$$\begin{cases} y'(t) = F(t, y(t)) \\ y(0) = y_0 \end{cases} \qquad t_0, \to t_k, \dots \qquad h_k = t_{k+1} - t_k$$

$$y_0, \to y_k, \dots \qquad y_k \approx y(t_k) \qquad (y(t) \in \mathbb{R}^n)$$

- $e_k = y_k - y(t_k)$ entran en juego todos los errores acumulados.

Errores locales: $\ell_k = y_k - u(t_k)$

con $u(t)$ sol. de $\begin{cases} u'(t) = F(t, u(t)) \\ u(t_{k-1}) = y_{k-1} \end{cases}$

suponer que en t_{k-1} es exacto. → $u(t_{k-1}) = y_{k-1}$

Método de orden p:

$$\ell_k = O\left((h_{k-1})^{p+1}\right)$$

("en general implica $e_k = O(h^p)$")

Método orden p: "$e_k = O(h^p)$"

$$|e_k| \leq C h^p$$

$h_1 \to e$

$h_2 = \frac{h_1}{10} \to e \approx \frac{e}{10^p}$

$h_1 = 10^{-1} \to e_k \leq C\, 10^{-p}$

$h_2 = 10^{-2} \to e'_k \leq C (10^{-2})^p$

(2) $y_{k+1} = y_k + h_k F(t_{k+1}, y_{k+1})$ $\qquad$ $\ell_{k+1} = O(h_k^2)$

$\ell_{k+1} = y_{k+1} - u(t_{k+1}) = y_k + h_k F(t_{k+1}, y_{k+1}) - u(t_{k+1})$

- Hay que hacer Taylor. Para que salga hay que hacerlo centrado en t_{k+1} y evaluarlo para atrás en t_k.

$u(t) = u(t_{k+1}) + u'(t_{k+1})(t - t_{k+1}) + O((t - t_{k+1})^2)$

$u(t_k) = u(t_{k+1}) - u'(t_{k+1}) h_k + O(h_k^2)$

$u(t_{k+1}) = u(t_k) + h_k u'(t_{k+1}) + O(h_k^2)$

$= y_k + h_k F(t_{k+1}, u(t_{k+1})) + O(h_k^2)$

porque $\begin{cases} u'(t) = F(t, u(t)) \quad \forall t \\ u(t_k) = y_k \end{cases}$

$O(h_k^2)$ es con $h_k \to 0$

$\ell_{k+1} = \cancel{y_k} + h_k F(t_{k+1}, y_{k+1}) - (\cancel{y_k} + h_k F(t_{k+1}, u(t_{k+1})) + O(h_k^2))$

$= h_k (F(t_{k+1}, y_{k+1}) - F(t_{k+1}, u(t_{k+1}))) + O(h_k^2)$

no tienen por qué ser iguales

$\|\ell_{k+1}\| \leq h_k \|F(t_{k+1}, y_{k+1}) - F(t_{k+1}, u(t_{k+1}))\| + O(h_k^2)$

H. Picard: $\leq h_k L \|\underbrace{y_{k+1} - u(t_{k+1})}_{\ell_{k+1}}\| + O(h_k^2)$

$\|\ell_{k+1}\| \leq h_k L \|\ell_{k+1}\| + O(h_k^2)$

$\leq \frac{1}{2} \|\ell_{k+1}\| + O(h_k^2)$ $\qquad$ (si $h_k \leq \frac{1}{2L}$)

$\Rightarrow \frac{1}{2} \|\ell_{k+1}\| \leq O(h_k^2) \Rightarrow \|\ell_{k+1}\| \leq 2\, O(h_k^2) = O(h_k^2)$

$\Rightarrow \|\ell_{k+1}\| = O(h_k^2)$

(11)

$$y_{k+1} = y_k + h_k \sum_{i=1}^{s} b_i K_i$$

$$K_i = f\left(t_k + c_i h_k, \; y_k + h_k \sum_{j=1}^{s} a_{ij} K_j\right)$$ ← sist de ecuaciones para hallar los K_i

Euler hacia adelante

$s = 1$, $b_1 = 1$, $c_1 = a_{11} = 0$

$$K_1 = f(t_k + c_1 h_k, \; y_k + h_k a_{11} K_1)$$
$$= f(t_k, y_k)$$

$$y_{k+1} = y_k + h_k K_1$$
$$= y_k + h_k f(t_k, y_k)$$

Euler hacia atrás

$s = b_1 = c_1 = a_{11} = 1$

implícita

$$K_1 = f(t_k + h_k, \; y_k + h_k K_1) \Rightarrow K_1 = f(t_{k+1}, y_{k+1})$$

$$y_{k+1} = y_k + h_k K_1$$

$$y_{k+1} = y_k + h_k f(t_{k+1}, y_{k+1})$$

Estabilidad absoluta

b) Problema test: $\begin{cases} y'(t) = \lambda y(t) \\ y(0) = 1 \end{cases}$ $\lambda < 0$ sol $y(t) = e^{\lambda t} \xrightarrow{t\to\infty} 0$

$F(t,y) = \lambda y$

dado h, el método es abs. est para h si con paso cte h $y_k \xrightarrow{k\to\infty} 0$

si se cumple $\forall h > 0$, incondicionalmente abs. est.

$K_1 = F(t_k, y_k)$ $\quad K_2 = F\left(t_k + \frac{h}{2}, y_k + \frac{h}{4}(k_1 + k_2)\right)$ $\quad y_{k+1} = y_k + h K_2$

$K_1 = \lambda y_k$

$K_2 = \lambda\left(y_k + \frac{h}{4}(\lambda y_k + K_2)\right)$

$= \lambda y_k + \frac{h\lambda^2}{4} y_k + \frac{h\lambda}{4} K_2$

$\left(1 - \frac{h\lambda}{4}\right) K_2 = \left(\lambda + \frac{h\lambda^2}{4}\right) y_k$

$\Rightarrow K_2 = \dfrac{\lambda + \frac{h\lambda^2}{4}}{1 - \frac{h\lambda}{4}} y_k$

$$y_{k+1} = y_k + \frac{\lambda h + \frac{\lambda^2 h^2}{4}}{1 - \frac{h\lambda}{4}} y_k = \left(\frac{1 + \frac{3h\lambda}{4} + \frac{(h\lambda)^2}{4}}{1 - \frac{h\lambda}{4}}\right) y_k$$

$$y_k = \left(\frac{1 + \frac{3h\lambda}{4} + \frac{(h\lambda)^2}{4}}{1 - \frac{h\lambda}{4}}\right)^k$$

abs. est. para h sii $y_k \xrightarrow{k\to\infty} 0$

$$y_k = \left(\frac{1 + \frac{3h\lambda}{4} + \frac{(h\lambda)^2}{4}}{1 - \frac{h\lambda}{4}} \right)^k \xrightarrow{k \to \infty} 0 \iff \frac{\left|1 + \frac{3h\lambda}{4} + \frac{(h\lambda)^2}{4}\right|}{\left|1 - \frac{h\lambda}{4}\right|} < 1$$

CV $h\lambda = z$. hallar para qué z se cumple.

si se cumple $\forall z < 0$ $(h > 0 \quad \lambda < 0)$

↪ incond. abs. est.

sino condicionalmente