30 de Octubre

Wednesday, October 30, 2024 9:44 AM

Descomposición SVD (En valores singulares).

Si $A \in M_n(\mathbb{R})$ simétrica $\Rightarrow A = PDP^T$

P ortogonal y D es diagonal

SVD: $A \in M_{m\times n}(\mathbb{R}) \Rightarrow \exists\, V \in M_n(\mathbb{R})$

$U \in M_m(\mathbb{R})$

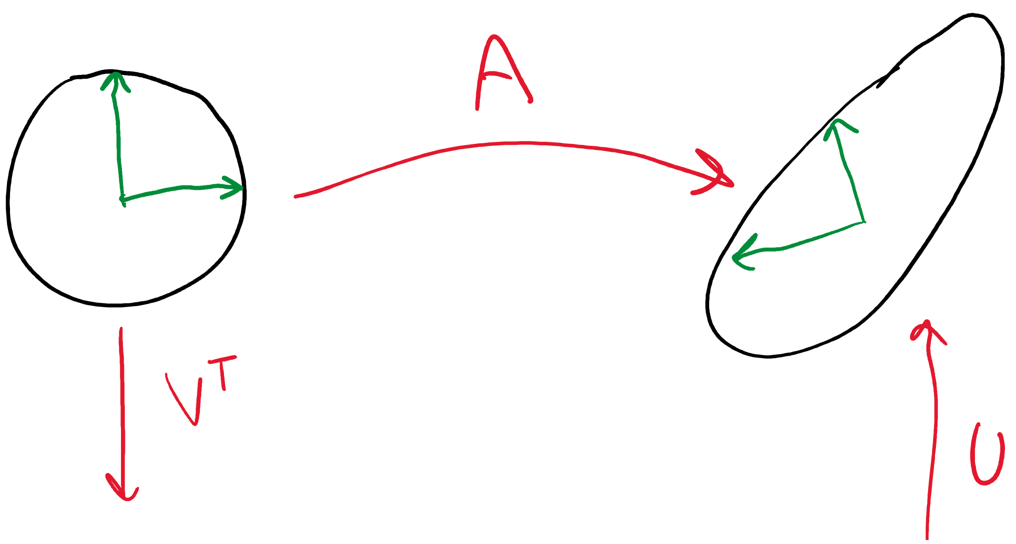
ortogonales

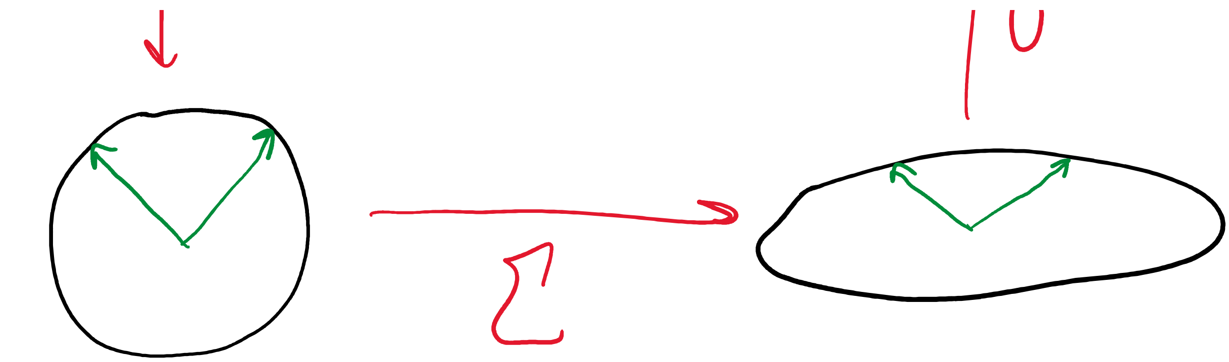
y $\Sigma \in M_{m\times n}(\mathbb{R})$ diagonal tq

$$A = U \Sigma V^T$$

$$\Sigma = \underbrace{\left.\begin{pmatrix} \sigma_1 & & & 0 \\ 0 & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ & 0 & & \end{pmatrix}\right\} m}_{n}$$

σ_i's valores singulares de A.





Mínimos cuadrados y SVD:

Si queremos resolver $Ax = y$

$$\Rightarrow U \Sigma V^T x = y$$

$$\Rightarrow \Sigma V^T x = U^T y$$

$$\begin{aligned} w &= V^T x \\ z &= U^T y \end{aligned} \Rightarrow \begin{aligned} &1)\ \Sigma w = z \quad \circledast \\ &2)\ x = V w \end{aligned}$$

$$\circledast \quad \begin{pmatrix} \sigma_1 & \cdots & 0 \\ 0 & \ddots & \sigma_n \\ & 0 & \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \\ \vdots \\ z_m \end{pmatrix}$$

$$\begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} \vdots \\ w_n \end{pmatrix} \qquad \begin{pmatrix} \vdots \\ z_m \end{pmatrix}$$

$$\uparrow$$

$$\Rightarrow \text{Si } w_i = \frac{z_i}{\sigma_i} \quad \forall i = 1, \dots, n$$

$$\Sigma = \begin{pmatrix} \Sigma_{(1:n)} \\ 0 \end{pmatrix}, \quad z = \begin{pmatrix} z_{(1:n)} \\ z_{(n+1:m)} \end{pmatrix}$$

$\Rightarrow$ I) Descompongo $\underline{A = U \Sigma V^T}$

II) Resuelvo $\Sigma_{(1:n)} w = (U^T y)_{(1:n)}$

III) $x = Vw$.

En octave

[U, S, V] = svd(A)

8) $\delta \geq 0$.

$$\begin{pmatrix} 1 & 1 & 1 \\ \delta & 0 & 0 \\ 0 & \delta & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{y}$$

a) $$A^TA = \begin{pmatrix} 1 & \delta & 0 & 0 \\ 1 & 0 & \delta & 0 \\ 1 & 0 & 0 & \delta \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix}$$

$$= \begin{pmatrix} 1+\delta^2 & 1 & 1 \\ 1 & 1+\delta^2 & 1 \\ 1 & 1 & 1+\delta^2 \end{pmatrix}$$

$$A^T y = \begin{pmatrix} 1 & \delta & 0 & 0 \\ 1 & 0 & \delta & 0 \\ 1 & 0 & 0 & \delta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Ec normales: $$\begin{pmatrix} 1+\delta^2 & 1 & 1 \\ 1 & 1+\delta^2 & 1 \\ 1 & 1 & 1+\delta^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow x_1 = x_2 = x_3 = \frac{1}{\delta^2+3}$$

$$\frac{1+\delta^2}{3+\delta^2} + \frac{1}{3+\delta^2} + \frac{1}{3+\delta^2} = 1$$

QR: $A = QR$

QR: $A = QR$

La sol. de $Ax = y$ en mínimos cuadrados es la sol. de

$$R_{(1:n)}\, x = (Q^t y)_{(1:n)}$$

Pseudoinversa o inversa Moore-Penrose

Si $A \in M_n(\mathbb{R})$ es invertible entonces la sol. de $Ax = y$ es $x = A^{-1}y$.

Si $A \in M_{m\times n}(\mathbb{R})$ entonces la pseudoinversa va a ser $A^+ \in M_{n\times m}(\mathbb{R})$ y cumple que si $x = A^+ y \Rightarrow x$ es solución de $Ax = y$ en el sentido de mínimos cuadrados.

Si $\Sigma \in M_{m\times n}$ es diagonal

$$\Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{pmatrix}$$ y $\Sigma x = y$ en mínimos

$$\Sigma = \begin{pmatrix} 0 & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix} \quad \text{y} \quad \Sigma x = y \quad \text{en mínimos cuadrados}$$

$$\Rightarrow \quad x_i = \frac{y_i}{\sigma_i} \qquad \text{Si defino } \sigma_i^+ = \begin{cases} 1/\sigma_i & \text{si } \sigma_i \neq 0 \\ 0 & \text{si } \sigma_i = 0 \end{cases}$$

Obs: $$X = \underbrace{\begin{pmatrix} \sigma_1^+ & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n^+ & \end{pmatrix}}_{\Sigma^+} \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

$\Rightarrow$ Dada $\Sigma \in M_{m\times n}(\mathbb{R})$ diagonal defino su pseudo inversa como la matriz $\Sigma^+ \in M_{n\times m}(\mathbb{R})$

$$\Sigma^+ = \begin{pmatrix} \sigma_1^+ & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n^+ & \end{pmatrix}$$

Si $A \in M_{m\times n}(\mathbb{R})$, SVD: $A = \underbrace{U}_{\substack{m\times m \\ \text{ortogonal}}} \underbrace{\Sigma}_{\substack{m\times n \\ \text{diagonal}}} \underbrace{V^T}_{\substack{n\times n \\ \text{ortogonal}}}$

Si quiero $Ax = y$ en mínimos cuadrados

$$\Rightarrow U \Sigma V^T x = y$$

$$\Rightarrow \Sigma V^T x = U^T y$$

$$\Rightarrow V^T x = \Sigma^+ U^T y$$

$$\Rightarrow x = \underbrace{V \Sigma^+ U^T}_{:= A^+} y$$

Defino la pseudoinversa de $A = U \Sigma V^T$ como

$$A^+ := V \Sigma^+ U^T \in M_{n \times m}(\mathbb{R})$$

<u>Prop.</u>

- Si $A \in M_n(\mathbb{R})$ invertible $\Rightarrow A^+ = A^{-1}$.

• Si $A \in M_{m\times n}(\mathbb{R})$ de rango completo $(rg(A)=n) \Rightarrow A^+ = \underbrace{(A^TA)^{-1}}_{\text{invertible}}A^T$.

$Ax=y$ en mínimos cuadrados sii. $\overbrace{A^TA}^{\text{invertible}}x = A^Ty$

$\Rightarrow x = (A^TA)^{-1}A^Ty$

• $x = A^+y$ es solución de $Ax=y$ en el sentido de mínimos cuadrados.

9) $A = \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$, $A^+ = \begin{pmatrix} 1 & 0 \\ 0 & \delta^+ \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1/\delta \end{pmatrix} & \delta \neq 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \delta = 0 \end{cases}$

$$AA^+ = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{si } \delta \neq 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{si } \delta = 0 \end{cases}$$