$\varphi: G_1 \to G_2$ es un morfismo de grupos si

$\varphi(g * g') = \varphi(g) \cdot \varphi(g')$ para todo $g, g' \in G_1$

* $\varphi(e_{G_1}) = e_{G_2}$

* $g \in G_1$ un elemento de orden finito

$\Rightarrow o(\varphi(g)) \mid o(g)$

Ejercicio 2. Sea $\varphi : G_1 \to G_2$ un homomorfismo de grupos finitos.

a. Sea $g \in G_1$. Probar que $o(\varphi(g))$ divide a $\mathrm{mcd}(|G_1|, |\mathrm{Im}(\varphi)|)$.

b. Probar que si $|G_1|$ y $|G_2|$ son coprimos, entonces φ es trivial.

c. Supongamos que φ es un isomorfismo de grupos. Sea $g \in G_1$. Probar que el orden de g en G_1 es igual al orden de $\varphi(g)$ en G_2.

d. Probar que $\mathbb{Z}_4$ y $\mathbb{Z}_2 \times \mathbb{Z}_2$ no son isomorfos.

c) $\varphi: G_1 \to G_2$ isomorfismo de grupos: $\begin{cases} \varphi: G_1 \to G_2 \text{ morfismo de grupos} \\ \varphi \text{ es biyectiva, es decir exist } \varphi^{-1} \\ \varphi^{-1}: G_2 \to G_1 \text{ morfismo de grupos} \end{cases}$

$g \in G_1$

$\varphi: G_1 \to G_2$ isomorfismo de grupos

queremos ver que $o_{G_1}(g) = o_{G_2}(\varphi(g))$

* $o_{G_2}(\varphi(g)) \mid o_{G_1}(g)$?

$\left.\begin{array}{l} \varphi: G_1 \to G_2 \text{ morfismo de grupos} \\ g \in G_1 \text{ de orden finito} \end{array}\right\} \Rightarrow o_{G_2}(\varphi(g)) \mid o_{G_1}(g)$

* $o_{G_1}(g) \mid o_{G_2}(\varphi(g))$?

$\left.\begin{array}{l} \varphi^{-1}: G_2 \to G_1 \text{ morfismo de grupos} \\ \varphi(g) \in G_2 \text{ de orden finito} \end{array}\right\} \Rightarrow o_{G_1}(\underbrace{\varphi^{-1}(\varphi(g))}_{=g}) \mid o_{G_2}(\varphi(g))$

$o_{G_1}(\varphi^{-1}(a)) \mid o_{G_2}(a)$, $a = \varphi(g)$

$o_{G_1}(g) \mid o_{G_2}(\varphi(g))$

$$\left.\begin{array}{l} * \ o_{G_2}(\varphi(g)) \mid o_{G_1}(g) \\ o_{G_1}(g) \mid o_{G_2}(\varphi(g)) \end{array}\right] \Rightarrow o_{G_2}(\varphi(g)) = o_{G_1}(g)$$

$\varphi: G_1 \to G_2$ isomorfismo de grupos $\Rightarrow o_{G_2}(\varphi(g)) = o_{G_1}(g)$

d) Probar que $\mathbb{Z}_4$ y $\mathbb{Z}_2 \times \mathbb{Z}_2$ no son isomorfos

$\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$

$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\bar{0},\bar{0}), (\bar{1},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{1})\}$

$(\bar{1},\bar{0}) + (\bar{1},\bar{0}) = (\bar{2},\bar{0})$
$= (\bar{0},\bar{0})$

$\mathbb{Z}_4$

g	$o(g)$
$\bar{0}$	1
$\bar{1}$	4
$\bar{2}$	2
$\bar{3}$	4

$\mathbb{Z}_2 \times \mathbb{Z}_2$

g	$o(g)$
$(\bar{0},\bar{0})$	1
$(\bar{1},\bar{0})$	2
$(\bar{0},\bar{1})$	2
$(\bar{1},\bar{1})$	2

Si $\varphi: \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ fuera un isomorfismo de grupos

$o(\varphi(\bar{1})) = o(\bar{1}) = 4$

pero en $\mathbb{Z}_2 \times \mathbb{Z}_2$ no existe ningún elemento de orden 4

entonces no existe un isomorfismo $\varphi: \mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$

Ejercicio 3. En cada caso, determinar si existe algún morfismo no trivial $f: G \to K$ (es decir, que no mande todos los elementos al neutro). Cuando exista, construir uno; y si no existe explicar por qué.

a. $G = \mathbb{Z}_7$ con la suma y $K = S_6$ con la composición.

b. $G = \mathbb{Z}_8$, $K = U(24)$. Sugerencia: G es cíclico.

c. $G = U(9)$, $K = \mathbb{Z}_{12}$. Sugerencia: G es cíclico.

d. $G = U(15)$, $K = \mathbb{Z}_6$. Sugerencia: hallar el orden de todos los elementos de G.

a) morfismo de grupos $f: \mathbb{Z}_7 \to S_6$ no trivial?

$|\mathbb{Z}_7| = 7$

$|S_6| = 6!$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ f(1) & f(2) & f(3) & f(4) & f(5) & f(6) \end{pmatrix} \leftarrow \text{permutación de } \{1,2,3,4,5,6\}$$

$|\mathbb{Z}_7|$ y $|S_6|$ son coprimos

$\Rightarrow$ el unico morfismo de grupos $f: \mathbb{Z}_7 \to S_6$ es el morfismo trivial.

b) morfismo de grupos $f: \mathbb{Z}_8 \to U(24)$ no trivial?

$|\mathbb{Z}_8| = 8$

$|U(24)| = \varphi(24) = 24 \cdot \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right) = 24 \cdot \frac{1}{2} \cdot \frac{2}{3} = 8$

$24 = 2^3 \cdot 3$

$\mathbb{Z}_8$ cíclico? $\mathbb{Z}_8 = \langle \bar{1} \rangle$

$\Rightarrow \mathbb{Z}_8$ es cíclico

un morfismo de grupos $f: \mathbb{Z}_8 \to U(24)$ queda determinado por $f(\bar{1})$:

sea $a \in \mathbb{Z}_8$

como $\mathbb{Z}_8 = \langle \bar{1} \rangle$, tenemos $a = \bar{1}^n$

entonces $f(a) = f(\bar{1}^n) = f(\bar{1} + \bar{1} + \dots + \bar{1})$

$= f(\bar{1}) f(\bar{1}) \cdots f(\bar{1})$

$= f(\bar{1})^n$

por ejemplo: $f(\bar{2}) = f(\bar{1} + \bar{1}) = f(\bar{1}) f(\bar{1}) = f(\bar{1})^2$

$f(\bar{1}) = ?$

$o(\bar{1}) = 8 \Rightarrow f(\bar{1})$ es un elemento de $U(24)$ tal que

$o(f(\bar{1})) \mid 8$

$24 = 2^3 \cdot 3$

$U(24)$

g	$o(g)$
$\bar{1}$	1
$\bar{5}$	2
$\bar{7}$	2
$\overline{11}$	2
$\overline{13}$	2
$\overline{17}$	2
$\overline{19}$	2
$\overline{23}$	2

$|U(24)| = 8$

$\bar{5}^2 = \bar{5} \times \bar{5} = \overline{25} = \bar{1}$

$\bar{7}^2 = \bar{7} \times \bar{7} = \bar{1}$

$\overline{11}^2 = \overline{11} \times \overline{11} = \overline{121} = \bar{1}$

$\overline{13}^2 = \overline{169} = \bar{1}$

$\overline{17}^2 = \bar{1}$

$\overline{19}^2 = \bar{1}$

$\overline{23}^2 = \bar{1}$

tenemos 7 morfismos de grupos no triviales $\mathbb{Z}_8 \to U(24)$

vamos a construir uno:

$$\boxed{\begin{array}{l} f: \mathbb{Z}_8 \to U(24) \\ f(\bar{1}) = \bar{5} \end{array}}$$

$f(\bar{2}) = f(\bar{1} + \bar{1}) = f(\bar{1})f(\bar{1}) = \bar{5} \times \bar{5} = \bar{1}$

$f(\bar{3}) = f(\bar{1} + \bar{1} + \bar{1}) = f(\bar{1})f(\bar{1})f(\bar{1}) = \bar{5} \times \bar{5} \times \bar{5} = \bar{5}$

$f(\bar{4}) = f(\bar{1} + \bar{1} + \bar{1} + \bar{1}) = f(\bar{1})^4 = \bar{1}$

$f(\bar{5}) = f(\bar{1}^5) = f(\bar{1})^5 = \bar{5}$

$f(\bar{6}) = f(\bar{1}^6) = f(\bar{1})^6 = \bar{1}$

$f(\bar{7}) = f(\bar{1}^7) = f(\bar{1})^7 = \bar{5}$

$f(\bar{0}) = f(\bar{8}) = f(\bar{1}^8) = f(\bar{1})^8 = \bar{5}^8 = \bar{1}$

c) $f: U(9) \to \mathbb{Z}_{12}$ morfismo de grupos no trivial?

* $|U(9)| = \varphi(9) = 9\left(1-\frac{1}{3}\right) = 9 \cdot \frac{2}{3} = 6$

  $|\mathbb{Z}_{12}| = 12$

  $\Rightarrow |U(9)|$ y $|\mathbb{Z}_{12}|$ no son coprimos ✓

* $U(9)$ es cíclico?

U(9)

g	$o(g)$
$\bar{1}$	1
$\bar{2}$	6 ← generador
$\bar{4}$	
$\bar{5}$	
$\bar{7}$	
$\bar{8}$	

$\bar{2}^2 = \bar{2} \times \bar{2} = \bar{4}$

$\bar{2}^3 = \bar{4} \times \bar{2} = \bar{8}$

$\bar{2}^4 = \bar{8} \times \bar{2} = \overline{16} = \bar{7}$

$\bar{2}^5 = \bar{7} \times \bar{2} = \overline{14} = \bar{5}$

$\bar{2}^6 = \bar{5} \times \bar{2} = \overline{10} = \bar{1}$

tenemos $o(\bar{2}) = 6 \Rightarrow U(9) = \langle \bar{2} \rangle$

entonces un morfismo $f: U(9) \to \mathbb{Z}_{12}$ queda determinado por $f(\bar{2})$

si $a \in U(9) \Rightarrow a = \bar{2}^n$ para algún n

$$\Rightarrow f(a) = f(\bar{2}^n) = f(\bar{2} \times \bar{2} \times \dots \times \bar{2})$$
$$= f(\bar{2}) + f(\bar{2}) + \dots f(\bar{2})$$
$$= n f(\bar{2})$$

* $f(\bar{2}) = ?$

$o(\bar{2}) = 6 \Rightarrow f(\bar{2})$ es un elemento de $\mathbb{Z}_{12}$ tal que

$$o(f(\bar{2})) \mid o(\bar{2})$$

$$\boxed{o(f(\bar{2})) \mid 6}$$

Prop: $f: G_1 \to G_2$ morfismo de grupos $\Rightarrow o(f(a)) \mid o(a)$ para todo $a \in G$

$f(\bar{2})$ puede ser igual a $g \in \mathbb{Z}_{12}$ sii $o_{\mathbb{Z}_{12}}(g) \mid o_{U(9)}(\bar{2})$

sii $o_{\mathbb{Z}_{12}}(g) \mid 6$

$\mathbb{Z}_{12}$		
g	$o(g)$	
$\bar{0}$	1	
$\bar{1}$	12	X porque $12 \nmid 6$
$\bar{2}$	6	✓
$\bar{3}$	4	X porque $4 \nmid 6$
$\bar{4}$	3	
$\bar{5}$		
$\bar{6}$		
$\bar{7}$		
$\bar{8}$		
$\bar{9}$		
$\bar{10}$		
$\bar{11}$		

$\bar{2}^2 = \bar{2} + \bar{2} = \bar{4}$

$\bar{2}^3 = \bar{2} + \bar{2} + \bar{2} = \bar{6}$

$\bar{2}^4 = \bar{4} + \bar{4} = \bar{8}$

$\bar{2}^5 = \bar{10}$

$\bar{2}^6 = \bar{12}$

$f: U(9) \to \mathbb{Z}_{12}$

$f(\bar{2}) = \bar{1}$??

$o_{\mathbb{Z}_{12}}(f(\bar{2})) \mid o_{U(9)}(\bar{2})$

$o_{\mathbb{Z}_{12}}(\bar{1}) \mid o_{U(9)}(\bar{2})$

$12 \mid 6$ imposible

$\Rightarrow f(\bar{2})$ no puede ser $\bar{1}$

$f(\bar{2}) = \bar{2}$??

$o_{\mathbb{Z}_{12}}(f(\bar{2})) \mid o_{U(9)}(\bar{2})$

$o_{\mathbb{Z}_{12}}(\bar{2}) \mid o_{U(9)}(\bar{2})$

$6 \mid 6$ ✓

$\Rightarrow f(\bar{2})$ puede ser $\bar{2}$

$f(\bar{2}) = \bar{3}$??

$o_{\mathbb{Z}_{12}}(\bar{3}) \mid o_{U(9)}(\bar{2})$

$4 \mid 6$ X

$\Rightarrow f(\bar{2})$ no puede ser $\bar{3}$

$f(\bar{2}) = \bar{4}$??

$o_{\mathbb{Z}_{12}}(\bar{4}) \mid o_{U(9)}(\bar{2})$

$3 \mid 6$ ✓

$\Rightarrow f(\bar{2})$ puede ser $\bar{4}$

existen morfismos de grupos no triviales $U(9) \to \mathbb{Z}_{12}$

por ejemplo:

$$\boxed{\begin{array}{c} f: U(9) \longrightarrow \mathbb{Z}_{12} \\ f(\bar{2}) = \bar{4} \end{array}} \longrightarrow f(\bar{2}^n) = n f(\bar{2})$$

$$U(9) = \{ \underset{}{\bar{2}}, \underset{\substack{\parallel \\ \bar{4}}}{\bar{2}^2}, \underset{\substack{\parallel \\ \bar{8}}}{\bar{2}^3}, \underset{\substack{\parallel \\ \bar{7}}}{\bar{2}^4}, \underset{\substack{\parallel \\ \bar{5}}}{\bar{2}^5}, \underset{\substack{\parallel \\ \bar{1}}}{\bar{2}^6} \}$$

$$f(\bar{4}) = f(\bar{2} \times \bar{2}) = f(\bar{2}) + f(\bar{2}) = \bar{4} + \bar{4} = \bar{8}$$

$$f(\bar{7}) = f(\bar{2}^4) = f(\bar{2}) + f(\bar{2}) + f(\bar{2}) + f(\bar{2}) = \bar{4} + \bar{4} + \bar{4} + \bar{4} = \bar{4}$$