redundant measurements, which is equal to the number of independent observation equations.

Observation equations:

$$l_1 + v_1 - (l_6 + v_6) + l_2 + v_2 \quad = 0 \qquad \text{loop: X-726-727-X}$$
$$-(l_2 + v_2) - (l_5 + v_5) + l_3 + v_3 = 0 \qquad \text{loop: X-727-707-X}$$
$$-(l_3 + v_3) + l_4 + v_4 - (l_1 + v_1) = 0 \qquad \text{loop: X-707-726-X}$$

rearranged as

$$v_1 + v_2 - v_6 \quad = -l_1 - l_2 + l_6 = 0.00023$$
$$-v_2 + v_3 - v_5 = l_2 - l_3 + l_5 = 0.00043$$
$$-v_1 - v_3 + v_4 = l_1 + l_3 - l_4 = 0.00106$$

In matrix form $\mathbf{A}_{r \times n} \mathbf{v}_{n \times 1} = \mathbf{f}_{r \times 1}$

$$
\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}_{r \times n}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}}_{\mathbf{v}_{n \times 1}} = \underbrace{\begin{bmatrix} 0.00023 \\ 0.00043 \\ 0.00106 \end{bmatrix}}_{\mathbf{f}_{r \times 1}}
$$

The weight w of each observation is assumed to be inversely proportional to the distance in kilometres of the level run, and by definition, weights are assumed to be inversely proportional to variances, hence

$$
w_i \propto \frac{1}{\text{dist}_i(\text{km})} \qquad \text{and} \qquad w_i \propto \frac{1}{s_i^2}
$$

where s_i^2 is an estimate of the variance of observation i and the weight matrix $\mathbf{W}$ is the inverse of the cofactor matrix $\mathbf{Q}$ containing estimates of variances and covariances, hence

$$
\mathbf{W} = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & w_6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Q} = \begin{bmatrix} s_1^2 & 0 & \cdots & 0 \\ 0 & s_2^2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & s_6^2 \end{bmatrix}
$$

and

$$
\mathbf{Q} = \begin{bmatrix} 0.099 & & & & & \\ & 0.130 & & & & \\ & & 0.215 & & & \\ & & & 0.264 & & \\ & & & & 0.365 & \\ & & & & & 0.113 \end{bmatrix}
$$

The solution is (see copy of MATLAB Command Window)

$$
\mathbf{W}_e = \mathbf{A}\mathbf{Q}\mathbf{A}^T = \begin{bmatrix} 0.3420 & -0.1300 & -0.0990 \\ -0.1300 & 0.7100 & -0.2150 \\ -0.0990 & -0.2150 & 0.5780 \end{bmatrix}
$$

$$\mathbf{k} = \left(\mathbf{AQA}^T\right)^{-1}\mathbf{f} = \mathbf{W}_e^{-1}\mathbf{f} = \begin{bmatrix} 0.002242401 \\ 0.001902112 \\ 0.002925522 \end{bmatrix}$$

The vector of residuals $\mathbf{v}$ and adjusted observations $\hat{\mathbf{l}}$ is

$$\mathbf{v} = \mathbf{QA}^T\mathbf{k} = \begin{bmatrix} -0.0000676 \\ 0.0000442 \\ -0.0002200 \\ 0.0007723 \\ -0.0006943 \\ -0.0002534 \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{l}} = \mathbf{l} + \mathbf{v} = \begin{bmatrix} 0.89643 \\ 0.10984 \\ 0.29333 \\ 1.18976 \\ 0.18349 \\ 1.00628 \end{bmatrix}$$

Checks

$$\hat{l}_1 - \hat{l}_6 + \hat{l}_2 \;\; = 0.00001 \qquad \text{loop: X-726-727-X}$$
$$-\hat{l}_2 - \hat{l}_5 + \hat{l}_3 \;\; = 0.00000 \qquad \text{loop: X-727-707-X}$$
$$-\hat{l}_3 + \hat{l}_4 - \hat{l}_1 \;\; = 0.00000 \qquad \text{loop: X-707-726-X}$$

Variance factor $\qquad\qquad\qquad \hat{\sigma}_0^2 = \dfrac{\mathbf{v}^T\mathbf{Wv}}{r} = 1.4782e-06$

Variances of adjusted observations (height differences) are the diagonal elements of the matrix

$\mathbf{\Sigma}_{\hat{l}\hat{l}} = \hat{\sigma}_0^2\mathbf{Q}_{\hat{l}\hat{l}}$ where $\mathbf{Q}_{\hat{l}\hat{l}} = \mathbf{Q} - \mathbf{Q}_{vv}$ and $\mathbf{Q}_{vv} = \mathbf{QA}^T\mathbf{W}_e\mathbf{AQ}$

The diagonal elements of $\mathbf{Q}_{\hat{l}\hat{l}} = \begin{bmatrix} 0.0880 & 0.1078 & 0.1356 & 0.2237 & 0.2704 & 0.1086 \end{bmatrix}$ giving

$$\sigma_1^2 = 1.3008e-07\,\text{m}^2 \quad \sigma_1 = 0.00036\ \text{m}$$
$$\sigma_2^2 = 1.5935e-07 \qquad\quad \sigma_2 = 0.00040\ \text{m}$$
$$\sigma_3^2 = 2.0044e-07 \qquad\quad \sigma_3 = 0.00045\ \text{m}$$
$$\sigma_4^2 = 3.3067e-07 \qquad\quad \sigma_4 = 0.00058\ \text{m}$$
$$\sigma_5^2 = 3.9971e-07 \qquad\quad \sigma_5 = 0.00063\ \text{m}$$
$$\sigma_6^2 = 1.6053e-07 \qquad\quad \sigma_6 = 0.00040\ \text{m}$$

Standard deviation of 1 kilometre of levelling $\sigma_0 = 0.00122$ m

Adjusted Heights

The "published values" of the BM's are: BM 707   RL 27.751 (AHD)

                                                            BM 726   RL 28.942

                                                            BM 727   RL 27.951

Using the adjusted height differences $\hat{\mathbf{l}}$ and the RL for BM 707 gives

                                                            BM 707   RL 27.751 (AHD)

                                                            BM 726   RL 28.941

                                                            BM 727   RL 27.934

                                                            TBM X   RL 28.044

It would appear that the published value of BM 727 is incorrect. The difference between the published value and the computed value above is approximately 0.017 metres.

YARRA BEND LEVEL NET EXERCISE – MATLAB RESULTS

Adjustment of Observations Only

MATLAB command window

```
>> A = [1 1 0 0 0 -1;0 -1 1 0 -1 0;-1 0 -1 1 0 0]

A =

    1    1    0    0    0   -1
    0   -1    1    0   -1    0
   -1    0   -1    1    0    0

>> q = [0.099 0.130 0.215 0.264 0.365 0.113];
>> Q = diag(q)

Q =

   0.0990        0        0        0        0        0
        0   0.1300        0        0        0        0
        0        0   0.2150        0        0        0
        0        0        0   0.2640        0        0
        0        0        0        0   0.3650        0
        0        0        0        0        0   0.1130

>> W = inv(Q)

W =

  10.1010        0        0        0        0        0
        0   7.6923        0        0        0        0
        0        0   4.6512        0        0        0
        0        0        0   3.7879        0        0
        0        0        0        0   2.7397        0
        0        0        0        0        0   8.8496

>> We = A*Q*A'

We =

   0.3420   -0.1300   -0.0990
  -0.1300    0.7100   -0.2150
  -0.0990   -0.2150    0.5780

>> format long
>> f = [0.00023; 0.00043; 0.00106]

f =

   0.00023000000000
   0.00043000000000
   0.00106000000000

>> k = inv(We)*f

k =

   0.00224240125147
   0.00190211184982
   0.00292552209621
```

YARRA BEND LEVEL NET EXERCISE – MATLAB RESULTS

Adjustment of Observations Only

MATLAB command window

```
>> format short
>> v = Q*A'*k

v =

  1.0e-003 *

   -0.0676
    0.0442
   -0.2200
    0.7723
   -0.6943
   -0.2534

>> Qvv = Q*A'*We*A*Q

Qvv =

    0.0110    0.0046    0.0162   -0.0177   -0.0031   -0.0049
    0.0046    0.0222   -0.0267    0.0040    0.0399   -0.0069
    0.0162   -0.0267    0.0794   -0.0450   -0.0726    0.0008
   -0.0177    0.0040   -0.0450    0.0403    0.0207    0.0030
   -0.0031    0.0399   -0.0726    0.0207    0.0946   -0.0054
   -0.0049   -0.0069    0.0008    0.0030   -0.0054    0.0044

>> Qll = Q-Qvv

Qll =

    0.0880   -0.0046   -0.0162    0.0177    0.0031    0.0049
   -0.0046    0.1078    0.0267   -0.0040   -0.0399    0.0069
   -0.0162    0.0267    0.1356    0.0450    0.0726   -0.0008
    0.0177   -0.0040    0.0450    0.2237   -0.0207   -0.0030
    0.0031   -0.0399    0.0726   -0.0207    0.2704    0.0054
    0.0049    0.0069   -0.0008   -0.0030    0.0054    0.1086

>> vfact = (v'*W*v)/3

vfact =

  1.4782e-006

>>
```

ADJUSTMENT OF INDIRECT OBSERVATIONS

(BM707 FIXED – RL 27.751 AHD)

Line	Height Diff	Dist (m)
1	0.89650	99
2	0.10980	130
3	0.29355	215
4	1.18899	264
5	0.18418	365
6	1.00653	113
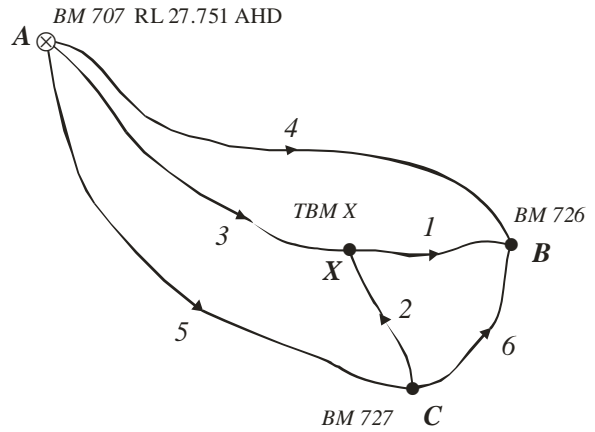
There are $n = 6$ levelled height differences (observations) and $u = 3$ unknown heights (*B, C* and *X*)

Observation equations:

$$\mathrm{RL}_X + l_1 + v_1 = \mathrm{RL}_B$$
$$\mathrm{RL}_C + l_2 + v_2 = \mathrm{RL}_X$$
$$\mathrm{RL}_A + l_3 + v_3 = \mathrm{RL}_X$$
$$\mathrm{RL}_A + l_4 + v_4 = \mathrm{RL}_B$$
$$\mathrm{RL}_A + l_5 + v_5 = \mathrm{RL}_C$$
$$\mathrm{RL}_C + l_6 + v_6 = \mathrm{RL}_B$$

rearranged as

$$
\begin{array}{rrrrcl}
v_1 & -B & & +X & = & -l_1 \\
v_2 & & +C & -X & = & -l_2 \\
v_3 & & & -X & = & -(A+l_3) \\
v_4 & -B & & & = & -(A+l_4) \\
v_5 & & -C & & = & -(A+l_5) \\
v_6 & -B & +C & & = & -l_6
\end{array}
$$

In matrix form $\mathbf{v}_{n\times 1} + \mathbf{B}_{n\times u}\mathbf{x}_{u\times 1} = \mathbf{f}_{n\times 1}$

$$
\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}}_{\mathbf{v}_{6\times 1}} + \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}}_{\mathbf{B}_{6\times 3}} \underbrace{\begin{bmatrix} B \\ C \\ X \end{bmatrix}}_{\mathbf{x}_{3\times 1}} = \underbrace{\begin{bmatrix} -l_1 \\ -l_2 \\ -(A+l_3) \\ -(A+l_4) \\ -(A+l_5) \\ -l_6 \end{bmatrix}}_{\mathbf{f}_{6\times 1}} = \begin{bmatrix} -0.89650 \\ -0.10980 \\ -28.04455 \\ -28.93999 \\ -27.93518 \\ -1.00653 \end{bmatrix}
$$

The weight w of each observation is assumed to be inversely proportional to the distance in kilometres of the level run, and by definition, weights are assumed to be inversely proportional to variances, hence

$$
w_i \propto \frac{1}{\text{dist}_i\,(\text{km})} \qquad \text{and} \qquad w_i \propto \frac{1}{s_i^2}
$$

where s_i^2 is an estimate of the variance of observation i and the weight matrix $\mathbf{W}$ is the inverse of the cofactor matrix $\mathbf{Q}$ containing estimates of variances and covariances

$$
\mathbf{W} = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & w_6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Q} = \mathbf{W}^{-1}
$$

and

$$
\mathbf{W} = \begin{bmatrix} 10.1010 & & & & & \\ & 7.6923 & & & & \\ & & 4.6512 & & & \\ & & & 3.7879 & & \\ & & & & 2.7397 & \\ & & & & & 8.8486 \end{bmatrix}
$$

The solution is (see copies of data and output files from the MATLAB program *least_squares.m*)

Normal equations $\left(\mathbf{B}^T\mathbf{W}\mathbf{B}\right)\mathbf{x} = \mathbf{B}^T\mathbf{W}\mathbf{f}$ or $\mathbf{N}\mathbf{x} = \mathbf{t}$

$$
\mathbf{N} = \mathbf{B}^T\mathbf{W}\mathbf{B} = \begin{bmatrix} 22.738500 & -8.849600 & -10.101000 \\ -8.849600 & 19.281600 & -7.692300 \\ -10.101000 & -7.692300 & 22.444500 \end{bmatrix} \quad \text{and} \quad \mathbf{t} = \mathbf{B}^T\mathbf{W}\mathbf{f} = \begin{bmatrix} 127.584723 \\ 66.782010 \\ 122.229879 \end{bmatrix}
$$

Solutions $\mathbf{x} = \mathbf{N}^{-1}\mathbf{t}$

$$
\mathbf{x} = \mathbf{N}^{-1}\mathbf{t} = \begin{bmatrix} 1.1005e\text{-}001 & 8.1399e\text{-}002 & 7.7426e\text{-}002 \\ 8.1399e\text{-}002 & 1.2028e\text{-}001 & 7.7857e\text{-}002 \\ 7.7426e\text{-}002 & 7.7857e\text{-}002 & 1.0608e\text{-}001 \end{bmatrix}\begin{bmatrix} 127.584723 \\ 66.782010 \\ 122.229879 \end{bmatrix} = \begin{bmatrix} 28.94076 \\ 27.93449 \\ 28.04433 \end{bmatrix}
$$

Variance factor $\qquad\qquad \hat{\sigma}_0^2 = \dfrac{\mathbf{v}^T\mathbf{W}\mathbf{v}}{n-u} = \dfrac{\mathbf{f}^T\mathbf{W}\mathbf{f} - \mathbf{x}^T\mathbf{t}}{n-u} = 1.4782e-06$

Variances of adjusted heights

$$
\Sigma_{xx} = \sigma_0^2\mathbf{Q}_{xx} = \sigma_0^2\mathbf{N}^{-1} = \begin{bmatrix} 1.6267e\text{-}7 & 1.2032e\text{-}7 & 1.1445e\text{-}7 \\ 1.2032e\text{-}7 & 1.7780e\text{-}7 & 1.1509e\text{-}7 \\ 1.1445e\text{-}7 & 1.1509e\text{-}7 & 1.5681e\text{-}7 \end{bmatrix}
$$

$$
\begin{aligned}
\sigma_B^2 &= 1.6267e-07\,\text{m}^2 & \sigma_B &= 0.00040\ \text{m} \\
\sigma_C^2 &= 1.7780e-07 & \sigma_C &= 0.00042\ \text{m} \\
\sigma_3^2 &= 1.5681e-07 & \sigma_X &= 0.00040\ \text{m}
\end{aligned}
$$

Variances of adjusted observations (height differences) are the diagonal elements of the matrix

$\Sigma_{\hat{\ell}\hat{\ell}} = \hat{\sigma}_0^2\mathbf{Q}_{\hat{\ell}\hat{\ell}}$ where $\mathbf{Q}_{\hat{\ell}\hat{\ell}} = \mathbf{Q} - \mathbf{Q}_{vv}$ and $\mathbf{Q}_{vv} = \mathbf{Q} - \mathbf{B}\mathbf{N}^{-1}\mathbf{B}^T$

The diagonal elements of

$\mathbf{Q}_{\hat{\ell}\hat{\ell}} = \begin{bmatrix} 6.1284e-2 & 7.0652e-2 & 1.0608e-1 & 1.1005e-1 & 1.2028e-1 & 6.7537e-2 \end{bmatrix}$ giving

$$
\begin{aligned}
\sigma_1^2 &= 9.0590e-08\,\text{m}^2 & \sigma_1 &= 0.00030\ \text{m} \\
\sigma_2^2 &= 1.0444e-07 & \sigma_2 &= 0.00032\ \text{m} \\
\sigma_3^2 &= 1.5681e-07 & \sigma_3 &= 0.00040\ \text{m} \\
\sigma_4^2 &= 1.6268e-07 & \sigma_4 &= 0.00040\ \text{m} \\
\sigma_5^2 &= 1.7780e-07 & \sigma_5 &= 0.00042\ \text{m} \\
\sigma_6^2 &= 9.9833e-08 & \sigma_6 &= 0.00032\ \text{m}
\end{aligned}
$$

Standard deviation of 1 kilometre of levelling $\sigma_0 = 0.00122$ m

Residuals and adjusted observations

$$\mathbf{v} = \mathbf{f} - \mathbf{Bx} = \begin{bmatrix} -0.000068 \\ 0.000044 \\ -0.000220 \\ 0.000772 \\ -0.000694 \\ -0.000253 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{l}} = \mathbf{l} + \mathbf{v} = \begin{bmatrix} 0.89643 \\ 0.10984 \\ 0.29333 \\ 1.18976 \\ 0.18349 \\ 1.00628 \end{bmatrix}$$

These results for the adjusted observations are identical to those obtained by the method of Adjustment of Observations Only.

Adjusted Heights

The "published values" of the BM's are: BM 707   RL 27.751 (AHD)
                                            BM 726   RL 28.942
                                            BM 727   RL 27.951

The adjusted heights of the BM's are: BM 726   RL 28.941 (AHD)
                                            BM 727   RL 27.934
                                            TBM X   RL 28.044

It would appear that the published RL of BM 727 is incorrect. The difference between the published value and the computed value above is approximately 0.017 metres.

These results for the adjusted heights are identical (nearest mm) to those obtained by the method of Adjustment of Observations Only.

YARRA BEND LEVEL NET EXERCISE – MATLAB RESULTS

Adjustment of Indirect Observations

Data file *YarraBendLevelNetExercise1.dat* for MATLAB program *least_squares.m*

```
% data file for Yarra Bend Level Net Exercise
% Adjustment of Indirect Observations
%   B(1)  B(2)  B(3)        f         w
    -1     0     1      -0.89650   10.1010
     0     1    -1      -0.10980    7.6923
     0     0    -1     -28.04455    4.6512
    -1     0     0     -28.93999    3.7879
     0    -1     0     -27.93518    2.7397
    -1     1     0      -1.00653    8.8496
```

Output file *YarraBendLevelNetExercise1.out* from MATLAB program *least_squares.m*

```
Least Squares Adjustment of Indirect Observations

Input Data

Coefficient matrix B of observation equations v + Bx = f
   -1.0000    0.0000    1.0000
    0.0000    1.0000   -1.0000
    0.0000    0.0000   -1.0000
   -1.0000    0.0000    0.0000
    0.0000   -1.0000    0.0000
   -1.0000    1.0000    0.0000

Vector of numeric terms f and weights w of observation equations v + Bx = f
   -0.896500    10.101000
   -0.109800     7.692300
  -28.044550     4.651200
  -28.939990     3.787900
  -27.935180     2.739700
   -1.006530     8.849600

Coefficient matrix N of Normal equations Nx = t
(upper triangular part)
   22.738500   -8.849600  -10.101000
   19.281600   -7.692300
   22.444500

Vector of numeric terms t of Normal equations Nx = t
  127.584723
   66.782010
  122.229879

Inverse of Normal equation coefficient matrix
(upper triangular part)
    1.1005e-001    8.1399e-002    7.7426e-002
    1.2028e-001    7.7857e-002
    1.0608e-001

Vector of solutions x
    2.894076234e+001
    2.793448573e+001
    2.804432997e+001

Vector of residuals v
 -0.000068
  0.000044
 -0.000220
  0.000772
 -0.000694
 -0.000253
```

YARRA BEND LEVEL NET EXERCISE – MATLAB RESULTS

Adjustment of Indirect Observations

```
Variance factor =      1.4782e-006

Cofactor matrix of unknowns Qxx
(upper triangular part)
     1.1005e-001    8.1399e-002     7.7426e-002
     1.2028e-001    7.7857e-002
     1.0608e-001

Cofactor matrix of residuals Qvv
(upper triangular part)
     3.7716e-002    3.2199e-002     2.8657e-002    -3.2627e-002    -3.5419e-003
    -2.9085e-002
     5.9348e-002   -2.8226e-002     3.9731e-003     4.2426e-002    -3.8453e-002
     1.0892e-001   -7.7426e-002    -7.7857e-002     4.3125e-004
     1.5395e-001   -8.1399e-002    -2.8653e-002
     2.4472e-001    3.8884e-002
     4.5462e-002

Cofactor matrix of adjusted observations Qll
(upper triangular part)
     6.1284e-002   -3.2199e-002    -2.8657e-002     3.2627e-002     3.5419e-003
     2.9085e-002
     7.0652e-002    2.8226e-002    -3.9731e-003    -4.2426e-002     3.8453e-002
     1.0608e-001    7.7426e-002     7.7857e-002    -4.3125e-004
     1.1005e-001    8.1399e-002     2.8653e-002
     1.2028e-001   -3.8884e-002
     6.7537e-002
```

FREE NET LEVEL ADJUSTMENT

(NO FIXED STATIONS)

(Adjustment of Indirect Observations with Inner Constraint)

Line	Height Diff	Dist (m)
1	0.89650	99
2	0.10980	130
3	0.29355	215
4	1.18899	264
5	0.18418	365
6	1.00653	113
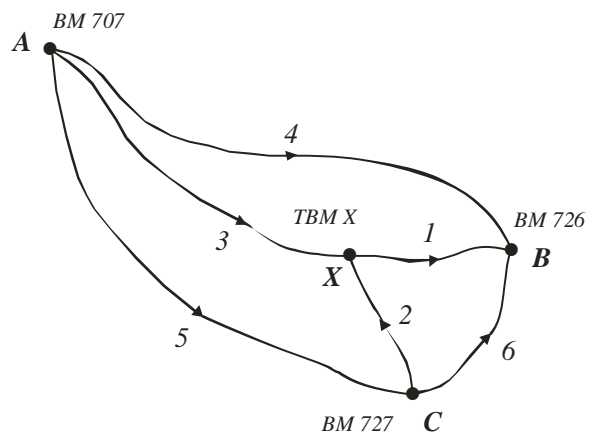
There are $n = 6$ levelled height differences (observations) and $u = 4$ unknown heights (*A, B, C* and *X*).

Observation equations:

$$\mathrm{RL}_X + l_1 + v_1 = \mathrm{RL}_B$$
$$\mathrm{RL}_C + l_2 + v_2 = \mathrm{RL}_X$$
$$\mathrm{RL}_A + l_3 + v_3 = \mathrm{RL}_X$$
$$\mathrm{RL}_A + l_4 + v_4 = \mathrm{RL}_B$$
$$\mathrm{RL}_A + l_5 + v_5 = \mathrm{RL}_C$$
$$\mathrm{RL}_C + l_6 + v_6 = \mathrm{RL}_B$$

rearranged as

$$
\begin{array}{lllllll}
v_1 & & -B & & +X & = & -l_1 \\
v_2 & & & +C & -X & = & -l_2 \\
v_3 & +A & & & -X & = & -l_3 \\
v_4 & +A & -B & & & = & -l_4 \\
v_5 & +A & & -C & & = & -l_5 \\
v_6 & & -B & +C & & = & -l_6 \\
\end{array}
$$

In matrix form $\mathbf{v}_{n \times 1} + \mathbf{B}_{n \times u} \mathbf{x}_{u \times 1} = \mathbf{f}_{n \times 1}$

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ X \end{bmatrix} = \begin{bmatrix} -l_1 \\ -l_2 \\ -l_3 \\ -l_4 \\ -l_5 \\ -l_6 \end{bmatrix}
$$

Note here that **B** is rank deficient. The rank of **B** is $rank(\mathbf{B}) \le u$ (and $u = 4$), and if we carry out elementary row transformations on the elements of **B** we obtain a reduced matrix of the form

$$
\begin{bmatrix} 1 & 0 & 0 & \vdots & -1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & -1 \\ \hdashline 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}
$$

and can identify that the largest non-zero determinant is of order 3. Hence $rank(\mathbf{B}) = 3$ which is one less than its possible value; therefore **B** is rank deficient. The elements of the reduced matrix **B** above are obtained using the MATLAB function `rref()` which reduces a matrix to row echelon form (rref = reduced row echelon form) using Gauss Jordan elimination with partial pivoting.

```
>> B = [0 -1 0 1; 0 0 1 -1; 1 0 0 -1; 1 -1 0 0; 1 0 -1 0; 0 -1 1 0]

B =

     0    -1     0     1
     0     0     1    -1
     1     0     0    -1
     1    -1     0     0
     1     0    -1     0
     0    -1     1     0

>> rref(B)

ans =

     1     0     0    -1
     0     1     0    -1
     0     0     1    -1
     0     0     0     0
     0     0     0     0
     0     0     0     0

>>
```

As before, the weight w of each observation is assumed to be inversely proportional to the distance in kilometres of the level run, and by definition, weights are assumed to be inversely proportional to variances, hence

$$w_i \propto \frac{1}{\text{dist}_i (\text{km})} \qquad \text{and} \qquad w_i \propto \frac{1}{s_i^2}$$

where s_i^2 is an estimate of the variance of observation i and the weight matrix $\mathbf{W}$ is the inverse of the cofactor matrix $\mathbf{Q}$ containing estimates of variances and covariances

$$\mathbf{W} = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & w_6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Q} = \mathbf{W}^{-1}$$

and

$$\mathbf{W} = \begin{bmatrix} 10.1010 & & & & & \\ & 7.6923 & & & & \\ & & 4.6512 & & & \\ & & & 3.7879 & & \\ & & & & 2.7397 & \\ & & & & & 8.8486 \end{bmatrix}$$

The normal equations are

$$\mathbf{Nx} = \mathbf{t}$$

where

$$\mathbf{N} = \mathbf{B}^T \mathbf{W} \mathbf{B} \quad \text{and} \quad \mathbf{t} = \mathbf{B}^T \mathbf{W} \mathbf{f}$$

but because of the rank deficiency of $\mathbf{B}$, the normal coefficient matrix $\mathbf{N}$ will also be rank deficient and $\mathbf{N}$ is <u>singular</u>, i.e., $|\mathbf{N}| = \mathbf{0}$ and $\mathbf{N}^{-1}$ does not exist; there is <u>no solution</u> of the normal equations by conventional means, so a constraint equation will be added. The general form of the set of c constraint equations is

$$\mathbf{Cx} = \mathbf{g}$$

$\mathbf{C}$ is the $c \times u$ matrix of coefficients

$\mathbf{g}$ is the $c \times 1$ vector of numeric terms (constants)

The constraint equations are added to the normal equations to give an augmented set of equation of the form

$$\left[\begin{array}{c|c} -\mathbf{N} & \mathbf{C}^T \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{k} \end{array}\right] = \left[\begin{array}{c} -\mathbf{t} \\ \hline \mathbf{g} \end{array}\right]$$

But, in our case, we are adding an INNER CONSTRAINT so the constraint equations have the form $\mathbf{Cx} = \mathbf{0}$ and the augmented system of equations becomes

$$\left[\begin{array}{c|c} -\mathbf{N} & \mathbf{C}^T \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{k} \end{array}\right] = \left[\begin{array}{c} -\mathbf{t} \\ \hline \mathbf{0} \end{array}\right]$$

Noting here that the (single) inner constraint equation is:

$$A + B + C + X = 0$$

The solution is (see copies of data and output files from the MATLAB program *constrained_ls.m*)

Normal equations $\left(\mathbf{B}^T\mathbf{WB}\right)\mathbf{x} = \mathbf{B}^T\mathbf{Wf}$ or $\mathbf{Nx} = \mathbf{t}$

$$\mathbf{N} = \mathbf{B}^T\mathbf{WB} = \begin{bmatrix} 11.1788 & -3.7879 & -2.7397 & -4.6512 \\ -3.7879 & 22.7375 & -8.8486 & -10.1010 \\ -2.7937 & -8.8486 & 19.2806 & -7.6923 \\ -4.6512 & -10.1010 & -7.6923 & 22.4445 \end{bmatrix} \text{ and } \mathbf{t} = \mathbf{B}^T\mathbf{Wf} = \begin{bmatrix} -6.373733 \\ 22.465703 \\ -9.246398 \\ -6.845572 \end{bmatrix}$$

Augmented system of equations $\left[\begin{array}{c|c} -\mathbf{N} & \mathbf{C}^T \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{k} \end{array}\right] = \left[\begin{array}{c} -\mathbf{t} \\ \hline \mathbf{g} \end{array}\right]$

$$\left[\begin{array}{cccc|c} -11.1788 & 3.7879 & 2.7397 & 4.6512 & 1 \\ 3.7879 & -22.7375 & 8.8486 & 10.1010 & 1 \\ 2.7937 & 8.8486 & -19.2806 & 7.6923 & 1 \\ 4.6512 & 10.1010 & 7.6923 & -22.4445 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 \end{array}\right]\left[\begin{array}{c} A \\ B \\ C \\ X \\ \hline k_1 \end{array}\right] = \left[\begin{array}{c} 6.373733 \\ -22.465703 \\ 9.246398 \\ 6.845572 \\ \hline 0.000000 \end{array}\right]$$

The solutions of the augmented system of equations is $\left[\begin{array}{c} \mathbf{x} \\ \hline \mathbf{k} \end{array}\right] = \left[\begin{array}{c|c} -\mathbf{N} & \mathbf{C}^T \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right]^{-1}\left[\begin{array}{c} -\mathbf{t} \\ \hline \mathbf{g} \end{array}\right]$

and the vector **x** contains the adjusted heights which are

$$\mathbf{x} = \begin{bmatrix} A \\ B \\ C \\ X \end{bmatrix} = \begin{bmatrix} -0.41664 \\ 0.77312 \\ -0.23316 \\ -0.12331 \end{bmatrix}$$

Variance factor
$$\hat{\sigma}_0^2 = \frac{\mathbf{v}^T \mathbf{W} \mathbf{v}}{n-u} = \frac{\mathbf{f}^T \mathbf{W} \mathbf{f} - \mathbf{x}^T \mathbf{t}}{n-u} = 2.2173e - 06$$

Variances of adjusted heights

$$\boldsymbol{\Sigma}_{xx} = \sigma_0^2 \mathbf{Q}_{xx} = \begin{bmatrix} 1.1222e-7 & -3.6825e-7 & -4.2734e-7 & -3.2661e-8 \\ -3.6825e-7 & 5.8151e-8 & -1.1297e-8 & -1.0030e-8 \\ -4.2734e-7 & -1.1297e-8 & 6.9016e-8 & -1.4984e-8 \\ -3.2661e-8 & -1.0030e-8 & -1.4984e-8 & 5.7674e-8 \end{bmatrix}$$

$$\sigma_A^2 = 1.1222e - 07\,\text{m}^2 \quad \sigma_A = 0.00033\ \text{m}$$
$$\sigma_B^2 = 5.8150e - 08 \quad\quad \sigma_B = 0.00024\ \text{m}$$
$$\sigma_C^2 = 6.9016e - 08 \quad\quad \sigma_C = 0.00026\ \text{m}$$
$$\sigma_X^2 = 5.7674e - 08 \quad\quad \sigma_X = 0.00024\ \text{m}$$

Variances of adjusted observations (height differences) are the diagonal elements of the matrix $\boldsymbol{\Sigma}_{\tilde{\ell}\tilde{\ell}} = \hat{\sigma}_0^2 \mathbf{Q}_{\tilde{\ell}\tilde{\ell}}$ where the diagonal elements of

$$\mathbf{Q}_{\tilde{\ell}\tilde{\ell}} = \begin{bmatrix} 6.1284e-02 & 7.0652e-02 & 1.0608e-01 & 1.1005e-01 & 1.2028e-01 & 6.7542e-02 \end{bmatrix}$$

giving

$$\sigma_1^2 = 1.3589e - 07\,\text{m}^2 \quad \sigma_1 = 0.00037\ \text{m}$$
$$\sigma_2^2 = 1.5666e - 07 \quad\quad \sigma_2 = 0.00040\ \text{m}$$
$$\sigma_3^2 = 2.3521e - 07 \quad\quad \sigma_3 = 0.00048\ \text{m}$$
$$\sigma_4^2 = 2.4401e - 07 \quad\quad \sigma_4 = 0.00049\ \text{m}$$
$$\sigma_5^2 = 2.6670e - 07 \quad\quad \sigma_5 = 0.00052\ \text{m}$$
$$\sigma_6^2 = 1.4976e - 07 \quad\quad \sigma_6 = 0.00039\ \text{m}$$

Standard deviation of 1 kilometre of levelling $\sigma_0 = 0.00149$ m

The vector of residuals **v** and adjusted observations $\hat{\mathbf{l}}$ is

$$\mathbf{v} = \begin{bmatrix} -0.000068 \\ 0.000044 \\ -0.000220 \\ 0.000772 \\ -0.000694 \\ -0.000253 \end{bmatrix} \qquad \text{and} \qquad \hat{\mathbf{l}} = \mathbf{l} + \mathbf{v} = \begin{bmatrix} 0.89643 \\ 0.10984 \\ 0.29333 \\ 1.18976 \\ 0.18349 \\ 1.00628 \end{bmatrix}$$

Note that these results for the adjusted observations are identical to those obtained by the method of Adjustment of Observations Only.

Adjusted Heights

The "published values" of the BM's are: BM 707   RL 27.751 (AHD)
                                            BM 726   RL 28.942
                                            BM 727   RL 27.951

If the value 28.16764 (equal to the published RL of BM 707 less the value for *A* in the vector **x** given above) is added to the values in vector **x**, the following adjusted RL's of the BM's are:

                                            BM 726   RL 28.941 (AHD)
                                            BM 727   RL 27.934
                                            TBM X   RL 28.044

It would appear that the published RL of BM 727 is incorrect. The difference between the published value and the computed value above is approximately 0.017 metres.

These results for the adjusted heights are identical (nearest mm) to those obtained by the method of Adjustment of Observations Only.

YARRA BEND LEVEL NET EXERCISE – MATLAB RESULTS

FREE NET LEVEL ADJUSTMENT (NO FIXED STATIONS)

(Adjustment of Indirect Observations with Inner Constraint)

Data file *YarraBendLevelNetExercise2.dat* for MATLAB program *constrained_ls.m*

```
% data file for Yarra Bend Level Net Exercise
% Adjustment of Indirect Observations
% Inner Constraint
%  B(1)  B(2)  B(3)  B(4)        f          w
     0    -1     0     1      -0.89650   10.1010
     0     0     1    -1      -0.10980    7.6923
     1     0     0    -1      -0.29355    4.6512
     1    -1     0     0      -1.18899    3.7879
     1     0    -1     0      -0.18418    2.7397
     0    -1     1     0      -1.00653    8.8486
     1     1     1     1       0.000      1
```

Output file *YarraBendLevelNetExercise2.out* from MATLAB program *constrained_ls.m*

```
Constrained Least Squares Adjustment of Indirect Observations

Input Data

Coefficient matrix B of observation equations v + Bx = f
    0.0000   -1.0000    0.0000    1.0000
    0.0000    0.0000    1.0000   -1.0000
    1.0000    0.0000    0.0000   -1.0000
    1.0000   -1.0000    0.0000    0.0000
    1.0000    0.0000   -1.0000    0.0000
    0.0000   -1.0000    1.0000    0.0000

Vector of numeric terms f and weights w of observation equations v + Bx = f
   -0.896500     10.101000
   -0.109800      7.692300
   -0.293550      4.651200
   -1.188990      3.787900
   -0.184180      2.739700
   -1.006530      8.848600

Coefficient matrix C of constraint equations Cx = g
    1.0000    1.0000    1.0000    1.0000

Vector of numeric terms g of constraint equations Cx = g
    0.000000

Coefficient matrix N of Normal equations Nx = t
(upper triangular part)
   11.178800   -3.787900   -2.739700   -4.651200
   22.737500   -8.848600  -10.101000
   19.280600   -7.692300
   22.444500

Vector of numeric terms t of Normal equations Nx = t
   -6.373733
   22.465703
   -9.246398
   -6.845572
```

YARRA BEND LEVEL NET EXERCISE – MATLAB RESULTS

FREE NET LEVEL ADJUSTMENT (NO FIXED STATIONS)

(Adjustment of Indirect Observations with Inner Constraint)


```
Vector of solutions x
   -4.166445073e-001
    7.731178210e-001
   -2.331587715e-001
   -1.233145421e-001

Vector of residuals v
   -0.000068
    0.000044
   -0.000220
    0.000772
   -0.000694
   -0.000253

Variance factor =      2.2173e-006

Cofactor matrix of unknowns Qxx
(upper triangular part)
    5.0611e-002    -1.6608e-002    -1.9273e-002    -1.4730e-002
    2.6226e-002    -5.0948e-003    -4.5235e-003
    3.1126e-002    -6.7578e-003
    2.6011e-002

Cofactor matrix of residuals Qvv
(upper triangular part)
    3.7716e-002     3.2198e-002     2.8657e-002    -3.2627e-002    -3.5407e-003
   -2.9087e-002
    5.9347e-002    -2.8226e-002     3.9720e-003     4.2427e-002    -3.8455e-002
    1.0892e-001    -7.7426e-002    -7.7857e-002     4.3128e-004
    1.5395e-001    -8.1398e-002    -2.8655e-002
    2.4472e-001     3.8887e-002
    4.5470e-002

Cofactor matrix of adjusted observations Qll
(upper triangular part)
    6.1284e-002    -3.2198e-002    -2.8657e-002     3.2627e-002     3.5407e-003
    2.9087e-002
    7.0653e-002     2.8226e-002    -3.9720e-003    -4.2427e-002     3.8455e-002
    1.0608e-001     7.7426e-002     7.7857e-002    -4.3128e-004
    1.1005e-001     8.1398e-002     2.8655e-002
    1.2028e-001    -3.8887e-002
    6.7542e-002
```