Práctico 3, sección 2.

Ejercicio 3: $f:\mathbb{R}\to\mathbb{R}$ una función monótona creciente e integrable.

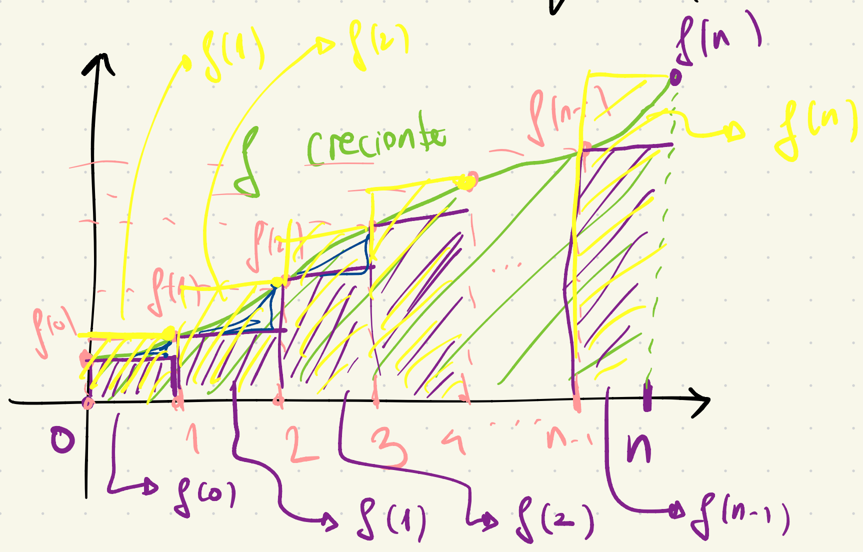
Probar que:

a) $$\sum_{k=0}^{n-1} f(k) \leq \int_0^n f(t)\,dt \leq \sum_{k=1}^{n} f(k)$$

$(\forall\, n\in\mathbb{N},\ n>0)$

b) $$\sum_{k=0}^{mn-1} \frac{f\left(\frac{k}{m}\right)}{m} \leq \int_0^n f(t)\,dt \leq \sum_{k=1}^{mn} \frac{f(k/m)}{m},$$

$\forall\ n,m\in\mathbb{N}^+.$



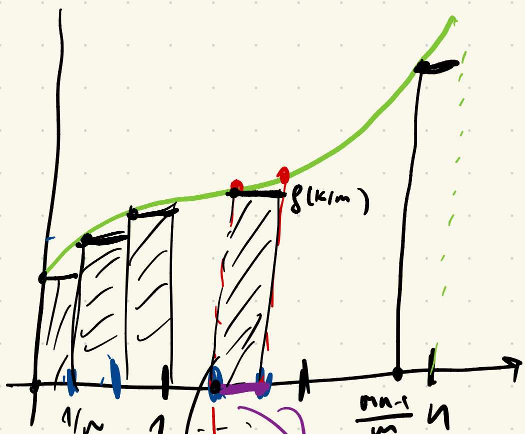
$$\Rightarrow \int_0^n f(t)\,dt \geq f(0)+f(1)+\dots+f(n-1) = \sum_{k=0}^{n-1} f(k) \Rightarrow \boxed{\sum_{k=0}^{n-1} f(k) \leq \int_0^n f(t)\,dt}$$

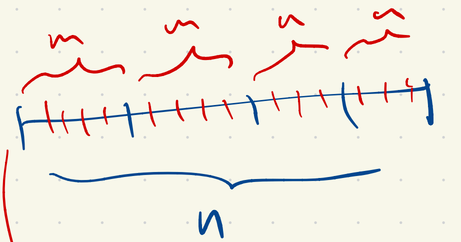
Análogamente: $\displaystyle\int_0^n f(t)\,dt \leq \underbrace{f(1)+f(2)+\dots+f(n)}_{\sum_{k=1}^{n} f(k)}$

$$\Rightarrow \boxed{\int_0^n f(t)\,dt \leq \sum_{k=1}^{n} f(k)}$$

Parte b)

$$\sum_{k=0}^{mn-1} \frac{f(k/m)}{m}$$





mn intervalitos con longitud $1/m$

$\frac{k}{m}$ $\qquad$ $\frac{k+1}{m}$

base $= 1/m$

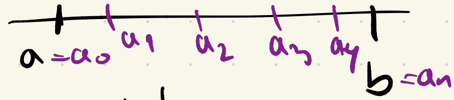
Altura $= f(k/m)$ $\Rightarrow$ Área $= \dfrac{f(k/m)}{m}$

$$\sum_{k=0}^{mn-1} \frac{f(k/m)}{m} \leq \int_0^n f(t)\,dt$$

Análogamente, se puede probar la otra desigualdad.

Sumas inferiores / superiores

$f : [a,b] \to \mathbb{R}$ acotada.



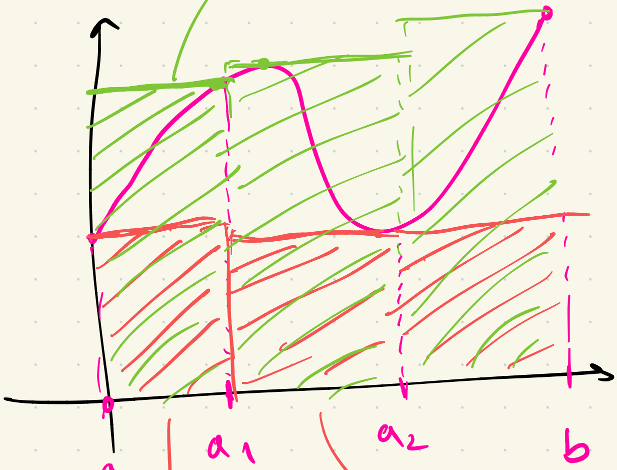
$P = \{ a_0 = a, a_1, a_2, \dots, a_{n-1}, a_n = b \}$ partición.

$$S_*(f,P) = \sum_{k=0}^{n-1} \inf(f, [a_k, a_{k+1}]) (a_{k+1} - a_k)$$

$$S^*(f,P) = \sum_{k=0}^{n-1} \sup(f, [a_k, a_{k+1}]) (a_{k+1} - a_k)$$

Ejemplo:

$\sup(f, [a_1, a_0]) (a_1 - a_0)$

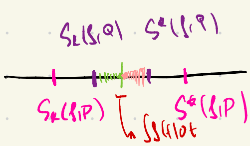


$\inf(f, [a_0, a_1]) (a_1 - a_0)$

$\inf(f, [a_1, a_2]) (a_2 - a_1)$

Definimos

$$I^*(f) = \inf \{ S^*(f,P) : P \text{ partición} \}$$



$$I_*(f) = \sup \{ S_*(f,P) : P \text{ partición} \}$$

Decimos que f es integrable si $I^*(f) = I_*(f)$.

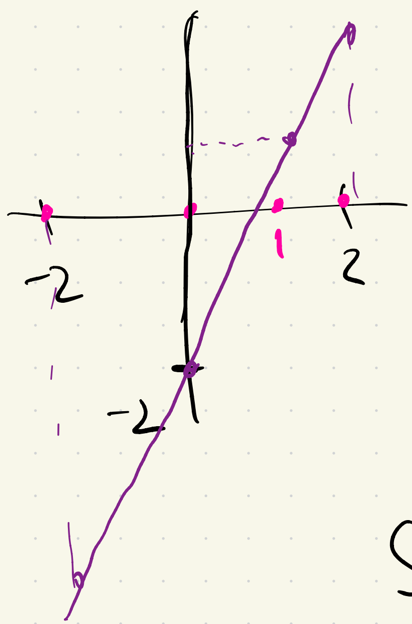
En ese caso, $\int_a^b f(t)\,dt := I^*(f) = I_*(f)$.

Sección 3.

Ejercicio 1: Calcular $S_*(f,P)$ y $S^*(f,P)$

a) $f: \mathbb{R} \to \mathbb{R} \,/\, f(x) = 3x - 2$.

a) $P = \{-2, 0, 1, 2\}$

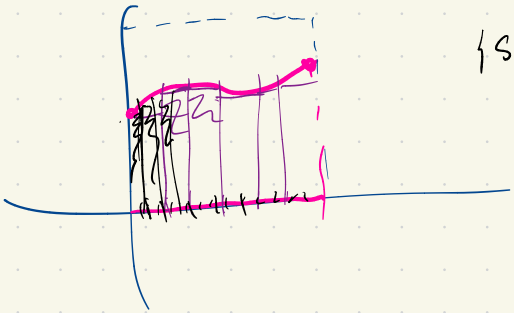


$S^*(f,P) = -2 \times 2$
$+ 1 \times 1$
$+ 4 \times 1$
$= -4 + 1 + 4 = 1$

$\Rightarrow \boxed{S^*(f,P) = 1}$

$S_*(f,P) = -8 \times 2$
$+ (-2) \times 1$
$+ 1 \times 1$

$$\Rightarrow S_*(f,P) = -16 - 2 + 1$$
$$= -17.$$



$\{ S_*(f,P) : P \text{ partición} \}$