INNER CONSTRAINTS FOR A 3-D SURVEY NETWORK

These notes follow closely the development of inner constraint equations by Dr Willie Tan, Department of Building, School of Design and Environment, National University of Singapore (Tan 2005).

Consider a set of *n* observation equations in *u* unknowns $(n > u)$ in the matrix form

$$\mathbf{v} + \mathbf{Bx} = \mathbf{f} \tag{1}$$

v is the $n \times 1$ vector of residuals
x is the $u \times 1$ vector of unknowns (or parameters)
B is the $n \times u$ matrix of coefficients (design matrix)
f is the $n \times 1$ vector of numeric terms (constants)
n is the number of equations (or observations)
u is the number of unknowns

Enforcing the least squares condition leads to the normal equations

$$\mathbf{Nx} = \mathbf{t} \tag{2}$$

where

$$\mathbf{N} = \mathbf{B}^T\mathbf{WB} \quad \text{and} \quad \mathbf{t} = \mathbf{B}^T\mathbf{Wf} \tag{3}$$

N is the $u \times u$ coefficient matrix of the normal equations
t is the $u \times 1$ vector of numeric terms
W is the $n \times n$ weight matrix

If **N** is non-singular, i.e., $|\mathbf{N}| \neq \mathbf{0}$ and $\mathbf{N}^{-1}$ exists: there is a solution for the unknowns **x** given by matrix inversion

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t} \tag{4}$$

If **N** is singular, i.e., $|\mathbf{N}| = \mathbf{0}$ and $\mathbf{N}^{-1}$ does not exist: there is no solution by conventional means (i.e., by the usual matrix inverse).

<u>The aim of these notes is to explain how we might obtain least squares solutions where the normal coefficient matrix is singular</u>. Such solutions, using INNER CONSTRAINTS are known as FREE NET ADJUSTMENTS.

In least squares adjustments of survey networks, singular sets of normal equations (singular normal coefficient matrices **N**) are rank deficient sets of equations and arise because design matrices **B** are rank deficient. This will invariably be due to datum defects, which could be: coordinate origin not defined (no fixed points), the network orientation not defined (no lines with fixed directions in space), no scale (no measured distances) or no height datum defined (no points with fixed heights). The datum defects are directly connected to the rank deficiency of the design matrix **B**.

A way to overcome this datum deficiency and find a solution is to impose constraints, and these constraints take the form of a set of CONSTRAINT EQUATIONS

$$\mathbf{Cx} = \mathbf{g} \tag{5}$$

C is the $c \times u$ matrix of coefficients
g is the $c \times 1$ vector of numeric terms (constants)
c is the number of constraint equations

Combining the constraint equations (5) with the observation equations (1) and enforcing the least squares condition

$$\varphi = \mathbf{v}^T\mathbf{W}\mathbf{v} - 2\mathbf{k}^T\left(\mathbf{Cx} - \mathbf{g}\right) \;\Rightarrow\; \text{minimum} \tag{6}$$

where **W** is the $n \times n$ weight matrix and **k** is the $c \times 1$ vector of *Lagrange multipliers,* gives rise to the set of equations

$$\left[\begin{array}{c:c} -\mathbf{N} & \mathbf{C}^T \\ \hdashline \mathbf{C} & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{k} \end{array}\right] = \left[\begin{array}{c} -\mathbf{t} \\ \hdashline \mathbf{g} \end{array}\right] \tag{7}$$

Equations (7) can be solved provided that **C** consists of linearly independent rows that overcome the datum problems. For a particular network with the maximum number of datum defects (e.g., a height network with no fixed points, a 2-D network of distances and directions with no fixed points or fixed orientation) there are many different constraints that can be imposed to enable a solution for **x**, and amongst this family of possible constraints there are MINIMAL CONSTRAINT sub-sets, i.e., sub-sets containing the minimum number of constraints required for a solution; the minimum number being equal to the rank deficiency. Within this group of minimal constraint sub-sets there will be a single set called INNER CONSTRAINTS. This set of inner constraints (inner constraint equations) has the unique property that $\mathbf{Cx} = \mathbf{0}$ and a solution of a network using inner constraints yields **x** such that the *norm* of **x**, denoted as $\|\mathbf{x}\|$, is a minimum and the *trace* of $\mathbf{Q}_{xx}$ denoted as $tr\left(\mathbf{Q}_{xx}\right)$ is also minimum. The trace of a (square) matrix is the sum of the elements of the leading diagonal, and $\mathbf{Q}_{xx}$ is the (square) cofactor matrix containing estimates of the variances $s^2_{x_k}$ of the adjusted quantities (the elements of **x**). Hence, a minimum trace solution has a set $\left\{ s^2_{x_1} \quad s^2_{x_2} \quad \cdots \quad s^2_{x_u} \right\}$ whose sum of the u elements is the smallest, i.e., a minimum variance solution. The norm of a column (or row) vector is the square root of the sum of squares of the elements, i.e., the "length" of the vector. A minimum norm solution would have a vector **x** having the smallest length.

Adjustments of survey networks where no datum is defined and INNER CONSTRAINTS are employed, are known as FREE NET ADJUSTMENTS, and those employing MINIMAL CONSTRAINTS are used are known as MINIMALLY CONSTRAINED ADJUSTMENTS.

In free net adjustments (where every point in the network is free to move) the inner constraints have the unique property that $\mathbf{Cx} = \mathbf{0}$, i.e., certain linear functions of the adjusted quantities **x** are equal to zero. These adjusted quantities can be heights in level networks, or coordinates and theodolite orientation constants in conventional 2-D and 3-D survey networks, or just coordinates in photogrammetric and GPS networks. In conventional adjustments (where there are enough fixed points to properly define the datum and orientation) or minimally constrained adjustments, or conventional adjustments with additional constraints, the constraint equations are conditions relating to actual points in the network. In free net adjustments the constraints are applied to, or relate to, a single "fictitious" point; the <u>centroid</u> of the network. The key here is to remember that the most convenient way to represent position is via height (in level networks) or coordinates in 2-D and 3-D networks, hence most network adjustments are actually variation of coordinate adjustments; height being a 1-D system. So, coordinates (approximate or fixed) are required for all points in the network – even if the coordinate datum or height datum is not fixed, or the orientation is not properly known. Once these coordinates are assigned then the centroid is defined, since it is just the average of the coordinates of all of the points. We can now decide on certain constraints on the centroid and between the centroid and all the points in the network that will overcome datum defects in 1-D, 2-D and 3-D survey networks. These three constraints are:

- the centroid remains unchanged;
- the average bearing from the centroid to all points remains unchanged;
- the average distance from the centroid to all points remains unchanged.

These constraints give rise to *positional* constraints, *rotational* constraints and a *scale* constraint.

In a free network adjustment of heights (a 1-D system) there is one datum defect and so one positional constraint is required and this is derived from the condition that the centroid remains unchanged. In a free net adjustment of a 2-D network (measured directions and distances) there are three datum defects, so two positional constraints and one rotational constraint are required. These are derived from the condition that the centroid remains unchanged and the average bearing from the centroid to all points remains unchanged. For 3-D networks (measured horizontal directions, zenith angles and distances) there are six datum defects, requiring three positional and three rotational constraints. Again, these are derived from the condition that the centroid remains unchanged and that the average bearing (in space) from the centroid to all points remains unchanged. For 2-D and 3-D networks without measured distances, there is an additional scale constraint required which can be derived from the condition that the average distance from the centroid to all points remains unchanged.

The positional, rotational and scale (inner) constraints for a 3-D survey network are derived as follows.

POSITIONAL CONSTRAINT

Let (X_k, Y_k, Z_k) be the coordinates of an arbitrary point P_k and there are *m* points in the network. The coordinates of the centroid *G* are

$$X_G = \frac{X_1 + X_2 + X_3 + \cdots + X_m}{m} = \frac{\sum_{k=1}^{m} X_k}{m} \;;\quad Y_G = \frac{\sum_{k=1}^{m} Y_k}{m} \;;\quad Z_G = \frac{\sum_{k=1}^{m} Z_k}{m} \tag{8}$$

If we impose the condition that *G* does not move, then there can be no change to the centroid coordinates during an iterative least squares solution, hence $\delta X_G = \delta Y_G = \delta Z_G = 0$. Now since $X_G = X_G(X_1, X_2, \ldots, X_m)$ the Total Increment Theorem gives

$$\begin{aligned}\delta X_G &= \frac{\partial X_G}{\partial X_1}\delta X_1 + \frac{\partial X_G}{\partial X_2}\delta X_2 + \cdots + \frac{\partial X_G}{\partial X_m}\delta X_m \\ &= \frac{1}{m}\delta X_1 + \frac{1}{m}\delta X_2 + \cdots + \frac{1}{m}\delta X_m \\ &= \frac{1}{m}\sum_{k=1}^{m}\delta X_k\end{aligned}$$

And similarly, $\delta Y_G = \frac{1}{m}\sum_{k=1}^{m}\delta Y_k$ and $\delta Z_G = \frac{1}{m}\sum_{k=1}^{m}\delta Z_k$. Remembering that $\delta X_G = \delta Y_G = \delta Z_G = 0$, the positional constraints are

$$\boxed{\sum_{k=1}^{m}\delta X_k = 0 \;;\qquad \sum_{k=1}^{m}\delta Y_k = 0 \;;\qquad \sum_{k=1}^{m}\delta Z_k = 0} \tag{9}$$

ROTATIONAL CONSTRAINTS

The three rotational constraints can be derived by imposing the condition that the average bearing from the centroid *G* to all points remains unchanged.

Figure 1 shows a line in 3-D space from *G* to an arbitrary point $P_k\left(X_k, Y_k, Z_k\right)$. The line GP_k has length s_k and projections $r_k = GP_k'$, $q_k = GP_k''$ and $p_k = GP_k'''$ on to the *X*-*Y*, *Y*-*Z* and *X*-*Z* planes respectively.

Consider the projection $r_k = GP_k'$ on the *X*-*Y* plane that has bearing θ_k. Plane trigonometry gives

$$\theta_k = \tan^{-1}\left(\frac{X_k - X_G}{Y_k - Y_G}\right) = \theta_k\left(X_k, Y_k, X_G, Y_G\right) \tag{10}$$

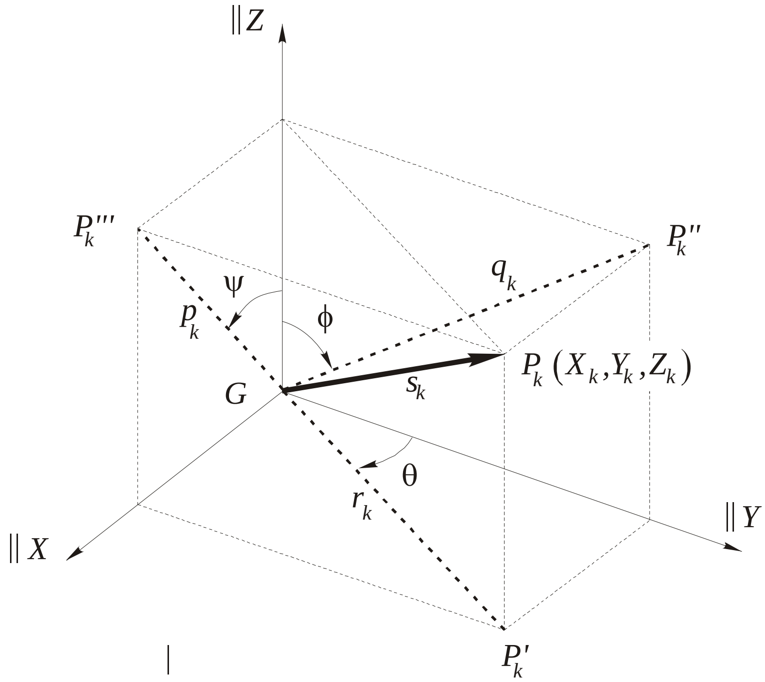

Figure 1. A line in space and its projection on the *X*-*Y*, *Y*-*Z* and *X*-*Z* planes.

Noting that X_G, Y_G in equation (10) are constant (since the centroid cannot move), the Total Increment Theorem gives

$$\delta\theta_k \simeq \frac{\partial\theta_k}{\partial X_k}\delta X_k + \frac{\partial\theta_k}{\partial Y_k}\delta Y_k \tag{11}$$

and the partial derivatives are

$$\frac{\partial\theta_k}{\partial X_k} = \frac{\cos\theta_k}{r_k} = \frac{Y_k - Y_G}{r_k^2}\ ; \qquad \frac{\partial\theta_k}{\partial Y_k} = -\frac{\sin\theta_k}{r_k} = -\frac{X_k - X_G}{r_k^2}$$

Hence

$$\delta\theta_k \simeq \frac{1}{r_k^2}\left\{\left(Y_k - Y_G\right)\delta X_k - \left(X_k - X_G\right)\delta Y_k\right\} \tag{12}$$

If the average bearing from *G* to all *m* points in the network is to remain unchanged then

$$\sum_{k=1}^{m}\delta\theta_k = 0 \tag{13}$$

<u>This is a rotational constraint in the *X*-*Y* plane.</u>

Let $\sum_{k=1}^{m}\delta\theta_k$ be the vector $\mathbf{s} = \left[\delta\theta_1 + \delta\theta_2 + \cdots + \delta\theta_m\right]$ and **s** contains a single scalar quantity. In equation (13) $\mathbf{s} = \left[0\right] = \mathbf{0}_{1\times 1}$. Now let **s** be the product of two vectors

$$\mathbf{s} = \mathbf{r}^T\mathbf{d} \tag{14}$$

where
$$\mathbf{r} = \begin{bmatrix} \dfrac{1}{r_1^2} \\ \dfrac{1}{r_2^2} \\ \vdots \\ \dfrac{1}{r_m^2} \end{bmatrix}_{m\times 1} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} \left(Y_1 - Y_G\right)\delta X_1 - \left(X_1 - X_G\right)\delta Y_1 \\ \left(Y_2 - Y_G\right)\delta X_2 - \left(X_2 - X_G\right)\delta Y_2 \\ \vdots \\ \left(Y_m - Y_G\right)\delta X_m - \left(X_m - X_G\right)\delta Y_m \end{bmatrix}_{m\times 1}$$

Pre-multiplying both sides of equation (14) by **r** gives

$$\mathbf{rs} = \mathbf{rr}^T \mathbf{d}$$

$\mathbf{rr}^T$ is a matrix of order $m \times m$ of full rank, since the elements of **r** are independent and the inverse $\left(\mathbf{rr}^T\right)^{-1}$ exists. Pre-multiplying both sides by $\left(\mathbf{rr}^T\right)^{-1}$ gives

$$\left(\mathbf{rr}^T\right)^{-1} \mathbf{rs} = \left(\mathbf{rr}^T\right)^{-1} \left(\mathbf{rr}^T\right) \mathbf{d} = \mathbf{Id} = \mathbf{d} \tag{15}$$

When **s** on the left-hand-side of equation (15) is zero (enforcing the condition of equation (13)) then $\text{L.H.S.} = \mathbf{0}_{m\times 1}$ and every element of $\mathbf{d}_{m\times 1}$ on the R.H.S. will be equal to zero, giving

$$\sum_{k=1}^{m} \left\{ \left(Y_k - Y_G\right) \delta X_k - \left(X_k - X_G\right) \delta Y_k \right\} = 0 \tag{16}$$

Expanding (16) gives

$$\begin{aligned}
\sum_{k=1}^{m} Y_k \delta X_k - \sum_{k=1}^{m} Y_G \delta X_k - \sum_{k=1}^{m} X_k \delta Y_k + \sum_{k=1}^{m} X_G \delta Y_k &= 0 \\
\sum_{k=1}^{m} \left(Y_k \delta X_k - X_k \delta Y_k\right) - Y_G \sum_{k=1}^{m} \delta X_k + X_G \sum_{k=1}^{m} \delta Y_k &= 0
\end{aligned}$$

But from equation (9), the positional constraints, $\sum_{k=1}^{m} \delta X_k = \sum_{k=1}^{m} \delta Y_k = 0$ hence the rotational constraint in the *X*-*Y* plane can be expressed as

$$\boxed{\sum_{k=1}^{m} \left(Y_k \delta X_k - X_k \delta Y_k\right) = 0} \tag{17}$$

[Equations (16) and (17) are identical to Tan 2005 (eq'ns (12) and (13), p. 93)]

Now, referring to Figure 1, the projection $q_k = GP_k''$ on the *Y*-*Z* plane makes an angle ϕ with the *Z*-axis. Plane trigonometry gives

$$\phi_k = \tan^{-1}\left(\frac{Y_k - Y_G}{Z_k - Z_G}\right) = \phi_k\left(Y_k, Z_k, Y_G, Z_G\right) \tag{18}$$

Noting that X_G, Y_G the Total Increment Theorem gives

$$\delta\phi_k \simeq \frac{\partial\phi_k}{\partial Y_k}\delta Y_k + \frac{\partial\phi_k}{\partial Z_k}\delta Z_k$$

and substituting the partial derivatives and simplifying gives

$$\delta\phi_k \simeq \frac{1}{q_k^2}\left\{\left(Z_k - Z_G\right)\delta Y_k - \left(Y_k - Y_G\right)\delta Z_k\right\}$$

If the average bearing from *G* (the centroid) to all *m* points in the network is to remain unchanged then the rotational constraint in the *Y*-*Z* plane is

$$\sum_{k=1}^{m}\delta\phi_k = 0 \tag{19}$$

In a similar manner as before, the rotational constraint in the *Y*-*Z* plane can be expressed as

$$\boxed{\sum_{k=1}^{m}\left(Z_k\delta Y_k - Y_k\delta Z_k\right) = 0} \tag{20}$$

Again, referring to Figure 1, the projection $p_k = GP_k'''$ on the *X*-*Z* plane makes an angle ψ with the *Z*-axis. Plane trigonometry gives

$$\psi_k = \tan^{-1}\left(\frac{X_k - X_G}{Z_k - Z_G}\right) = \psi_k\left(X_k, Z_k, X_G, Z_G\right) \tag{21}$$

In an identical manner as before, the rotational constraint in the *X*-*Z* plane is

$$\sum_{k=1}^{m}\delta\psi_k = 0 \tag{22}$$

which becomes

$$\boxed{\sum_{k=1}^{m}\left(Z_k\delta X_k - X_k\delta Z_k\right) = 0} \tag{23}$$

Equations (13), (19) and (22) are rotational constraints expressed in angular quantities. Equations (17), (20) and (23) are the same rotational constraints expressed in linear quantities (coordinates and coordinate corrections)

SCALE CONSTRAINT

The distance from the centroid *G* to an arbitrary point $P_k\left(X_k, Y_k, Z_k\right)$ is given by

$$s_k = \sqrt{\left(X_k - X_G\right)^2 + \left(Y_k - Y_G\right)^2 + \left(Z_k - Z_G\right)^2} = s_k\left(X_k, Y_k, Z_k, X_G, Y_G, Z_G\right) \tag{24}$$

Noting that X_G, Y_G, Z_G in equation (24) are constant (since the centroid cannot move), the Total Increment Theorem gives

$$\delta s_k \simeq \frac{\partial s_k}{\partial X_k}\delta X_k + \frac{\partial s_k}{\partial Y_k}\delta Y_k + \frac{\partial s_k}{\partial Z_k}\delta Z_k \tag{25}$$

The partial derivatives are

$$\frac{\partial s_k}{\partial X_k} = \frac{X_k - X_G}{s_k}\,; \qquad \frac{\partial s_k}{\partial Y_k} = \frac{Y_k - Y_G}{s_k}\,; \qquad \frac{\partial s_k}{\partial Z_k} = \frac{Z_k - Z_G}{s_k}$$

and substituting into equation (25) and simplifying gives

$$\delta s_k \simeq \frac{1}{s_k}\left\{\left(X_k - X_G\right)\delta X_k + \left(Y_k - Y_G\right)\delta Y_k + \left(Z_k - Z_G\right)\delta Z_k\right\} \tag{26}$$

If the average distance from *G* (the centroid) to all *m* points in the network is to remain unchanged then

$$\sum_{k=1}^{m} \delta s_k = 0 \tag{27}$$

In the same manner as for the development of rotational constraints, matrix manipulations can be used to turn the sum of small changes in distances to a sum of small changes in coordinates and equations (26) and (27) imply

$$\sum_{k=1}^{m}\left\{\left(X_k - X_G\right)\delta X_k + \left(Y_k - Y_G\right)\delta Y_k + \left(Z_k - Z_G\right)\delta Z_k\right\} = 0 \tag{28}$$

Expanding (28) gives

$$\sum_{k=1}^{m} X_k \delta X_k - \sum_{k=1}^{m} X_G \delta X_k + \sum_{k=1}^{m} Y_k \delta Y_k - \sum_{k=1}^{m} Y_G \delta Y_k + \sum_{k=1}^{m} Z_k \delta Z_k - \sum_{k=1}^{m} Z_G \delta Z_k = 0$$

$$\sum_{k=1}^{m}\left(X_k \delta X_k + Y_k \delta Y_k + Z_k \delta Z_k\right) - X_G \sum_{k=1}^{m} \delta X_k - Y_G \sum_{k=1}^{m} \delta Y_k - Z_G \sum_{k=1}^{m} \delta Z_k = 0$$

But from equation (9), the positional constraints, $\sum_{k=1}^{m}\delta X_k = \sum_{k=1}^{m}\delta Y_k = \sum_{k=1}^{m}\delta Z_k = 0$ hence the scale constraint can be expressed as

$$\sum_{k=1}^{m}\left(X_k\delta X_k + Y_k\delta Y_k + Z_k\delta Z_k\right) = 0 \qquad (29)$$

[Equations (28) and (29) are identical to Tan 2005 (eq'ns (20) and (21), p. 93)]

SUMMARY OF CONSTRAINTS

Positional constraints

$$\sum_{k=1}^{m}\delta X_k = 0\,; \qquad \sum_{k=1}^{m}\delta Y_k = 0\,; \qquad \sum_{k=1}^{m}\delta Z_k = 0$$

Rotational constraints

$$\sum_{k=1}^{m}\left(Y_k\delta X_k - X_k\delta Y_k\right) = 0$$
$$\sum_{k=1}^{m}\left(Z_k\delta Y_k - Y_k\delta Z_k\right) = 0$$
$$\sum_{k=1}^{m}\left(Z_k\delta X_k - X_k\delta Z_k\right) = 0$$

Scale constraint

$$\sum_{k=1}^{m}\left(X_k\delta X_k + Y_k\delta Y_k + Z_k\delta Z_k\right) = 0$$

Equation (7) is the matrix equation for a constrained least squares solution of a survey network, where **C** is the coefficient matrix of the constraint equations $\mathbf{Cx} = \mathbf{g}$. In the case of INNER CONSTRAINTS, $\mathbf{Cx} = \mathbf{0}$ and equation (7) becomes

$$\begin{bmatrix} -\mathbf{N} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} -\mathbf{t} \\ \mathbf{0} \end{bmatrix} \qquad (30)$$

In the case of a 3-D network, with no measured distances (no scale) and no fixed points there will be seven datum defects requiring seven constraint equations to overcome the defects in a free net adjustment. The seven constraints combine (i) three positional constraints, (ii) three rotational constraints and (iii) one scale constraint. If this network solution has the coordinate corrections in $\mathbf{x}$ in the following sequence

$$\mathbf{x}^T = \begin{bmatrix} \delta X_1 & \delta Y_1 & \delta Z_1 & \delta X_2 & \delta Y_2 & \delta Z_2 & \cdots & \delta X_m & \delta Y_m & \delta Z_m \end{bmatrix}$$

the (inner) constraint equations $\mathbf{Cx} = \mathbf{0}$ have the form

$$\left[\begin{array}{cccccccccc} 1 & 0 & 0 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 1 \\ \hdashline Y_1 & -X_1 & 0 & Y_2 & -X_2 & 0 & \cdots & Y_m & -X_m & 0 \\ 0 & Z_1 & -Y_1 & 0 & Z_2 & -Y_2 & \cdots & 0 & Z_m & -Y_m \\ Z_1 & 0 & -X_1 & Z_2 & 0 & -X_2 & \cdots & Z_m & 0 & -X_m \\ \hdashline X_1 & Y_1 & Z_1 & X_2 & Y_2 & Z_2 & \cdots & X_m & Y_m & Z_m \end{array}\right] \begin{bmatrix} \delta X_1 \\ \delta Y_1 \\ \delta Z_1 \\ \delta X_2 \\ \delta Y_2 \\ \delta Z_2 \\ \vdots \\ \delta X_m \\ \delta Y_m \\ \delta Z_m \end{bmatrix} = \mathbf{0} \tag{31}$$

In subsequent notes on this topic, several examples of free net adjustments will be shown.

REFERENCES

Tan, W., 2005, 'Inner constraints for 3-D survey networks', *Spatial Science*, Vol. 50, No. 1, June 2005, pp. 91-94.