Ejercicio 4: Calcular los siguientes límites usando el polinomio de Taylor.

a) $$\lim_{x\to 0} \frac{\overbrace{x-\log(1+x)}^{f(x)}}{x^2}$$

$P_n(f,a)(x)$

$= f(a) + f'(a)(x-a) + \dots \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$f(x) = P_n(f,0)(x) + \underbrace{R_n(x)}_{\text{Error del polinomio}}$$

Teo: $$\lim_{x\to a} \frac{R_n(x)}{(x-a)^n} = 0$$

Consecuencia: $$\lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \frac{\overbrace{P_2(f,0)(x) + R_2(x)}^{f(x)=}}{x^2}$$

$$= \lim_{x\to 0} \frac{P_2(f,0)(x)}{x^2} + \underbrace{\frac{R_2(x)}{x^2}}_{\to 0}$$

$$= \lim_{x\to 0} \frac{P_2(f,0)(x)}{x^2}$$

Calcular $P_2(f,0) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2$

$f(x) = x - \log(1+x)$

$\Rightarrow f(0) = 0 - \log(1+0) = 0 - 0 = 0$

$f(0) = 0$

$$f'(x) = (x - \log(1+x))' = 1 - \frac{1}{1+x}$$

$$\Rightarrow f'(0) = 1 - \frac{1}{1+0} = 0$$

$$f'(0) = 0$$

$$\left((1+x)^{-1}\right)' = -1\,(1+x)^{-2} = \frac{-1}{(1+x)^2}$$

$$\Rightarrow f''(x) = \left(1 - \frac{1}{1+x}\right)' = 0 - \frac{(-1)}{(1+x)^2}$$

$$= \frac{1}{(1+x)^2} \Rightarrow f''(0) = \frac{1}{(1+0)^2} = 1$$

$$\Rightarrow P_2(f,0)(x) = 0 + 0 \cdot x + \frac{1}{2}x^2$$

$$= \frac{x^2}{2} = P_2(f,0)(x)$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{x^2} = \lim_{x\to 0} \frac{x^2/2}{\cancel{x^2}} = \frac{1}{2}$$

b) $\lim_{x\to 0} \dfrac{\overbrace{\log(1+x) - \sin(x)}^{f(x)}}{\underbrace{x^2 + 4x^3}_{g(x)}}$

Idee: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f(x)}{(x-a)^n} \cdot \dfrac{(x-a)^n}{g(x)}$

$$= \lim_{x\to a} \frac{P_n(f,a)(x)}{\cancel{(x-a)^n}} \cdot \frac{\cancel{(x-a)^n}}{P_n(g,a)(x)}$$

↑ (el menor n tal que $P_n(g,a)(x) \neq 0$

$$= \lim_{x\to a} \frac{P_n(f,a)(x)}{P_n(g,a)(x)}$$

$$g(x) = x^2 + 4x^3$$

$$P_1(g,0)(x) = 0$$

$$\Rightarrow P_2(g,0)(x) = x^2$$ ← Nos sirve $n=2$

Ahora hallemos $P_2(f,0)(x)$:

$$f(x) = \log(1+x) - \sin(x) \Rightarrow f(0) = \log(1) - \sin(0) = 0$$

$$\Rightarrow f'(x) = \frac{1}{1+x} - \cos(x) \Rightarrow f'(0) = \frac{1}{1+0} - \cos(0) = 1 - 1 = 0$$

$$\Rightarrow f''(x) = \frac{-1}{(1+x)^2} + \sin(x) \Rightarrow f''(0) = \frac{-1}{(1+0)^2} + \sin(0) = -1$$

$$\Rightarrow P_2(f,0)(x) = 0 + 0x + \frac{(-1)}{2}x^2$$

$$= \frac{-x^2}{2}$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{P_2(f,0)(x)}{P_2(g,0)(x)}$$

$$= \lim_{x\to 0} \frac{-\cancel{x^2}/2}{\cancel{x^2}} = -\frac{1}{2}$$

Vale: $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m$

$$\Rightarrow P_n(f,0)(x) = a_0 + a_1 x + \dots + a_n x^n$$

Ejercicio 6:

a) $f(x) = a(e^x - 1) - b x^2 - x$

- Quiero $f(0) = 0$

$$\overset{\shortparallel}{a(e^0-1) - b\,0^2 - 0} = 0$$

- Quiero $f'(0)=0$

$$f'(x)=\left(a(e^x-1)-bx^2-x\right)'=ae^x-2bx-1$$
$$= e^x-2bx-1$$

$$f'(0)=ae^0-2b\cdot 0-1$$
$$=\boxed{a-1=0} \iff \underline{\underline{a=1}} \checkmark$$

- Quiero $f''(0)=0$

$$f''(x)=\left(e^x-2bx-1\right)'=e^x-2b$$
$$\Rightarrow f''(0)=e^0-2b=1-2b=0$$
$$\iff 2b=1 \iff b=1/2 \checkmark$$

$$f^{(3)}(x)=\left(e^x-2\left(\tfrac{1}{2}\right)\right)'=(e^x-1)'=e^x$$

$$\Rightarrow f^{(3)}(0)=e^0=1\neq 0$$

$\Rightarrow$ f es un infinitésimo de orden $\underline{2}$.

$\left(\iff P_2(f,0)(x)=0\right)$

La parte principal es $\lim_{x\to 0}\frac{f(x)}{x^3}=\frac{f^{(3)}(0)}{3!}$

c) $\overbrace{e^x \sin(x) - (ax + bx^2 + cx^3)}^{= f(x)}$

$$e^x \approx \overbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{3!}}^{P_3(e^x, 0)} + \dots$$

$$\sin(x) = \underbrace{x - \frac{x^3}{3!}}_{P_3(\sin(x), 0)} + \frac{x^5}{5!} - \dots$$

$$e^x \sin(x) \approx \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!}\right)\left(x - \frac{x^3}{3!}\right) + \dots$$

$$\underbrace{x + x^2 + \frac{x^3}{2} + \cancel{\frac{x^4}{3!}} - \frac{x^3}{3!} - \cancel{\frac{x^4}{3!}} - \cancel{\frac{x^5}{2 \cdot 3!}} - \cancel{\frac{x^6}{3! \cdot 3!}}}$$

$$P_3(e^x \sin(x), 0)(x) = x + x^2 + \frac{x^3}{2} - \frac{x^3}{3!} \quad 3! = 6$$

$$= x + x^2 + \frac{3x^3 - x^3}{6}$$

$$= x + x^2 + \frac{2x^3}{6}$$

$$= \boxed{x + x^2 + \frac{x^3}{3}}$$

$$P_3(f,0) = x + x^2 + \frac{x^3}{3} - (ax + bx^2 + cx^3)$$

$$= \underbrace{(1-a)}_{=0}x + \underbrace{(1-b)}_{=0}x^2 + \underbrace{\left(\frac{1}{3} - c\right)}_{=0}x^3 = 0$$

$$\longleftrightarrow \begin{cases} 1-a=0 \\ 1-b=0 \\ \frac{1}{3}-c=0 \end{cases} \longleftrightarrow \begin{cases} a=1 \\ b=1 \\ c=1/3 \end{cases}$$

Ejercicio 9: Hallar el polinomio de Mc Laurin de orden n.

a) $f(x) = \dfrac{1}{2-x}$

$$P_n(f,0)(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$$+ \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) = \frac{1}{2-x} \rightsquigarrow f(0) = \frac{1}{2-0} = \frac{1}{2}$$

$$= (2-x)^{-1}$$

$$\Rightarrow f'(x) = (-1)\,(2-x)^{-1-1}\,(2-x)'$$
$$= (-1)\,(2-x)^{-2}\,(-1)$$
$$= (2-x)^{-2} = \frac{1}{(2-x)^2}$$

$$\rightsquigarrow f^{(1)}(0) = \frac{1}{(2-0)^2} = \frac{1}{2^2}$$

$$f^{(2)}(x) = \left((2-x)^{-2}\right)' = (-2)\,(2-x)^{-2-1}\,(2-x)'$$
$$= (-2)\,(2-x)^{-3}\,(-1)$$
$$= 2\,(2-x)^{-3} = \frac{2}{(2-x)^3}$$

$$f^{(3)}(x) = \left(2\,(2-x)^{-3}\right)' = 2\,(-3)\,(2-x)^{-3-1}(2-x)'$$
$$= 2\,(-3)\,(2-x)^{-4}\,(-1)$$
$$= 2\cdot 3\,(2-x)^{-4} = \frac{6}{(2-x)^4}$$

$$f^{(4)}(x) = 2\cdot 3\,(-4)\,(2-x)^{-5}\,(-1)$$
$$= 2\cdot 3\cdot 4\,(2-x)^{-5} = 4!\,(2-x)^{-5}$$

En general: $f^{(n)}(x) = n!\,(2-x)^{-(n+1)}$

$$= \frac{n!}{(2-x)^{n+1}}$$

$$\Rightarrow f^{(n)}(0) = \frac{n!}{(2-0)^{n+1}} = \boxed{\frac{n!}{2^{n+1}}}$$

$$P_n(f,0)(x) = f(0) + f^{(1)}(0)\, x^1 + \frac{f^{(2)}(0)x^2}{2!} + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{1}{2} + \frac{1!}{2^{1+1}} x + \frac{\cancel{2!}/2^{2+1}}{\cancel{2!}} x^2 + \cdots + \frac{\cancel{n!}/2^{n+1}}{\cancel{n!}} x^n$$

$$= \frac{1}{2} + \frac{1}{2^2} x + \frac{1}{2^3} x^2 + \cdots + \frac{1}{2^{n+1}} x^n$$

$$= \boxed{\sum_{i=0}^{n} \frac{1}{2^{i+1}} x^i}$$

c) $f(x) = \log(1-x) \rightsquigarrow f(0) = \log(1) = 0$

$$f^{(1)}(x) = \frac{1}{1-x}(1-x)' = \frac{-1}{1-x} = -(1-x)^{-1}$$

$$\Rightarrow f^{(2)}(x) = \left(-(1-x)^{-1}\right)' = -\left((1-x)^{-1}\right)'$$

$$= -\left((-1)(1-x)^{-2}(1-x)'\right)$$

$$= -\left((-1)(1-x)^{-2}(-1)\right)$$

$$= -(1-x)^{-2} = \frac{-1}{(1-x)^2}$$

$$f^{(3)}(x) = -\left((1-x)^{-2}\right)' = -\left((-2)(1-x)^{-3}(1-x)'\right)$$

$$= -2(1-x)^{-3}$$

$$f^{(4)}(x) = -2.3(1-x)^{-4} = -3!(1-x)^{-4}$$

$$f^{(n)}(x) = -(n-1)!(1-x)^{-n}$$

$$\Rightarrow f^{(n)}(0) = -(n-1)!(1-0)^{-n}$$

$n! = (n-1)!\,n$

$$= -(n-1)!\,1^{-n} = -(n-1)!$$

$$\Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{-(n-1)!}{n!} = \frac{-(\cancel{1}.\cancel{2}.\cancel{3}\cdots(n\cancel{-1}))}{\cancel{1}.2.\cancel{3}\cdots\cancel{(n-1)}n}$$

$$\Rightarrow \frac{f^{(n)}(0)}{n!} = \frac{-1}{n}$$ $\rightsquigarrow$ Con esto podemos escribir el Taylor...