Práctico 7 - Residuos

Def: Decimos que f es meromorfa en Ω si $f \in H(\Omega \setminus A)$ con A sin puntos de acumulación en Ω y f tiene polos en A.

Ejemplos: $\dfrac{P(z)}{Q(z)}$ con P y Q polinomios

$\dfrac{f(z)}{Q(z)}$ con $f \in H(\Omega)$ y Q polinomio

Teorema 33. **Teorema de los residuos.** *Sea f meromorfa en Ω, $f \in H(\Omega \setminus A)$. Si Γ es un ciclo en $\Omega \setminus A$ tal que $\operatorname{Ind}_\Gamma(a) = 0\ \forall a \notin \Omega$, entonces*

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{a\in A} C_1(a)\operatorname{Ind}_\Gamma(a).$$

Orden de a como polo de f

$$\lim_{z\to a}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial z^{m-1}}[(z-a)^m f(z)] = c_1.$$

1. Sea $f(z) = \dfrac{z^2+2}{(z^2+9)(z^2+4)}$. $= \dfrac{z^2+2}{(z-3i)(z+3i)(z-2i)(z+2i)}$

   a) Encontrar los residuos de $f(z)$ en cada uno de sus polos.

Polos de orden 1 en $\pm 3i$, $\pm 2i$

$$\operatorname{Res}(f, 2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{z^2+2}{(z^2+9)(z+2i)}$$

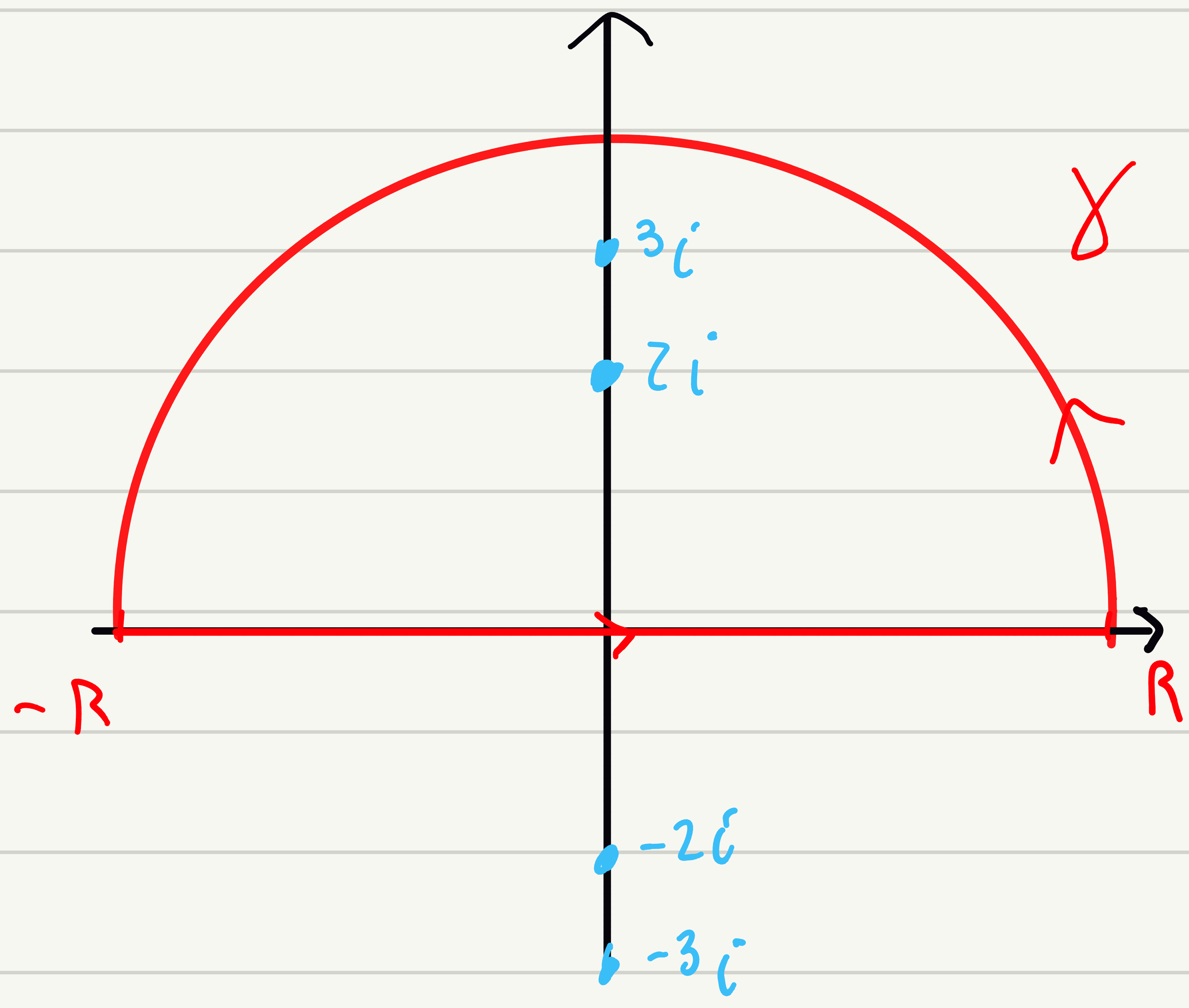
$$= \frac{-4+2}{(-4+9)(4i)} = \frac{-2}{20i} = \frac{-1}{10i} = \frac{i}{10}$$

$$\operatorname{Res}(f, -2i) = \lim_{z\to -2i}\frac{z^2+2}{(z^2+9)(z-2i)} = \frac{-2}{5(-4i)} = \frac{-i}{10}$$

$$\operatorname{Res}(f, 3i) = \lim_{z\to 3i}\frac{z^2+2}{(z+3i)(z^2+4)} = \frac{-9+2}{6i(-9+4)} = \frac{-7}{-30i} = \frac{-7i}{30}$$

$$\operatorname{Res}(f, -3i) = \lim_{z\to -3i}\frac{z^2+2}{(z-3i)(z^2+4)} = \frac{-7}{-6i\cdot 5} = \frac{7i}{30}$$

b) Calcular $\int_\gamma f(z)dz$ siendo γ la concatenación del segmento $[-R, R]$ con una semicircunferencia de radio R en el semiplano superior.



$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \operatorname{Res}(f, 2i) + \operatorname{Res}(f, 3i)$$

$$= \frac{i}{10} - \frac{7}{30} i$$

$$= i\left(\frac{3}{30} - \frac{7}{30}\right) = i\left(\frac{-4}{30}\right)$$

$$= -i\ \frac{2}{15}$$

$$\Rightarrow \int_\gamma f(z)dz = \frac{4\pi}{15}$$

Lema 1 ***Lema de deformación de curvas.***
Sea $f : \Omega \to \mathbb{C}$ continua. Sea para todo $R > 0$ suficientemente grande $S_R \subset \Omega$ el arco de circunferencia $z = z(t) = Re^{it},\ \theta_1 \le t \le \theta_2$.

1. *Si* $\lim_{z\to\infty} zf(z) = L$ *entonces*

$$\lim_{R\to+\infty} \int_{S_R} f(z)dz = iL(\theta_2 - \theta_1)$$

c) Calcular, justificando todos los pasos: $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2+2}{(x^2+9)(x^2+4)}\, dx.$

$$\int_{\gamma_R} f(z)dz = \frac{4\pi}{15} = \int_{[-R,R]} f(z)dz + \int_{S_R} f(z)dz$$

$$S_R(t) = Re^{it}, \quad t \in [0, \pi]$$

$$\beta(t) = t, \quad t \in [-R, R]. \qquad \int_{[-R,R]} f = \int_\beta f = \int_{-R}^{R} f(x)\cdot 1\, dx = \int_{-R}^{R} f(x)dx$$

$$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} z\,\frac{z^2+2}{(z^2+9)(z^2+4)} = 0$$

$$\Rightarrow \lim_{R\to+\infty} \int_{S_R} f(z)\,dz = 0$$

$$\lim_{R\to+\infty}\int_{\gamma_R} f = \lim_{R\to+\infty}\int_{[-R,R]} f + \int_{S_R} f = \lim_{R\to+\infty}\int_{[-R,R]} f + \lim_{R\to+\infty}\int_{S_R} f \to 0$$

$$= \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to+\infty} \frac{4\pi}{15} = \frac{4\pi}{15}$$

5. Sea $R(x,y)$ una función racional de 2 variables tal que no se anula el denominador en la circunferencia unitaria.

   a) Probar que

$$\int_0^{2\pi} R(cos\theta, sen\theta)d\theta = -i\int_\gamma R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{z}$$

   siendo γ la circunferencia unitaria recorrida en sentido antihorario.

$\gamma(t) = e^{it} \Rightarrow \gamma'(t) = ie^{it}$, $t \in [0, 2\pi]$

$$-i\int_\gamma R(\dots)\frac{dz}{z} = \cancel{-i}\int_0^{2\pi} R\left(\frac{1}{2}\left(e^{it}+e^{-it}\right), \frac{1}{2i}\left(e^{it}-e^{-it}\right)\right)\frac{1}{\cancel{e^{it}}}\cancel{ie^{it}}\,dt$$

$$= \int_0^{2\pi} R(\cos t, \sin t)\,dt$$

2) $\displaystyle\int_0^{2\pi} \frac{\cos x}{\cos x + \cosh a}\,dx$ $\quad a > 0 \quad$ $R(x,y) = \dfrac{x}{x + \cosh a}$

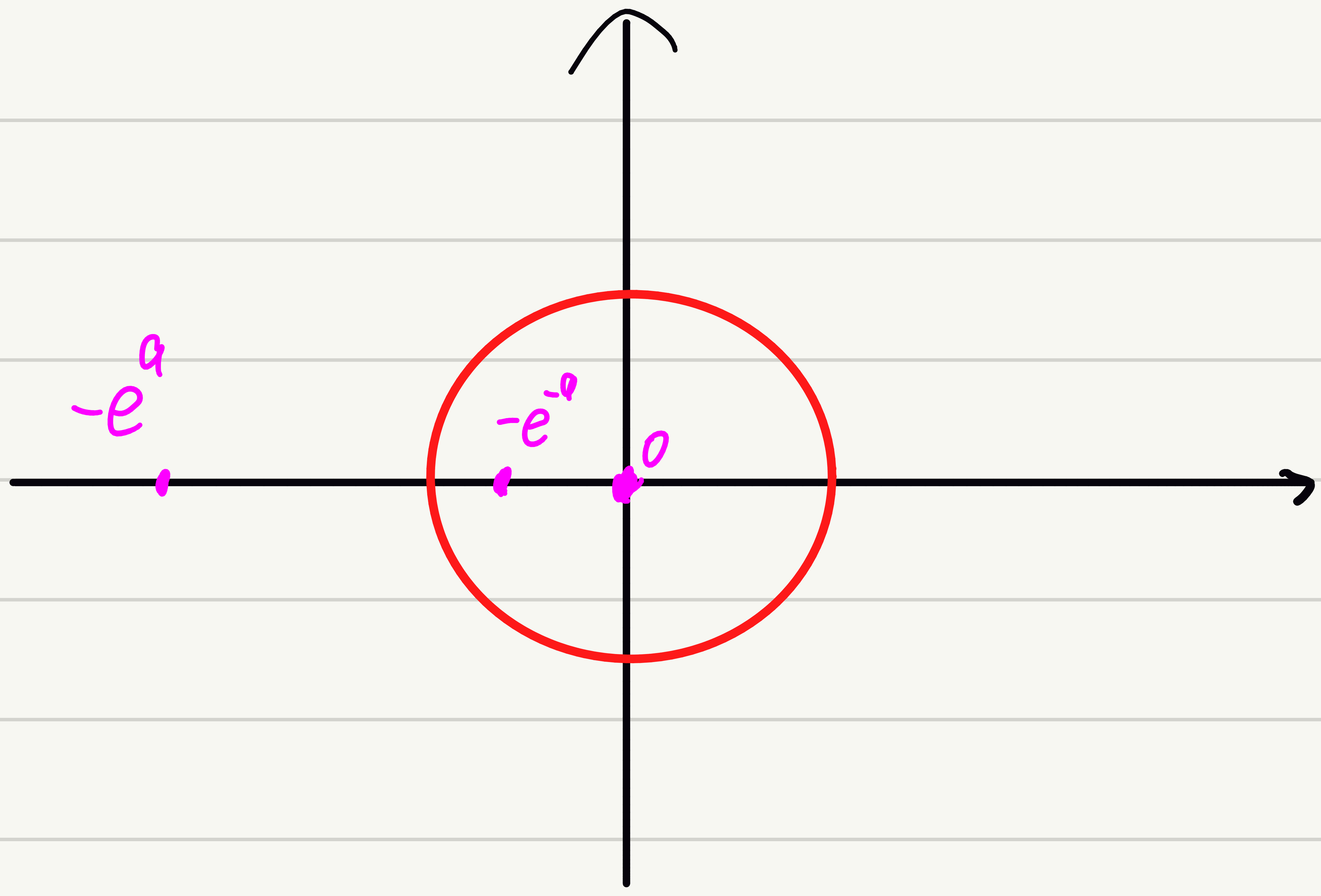
$$f(z) = \frac{-i}{z} R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)$$

$$= \frac{-i}{z}\,\frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{\frac{1}{2}\left(z+\frac{1}{z}\right)+\cosh a} = \frac{-i}{z}\,\frac{z^2+1}{z^2+1+2z\cosh a}$$

$(z+\alpha)(z+\beta) = z^2 + z(\alpha+\beta) + \alpha\cdot\beta$

$$z^2 + 2z\cosh a + 1 = z^2 + \cancel{2}z\left(\frac{e^a+e^{-a}}{\cancel{2}}\right) + 1 = z^2 + z\left(e^a+e^{-a}\right) + e^a\cdot e^{-a}$$

$$= (z+e^a)(z+e^{-a}) = 0 \Leftrightarrow \underbrace{z = -e^a}_{<-1} \text{ ó } z = -e^{-a} \in (-1,1)$$



$$\text{Res}(f,0) = \lim_{z\to 0} z f(z) = \lim_{z\to 0} -i \frac{z^2+1}{z^2+1+2z\cosh a}$$

$$= \frac{-i}{1} = -i$$

$$\text{Res}(f,-e^{-a}) = \lim_{z\to -e^{-a}} (z+e^{-a}) f(z) = \lim_{z\to -e^{-a}} -i \frac{z^2+1}{z(z+e^{a})}$$

$$= +i \frac{e^{-2a}+1}{+e^{-a}\cdot(e^{a}-e^{-a})} = i \frac{\frac{1}{2}(e^{-a}+e^{a})}{\frac{1}{2}(e^{a}-e^{-a})} = i \frac{\cosh a}{\sinh a} = i\cdot\coth(a)$$

$$\Rightarrow \frac{1}{2\pi i}\oint_{|z|=1} f(z)\,dz = i(-1+\coth(a)) \Longrightarrow$$

$$\oint_{|z|=1} f(z)\,dz = 2\pi(1-\coth(a)) = \int_0^{2\pi} \frac{\cos x}{\cos x+\cosh a}\,dx$$

$$f(z) = a_0 + a_1(z-1) + a_2(z-1)^2 + \cdots$$

$$\frac{f(z)}{(z-1)^2} = \frac{a_0}{(z-1)^2} + \frac{a_1}{z-1} \,\Big|\, + a_2 + a_3(z-1) + a_4(z-1)^2 + \cdots$$