

# Morfología Matemática

# Morfología matemática

- Imágenes binarias.
- Estudio de las formas (topología y/o estructura) de los objetos observados.
- Se utilizan procedimientos que incluyen la aplicación de un elemento estructurante para extraer del objeto sus características morfológicas principales.

# Definiciones básicas

- Objeto:  $X$
- Elemento estructurante:  $B$
- Ambos en un espacio euclidiano 2D.
- $B_x$  : Traslación de  $B$  tal que su origen está en  $x = (x_1, x_2)$

# Definiciones básicas

- Reflexión de B:

$$\hat{B} = \{x \mid x = -b, \text{ para } b \in B\}$$

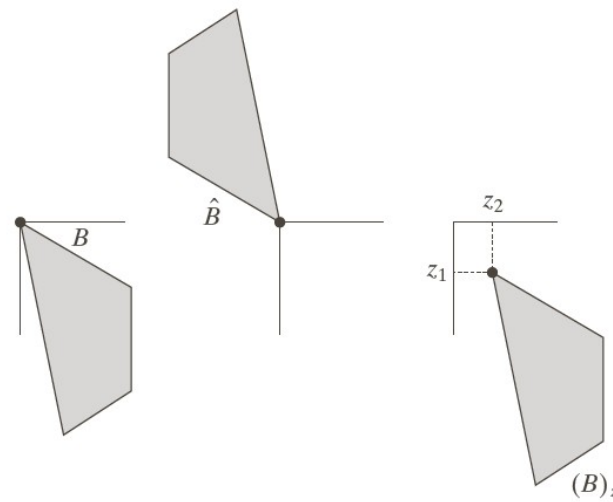
- Complemento del conjunto A:

$$A^c = \{x \mid x \notin A\}$$

- Diferencia de dos conjuntos:

$$A - B = \{x \mid x \in A, x \notin B\} = A \cap B^c$$

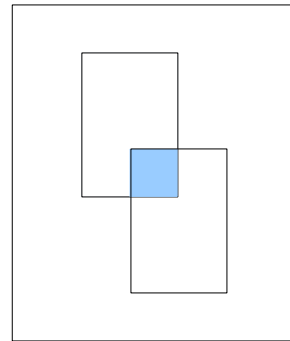
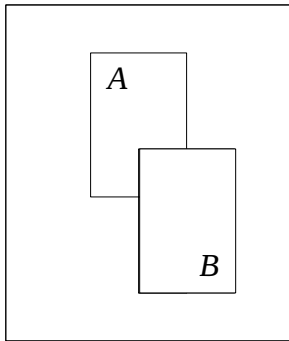
# Conjunto, reflexión, traslación



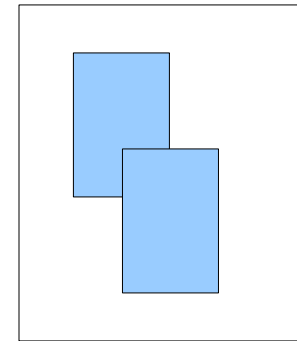
a b c

**FIGURE 9.1**  
(a) A set, (b) its reflection, and  
(c) its translation by  $z$ .

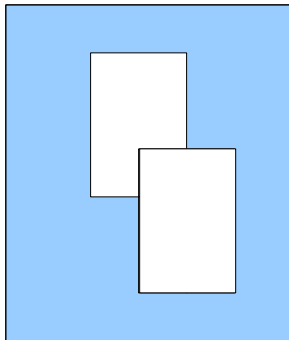
# Operaciones básicas de la teoría de conjuntos



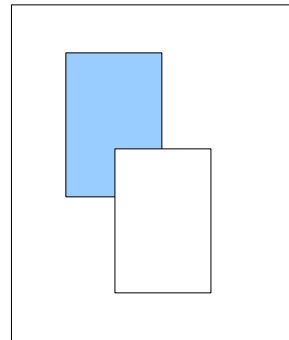
$$(A \cap B)$$



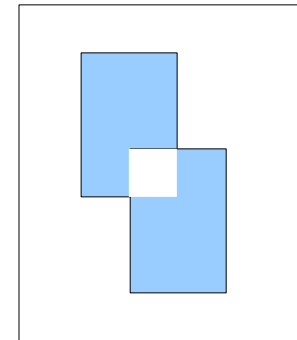
$$(A \cup B)$$



$$(A \cup B)^c$$

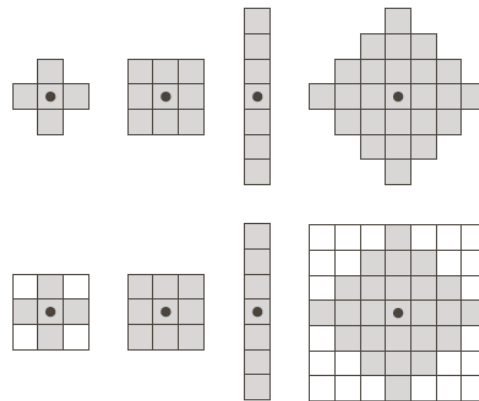


$$(A - B)$$



$$(A / B)$$

# Elemento estructurante



**FIGURE 9.2** First row: Examples of structuring elements. Second row: Structuring elements converted to rectangular arrays. The dots denote the centers of the SEs.

# Transformaciones en todo o nada (TTN)

- Consideremos un objeto y  $X$  el conjunto de puntos que lo constituye.
- La estructura de  $X$  está dada por el conjunto de relaciones entre los puntos que lo forman.
- La TTN genera una imagen transformada que expresa la información estructural de  $X$ .



# Transformación en todo o nada

- Se desplaza  $B$  por todo punto de  $X$  y se hace una pregunta relativa a la inclusión, intersección o unión de  $B$  en  $X$  o con  $X$ .
- El resultado es una imagen formada por el conjunto de respuestas, positivas o negativas a la pregunta.
- La imagen transformada es el conjunto de respuestas positivas.

# Condiciones de las operaciones en todo o nada.

- Invariancia por traslación  $T(X) = (T(X))_{tx}$
- Compatibilidad con las homotopías
- Conocimiento local uniforme
- Continuidad monótona

# Operaciones básicas

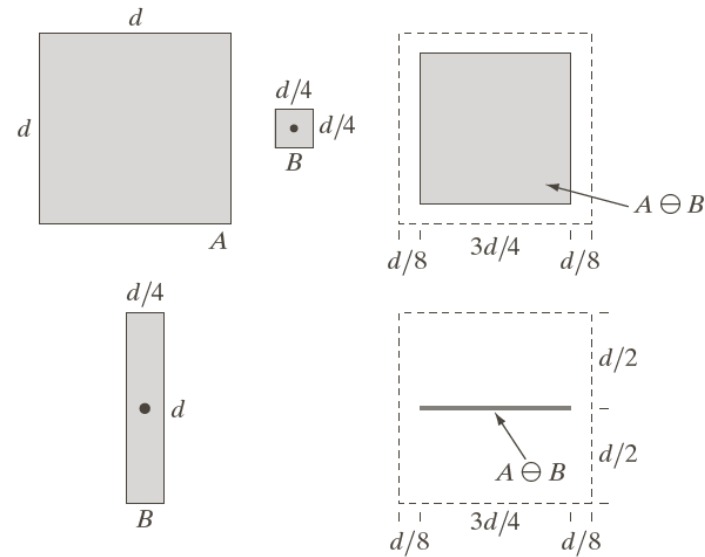
- Se trata de las dos operaciones básicas de la MM:
  - Erosión
  - Dilatación.

# Erosión

- Erosión de  $X$  por  $B$ : Conjunto de todos los puntos  $x$  tal que  $B_x$  esté incluida en  $X$

$$X \ominus B \equiv \left\{ x : B_x \subset X \right\}$$

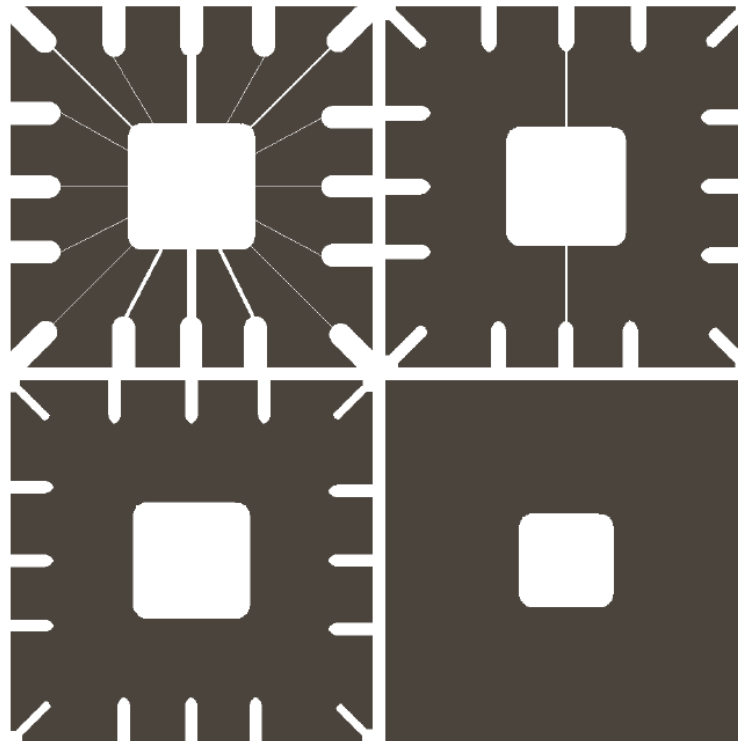
# Erosión



a	b	c
d	e	

**FIGURE 9.4** (a) Set  $A$ . (b) Square structuring element,  $B$ . (c) Erosion of  $A$  by  $B$ , shown shaded. (d) Elongated structuring element. (e) Erosion of  $A$  by  $B$  using this element. The dotted border in (c) and (e) is the boundary of set  $A$ , shown only for reference.

# Erosión



a	b
c	d

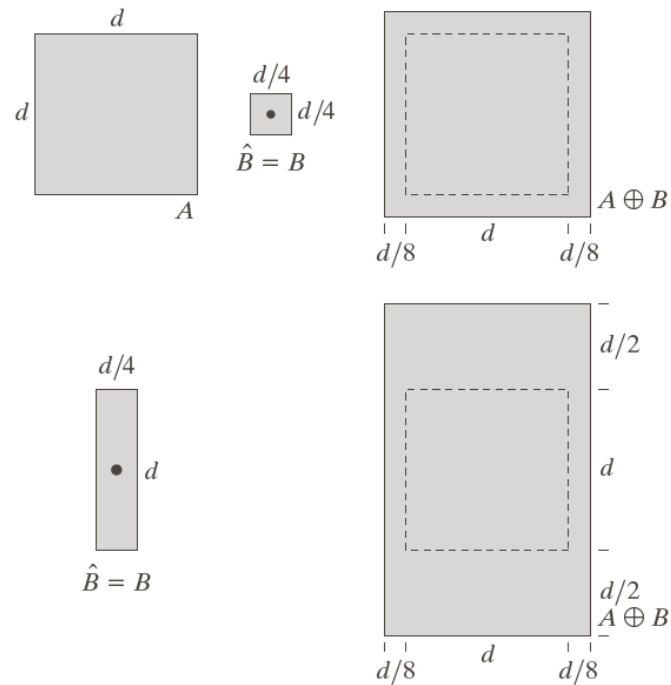
**FIGURE 9.5** Using erosion to remove image components. (a) A  $486 \times 486$  binary image of a wire-bond mask. (b)–(d) Image eroded using square structuring elements of sizes  $11 \times 11$ ,  $15 \times 15$ , and  $45 \times 45$ , respectively. The elements of the SEs were all 1s.

# Dilatación

- Dilatación de  $X$  por  $B$  : Conjunto de todos los puntos  $x$  tal que la intersección de  $B_x$  y  $X$  sea no nula.

$$X \oplus B \equiv \left\{ x : B_x \cap X \neq \mathbf{0} \right\}$$

# Dilatación



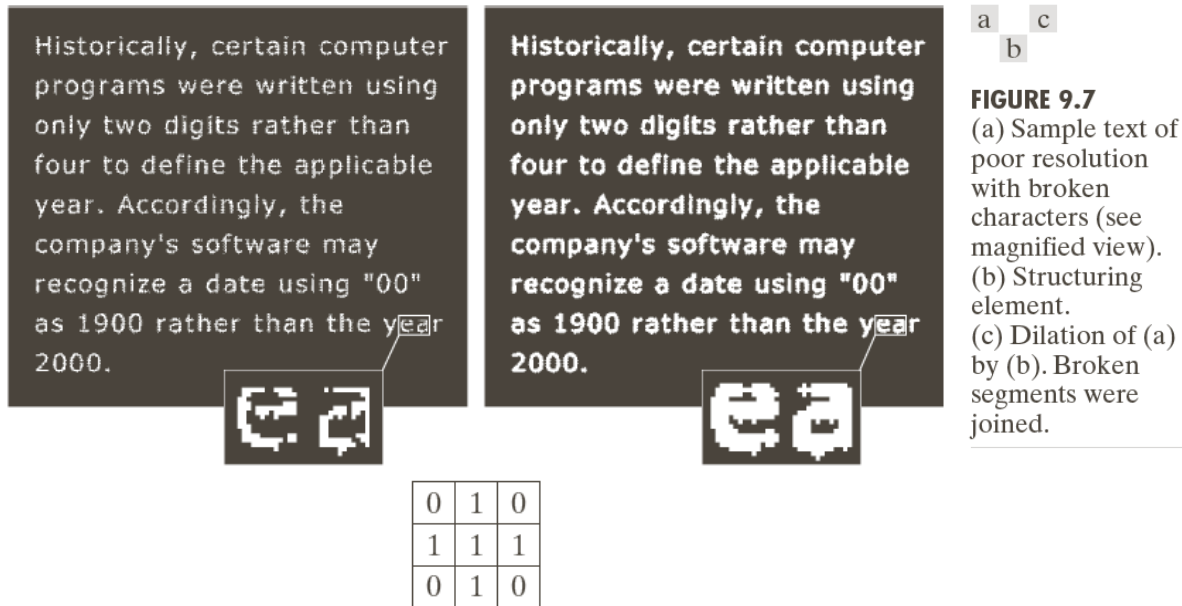
a	b	c
d	e	

**FIGURE 9.6**

(a) Set  $A$ .  
 (b) Square structuring element (the dot denotes the origin).  
 (c) Dilation of  $A$  by  $B$ , shown shaded.  
 (d) Elongated structuring element. (e) Dilation of  $A$  using this element. The dotted border in (c) and (e) is the boundary of set  $A$ , shown only for reference



# Dilatación



# Propiedades

- Invariante respecto a traslaciones
- No son operaciones inversas.
- Idempotencia?
- Distributividad:

$$X \oplus (B \cup B') = (X \oplus B) \cup (X \oplus B')$$

$$X \ominus (B \cup B') = (X \ominus B) \cap (X \ominus B')$$

# Propiedades

- Conocimiento local:

$$(X \cap Z) \ominus B = (X \ominus B) \cap (Z \ominus B)$$

- Iteración:

$$(X \ominus B) \ominus B' = X \ominus (B \oplus B')$$

$$(X \oplus B) \oplus B' = X \oplus (B \oplus B')$$

# Propiedades

- Incremento:

$$\begin{array}{l} \textit{Si } X \subset X' \Rightarrow X \ominus B \subset X' \ominus B \quad \forall B \\ X \oplus B \subset X' \oplus B \quad \forall B \end{array}$$

$$\textit{Si } B \subset B' \Rightarrow X \ominus B \subset X \ominus B' \quad \forall X$$

# Propiedades

- Dualidad: Sea  $X^c$  el complemento de  $X$ .

$$X^c \oplus B = (X \ominus B)^c$$

# Transformaciones morfológicas

- Combinando las operaciones básicas se pueden construir transformaciones muy útiles en procesamiento de imágenes binarias.

# Apertura

- Suaviza contornos. Elimina pequeñas islas y promontorios de  $X$ .

$$X \circ B = (X \ominus B) \oplus B$$

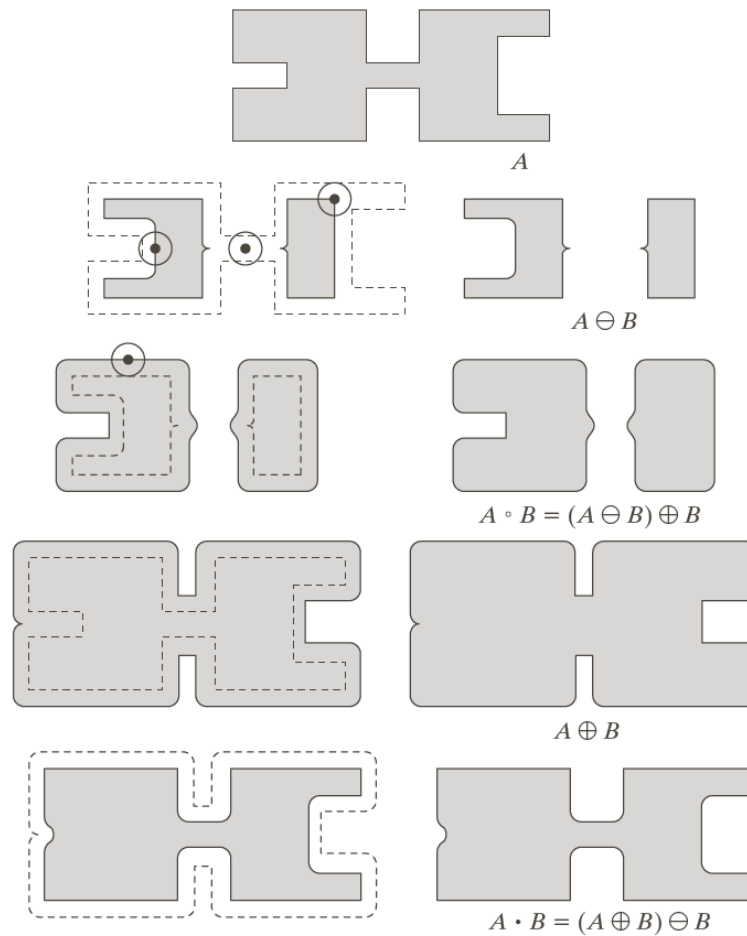
# Cierre

- Llena canales estrechos y pequeños lagos.

$$X \bullet B = (X \oplus B) \ominus B$$



# Apertura y cerradura



a
b c
d e
f g
h i

**FIGURE 9.10** Morphological opening and closing. The structuring element is the small circle shown in various positions in (b). The SE was not shaded here for clarity. The dark dot is the center of the structuring element.

# Propiedades

- Apertura y cerradura son operaciones duales  $(A \circ B)^c = (A^c \cdot \hat{B})$   $(A \cdot B)^c = (A^c \circ \hat{B})$

- Propiedades de la apertura:

$$(A \circ B) \subset A$$

$$\text{si } C \subset D \text{ entonces } (C \circ B) \subset (C \circ D)$$

- Propiedades de la cerradura:

$$A \subset (A \cdot B)$$

$$\text{si } C \subset D \text{ entonces } (C \cdot B) \subset (C \cdot D)$$

- Ambas son idempotentes:

$$(A \cdot B) \cdot B = A \cdot B$$

$$(A \circ B) \circ B = A \circ B$$

# ejemplo



a	b
d	c
e	f

**FIGURE 9.11**

(a) Noisy image.  
 (b) Structuring element.  
 (c) Eroded image.  
 (d) Opening of  $A$ .  
 (e) Dilation of the opening.  
 (f) Closing of the opening.  
 (Original image courtesy of the National Institute of Standards and Technology.)

# Elementos estructurantes

$$\begin{array}{ccc} 1 & 1 & 1 \\ G = 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ L = x & 1 & x \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{ccc} 1 & x & x \\ C = 1 & 0 & x \\ 1 & x & x \\ x & x & x \\ E = 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}$$

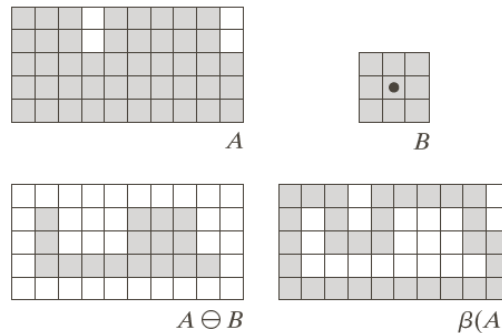
$1$ : *objeto*

$0$ : *fondo*

# Contorno

- Conjunto de pixels del borde:

$$\beta(X) = X - (X \ominus B)$$



a	b
c	d

**FIGURE 9.13** (a) Set  $A$ . (b) Structuring element  $B$ . (c)  $A$  eroded by  $B$ . (d) Boundary, given by the set difference between  $A$  and its erosion.

# Extracción de contornos



a b

**FIGURE 9.14**  
(a) A simple binary image, with 1s represented in white. (b) Result of using Eq. (9.5-1) with the structuring element in Fig. 9.13(b).

# Relleno de regiones

- Proceso iterativo comenzado en un punto  $p$  del contorno.

$$X_k = (X_{k-1} \oplus B) \cap A^c \quad k = 1, 2, 3, \dots$$

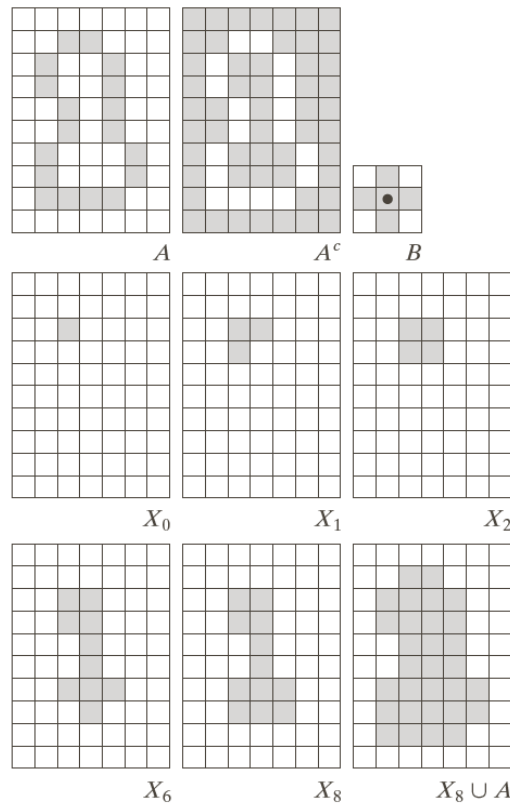
$$X_0 = p$$

$$x \quad 1 \quad x$$

$$B = 1 \quad 1 \quad 1$$

$$x \quad 1 \quad x$$

# Relleno de regiones



a	b	c
d	e	f
g	h	i

**FIGURE 9.15** Hole filling. (a) Set  $A$  (shown shaded). (b) Complement of  $A$ . (c) Structuring element  $B$ . (d) Initial point inside the boundary. (e)–(h) Various steps of Eq. (9.5-2). (i) Final result [union of (a) and (h)].



# Extracción de componentes conexas

- Proceso Iterativo empezando en un punto  $p$  del contorno.

$$X_k = (X_{k-1} \oplus B) \cap A \quad k = 1, 2, 3, \dots$$

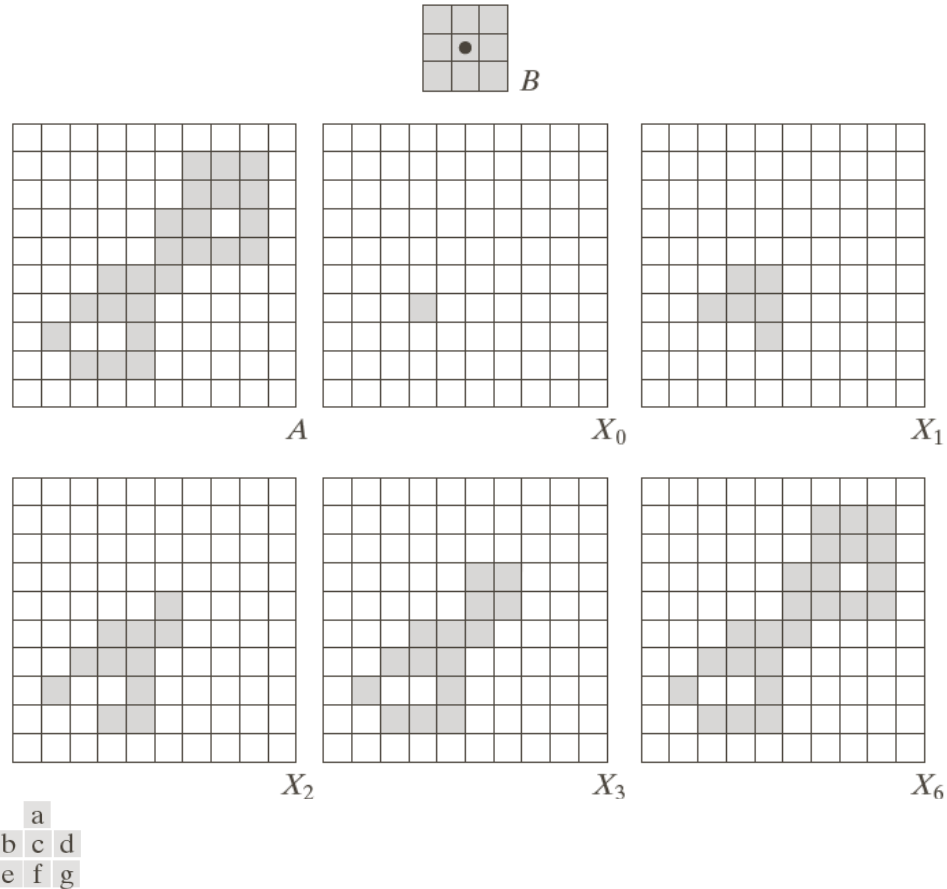
$$X_0 = p$$

$$1 \quad 1 \quad 1$$

$$B = 1 \quad 1 \quad 1$$

$$1 \quad 1 \quad 1$$

# Componentes conexas



**FIGURE 9.17** Extracting connected components. (a) Structuring element. (b) Array containing a set with one connected component. (c) Initial array containing a 1 in the region of the connected component. (d)–(g) Various steps in the iteration of Eq. (9.5-3).

# Cerco convexo

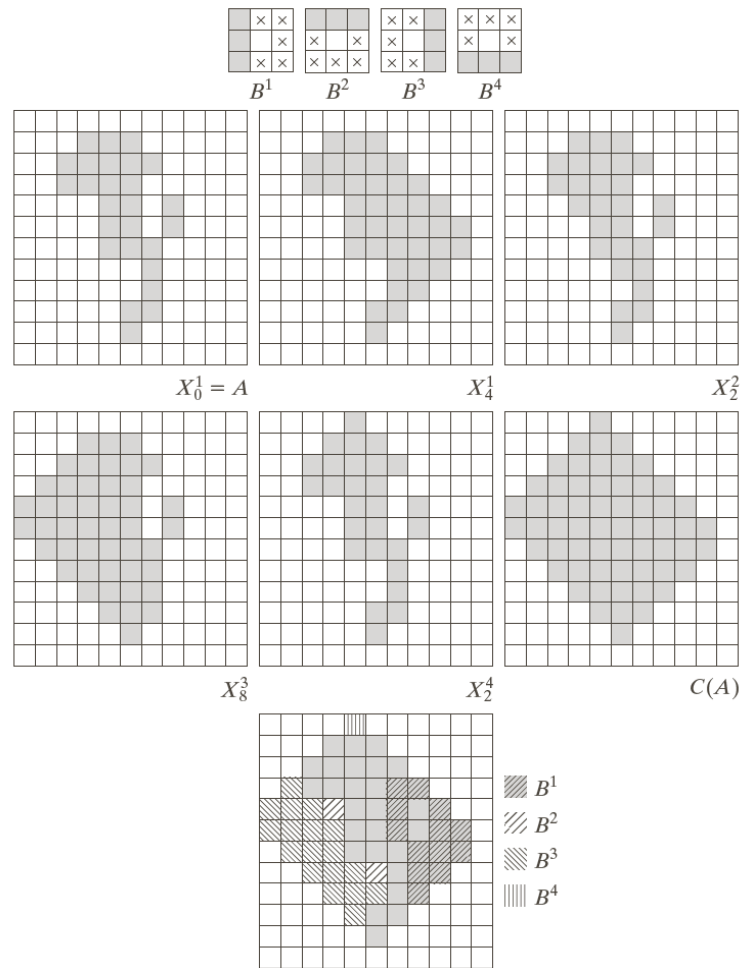
- Se utiliza un ES apropiado, por ejemplo C, que se va rotando.

$$X_1^1 = X$$

$$X_{i+1}^1 = (X_i^1 \otimes B^1)$$

$$X_{CH} = \bigcup_{j=1}^4 X_{\infty}^j$$

# Cerco convexo

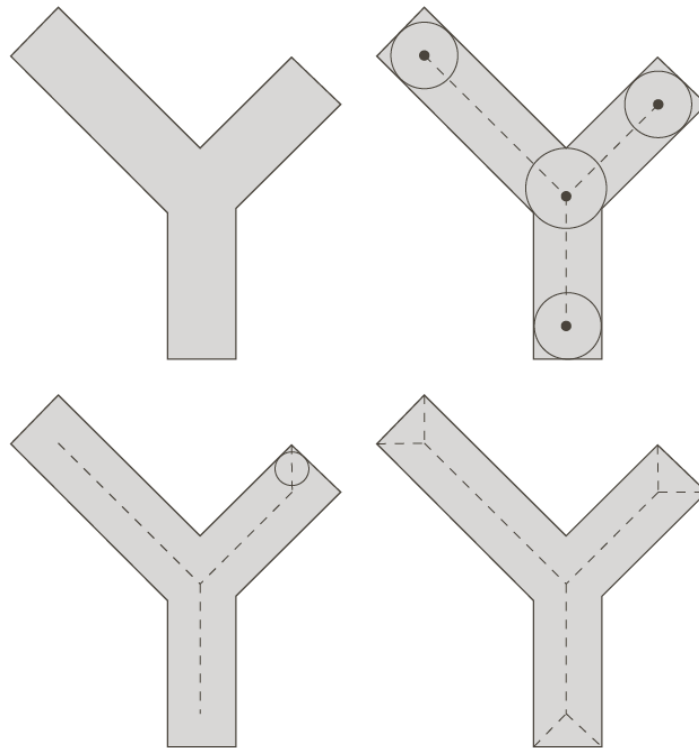


**FIGURE 9.19**  
 (a) Structuring elements. (b) Set  $A$ . (c)–(f) Results of convergence with the structuring elements shown in (a). (g) Convex hull. (h) Convex hull showing the contribution of each structuring element.

# Esqueleto

- Medial Axis Transformation (MAT): Para una región  $\mathbf{R}$  con un borde  $\mathbf{B}$ : para cada punto  $p$  de  $\mathbf{R}$  se busca el vecino  $v$  más próximo en  $\mathbf{B}$ . Los  $p$  que tengan más de 1 vecino  $v$  conforman el “eje mediano” de  $\mathbf{B}$ . (González)

# Esqueleto



a	b
c	d

**FIGURE 9.23**

(a) Set  $A$ .  
(b) Various positions of maximum disks with centers on the skeleton of  $A$ .  
(c) Another maximum disk on a different segment of the skeleton of  $A$ .  
(d) Complete skeleton.

# Esqueleto

- Conjunto de los centros de discos tales que:
  - Tienen radio máximo
  - Están contenidos en el objeto y
  - Intersectan el borde en al menos 2 puntos.
- Se aproxima un círculo por un elemento estructurante cuadrado.
- Se puede pensar como erosionar hasta que queden los puntos últimos.

# Esqueleto

$$S(A) = \bigcup_{k=0}^K S_k(A)$$

$$S_k(A) = \bigcup_{k=0}^K \left\{ (A \ominus kG) - \left[ (A \ominus kG) \circ B \right] \right\}$$

$A \ominus kG$  : Erosión iterada k veces.

K : último paso antes que A se erosione a un conjunto vacío.



Operation	Equation	Comments (The Roman numerals refer to the structuring elements in Fig. 9.33.)
Closing	$A \bullet B = (A \oplus B) \ominus B$	Smooths contours, fuses narrow breaks and long thin gulfs, and eliminates small holes. (I)
Hit-or-miss transform	$A \otimes B = (A \ominus B_1) \cap (A^c \ominus B_2)$ $= (A \ominus B_1) - (A \oplus \hat{B}_2)$	The set of points (coordinates) at which, simultaneously, $B_1$ found a match (“hit”) in $A$ and $B_2$ found a match in $A^c$
Boundary extraction	$\beta(A) = A - (A \ominus B)$	Set of points on the boundary of set $A$ . (I)
Hole filling	$X_k = (X_{k-1} \oplus B) \cap A^c$ ; $k = 1, 2, 3, \dots$	Fills holes in $A$ ; $X_0 =$ array of 0s with a 1 in each hole. (II)
Connected components	$X_k = (X_{k-1} \oplus B) \cap A$ ; $k = 1, 2, 3, \dots$	Finds connected components in $A$ ; $X_0 =$ array of 0s with a 1 in each connected component. (I)
Convex hull	$X_k^i = (X_{k-1}^i \otimes B^i) \cup A$ ; $i = 1, 2, 3, 4$ ; $k = 1, 2, 3, \dots$ ; $X_0^i = A$ ; and $D^i = X_{\text{conv}}^i$	Finds the convex hull $C(A)$ of set $A$ , where “conv” indicates convergence in the sense that $X_k^i = X_{k-1}^i$ . (III)
Thinning	$A \otimes B = A - (A \otimes B)$ $= A \cap (A \otimes B)^c$ $A \otimes \{B\} =$ $((\dots((A \otimes B^1) \otimes B^2) \dots) \otimes B^n)$ $\{B\} = \{B^1, B^2, B^3, \dots, B^n\}$	Thins set $A$ . The first two equations give the basic definition of thinning. The last equations denote thinning by a sequence of structuring elements. This method is normally used in practice. (IV)
Thickening	$A \odot B = A \cup (A \otimes B)$ $A \odot \{B\} =$ $((\dots(A \odot B^1) \odot B^2 \dots) \odot B^n)$	Thickens set $A$ . (See preceding comments on sequences of structuring elements.) Uses IV with 0s and 1s reversed.
Skeletons	$S(A) = \bigcup_{k=0}^K S_k(A)$ $S_k(A) = \bigcup_{k=0}^K \{A \ominus kB\}$ $- [(A \ominus kB) \odot B]$ Reconstruction of $A$ : $A = \bigcup_{k=0}^K (S_k(A) \oplus kB)$	Finds the skeleton $S(A)$ of set $A$ . The last equation indicates that $A$ can be reconstructed from its skeleton subsets $S_k(A)$ . In all three equations, $K$ is the value of the iterative step after which the set $A$ erodes to the empty set. The notation $(A \ominus kB)$ denotes the $k$ th iteration of successive erosions of $A$ by $B$ . (I)

Operation	Equation	Comments (The Roman numerals refer to the structuring elements in Fig. 9.33.)
Pruning	$X_1 = A \otimes \{B\}$ $X_2 = \bigcup_{k=1}^8 (X_1 \otimes B^k)$ $X_3 = (X_2 \oplus H) \cap A$ $X_4 = X_1 \cup X_3$	$X_4$ is the result of pruning set $A$ . The number of times that the first equation is applied to obtain $X_1$ must be specified. Structuring elements $V$ are used for the first two equations. In the third equation $H$ denotes structuring element I.
Geodesic dilation of size 1	$D_G^{(1)}(F) = (F \oplus B) \cap G$	$F$ and $G$ are called the <i>marker</i> and <i>mask</i> images, respectively.
Geodesic dilation of size $n$	$D_G^{(n)}(F) = D_G^{(1)}[D_G^{(n-1)}(F)]$ ; $D_G^{(0)}(F) = F$	
Geodesic erosion of size 1	$E_G^{(1)}(F) = (F \ominus B) \cup G$	
Geodesic erosion of size $n$	$E_G^{(n)}(F) = E_G^{(1)}[E_G^{(n-1)}(F)]$ ; $E_G^{(0)}(F) = F$	
Morphological reconstruction by dilation	$R_G^D(F) = D_G^{(k)}(F)$	$k$ is such that $D_G^{(k)}(F) = D_G^{(k+1)}(F)$
Morphological reconstruction by erosion	$R_G^E(F) = E_G^{(k)}(F)$	$k$ is such that $E_G^{(k)}(F) = E_G^{(k+1)}(F)$
Opening by reconstruction	$O_R^{(n)}(F) = R_F^D[(F \ominus nB)]$	$(F \ominus nB)$ indicates $n$ erosions of $F$ by $B$ .
Closing by reconstruction	$C_R^{(n)}(F) = R_F^E[(F \oplus nB)]$	$(F \oplus nB)$ indicates $n$ dilations of $F$ by $B$ .
Hole filling	$H = [R_F^D(F)]^c$	$H$ is equal to the input image $I$ , but with all holes filled. See Eq. (9.5-28) for the definition of the marker image $F$ .
Border clearing	$X = I - R_I^D(F)$	$X$ is equal to the input image $I$ , but with all objects that touch (are connected to) the boundary removed. See Eq. (9.5-30) for the definition of the marker image $F$ .

(Continued)