Clase 40:

Coordenadas esférica

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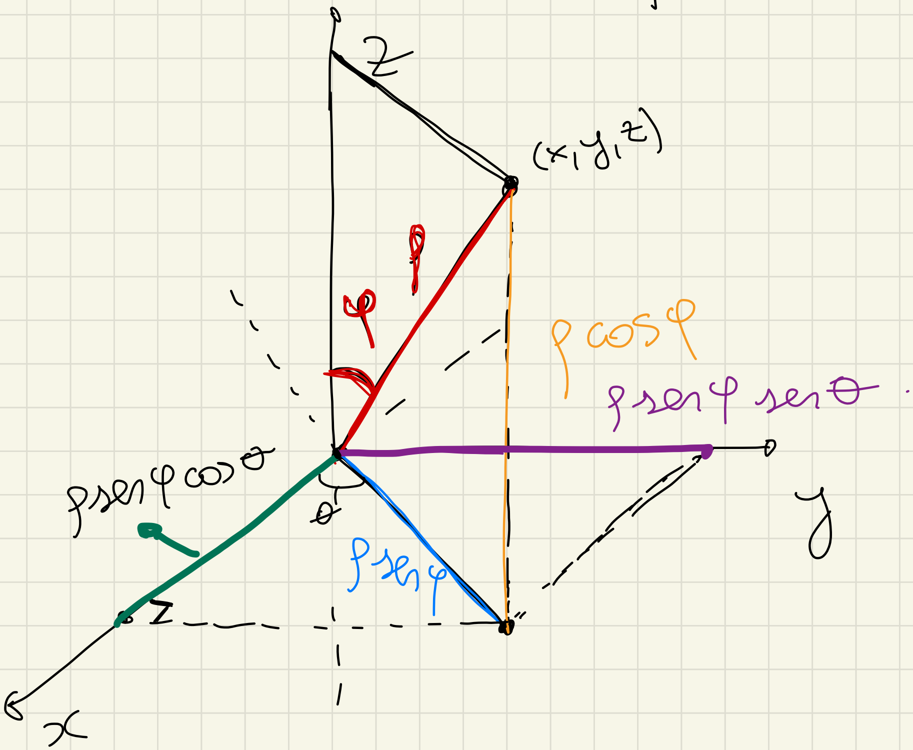
Teorema de cambio de variable

Sea $f : \underset{\underset{\mathbb{R}^3}{\cap}}{D} \longrightarrow \mathbb{R}$ continua y

$g : U \longrightarrow V$ un cambio de variable con $D \subseteq V$

$$\boxed{\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{g^{-1}(D)} f(g(u,v,w))\,|\det J_g(u,v,w)|\,du\,dv\,dw}$$

Coordenadas esféricas



$$x = \rho\,\text{sen}\,\varphi\cos\theta \qquad\qquad \rho > 0$$

$$y = \rho\,\text{sen}\,\varphi\,\text{sen}\,\theta \qquad\qquad \theta \in [0, 2\pi)$$

$$z = \rho\cos\varphi \qquad\qquad \varphi \in [0, \pi)$$

$$J_g(\rho,\theta,\varphi) = \begin{pmatrix} \operatorname{sen}\varphi\cos\theta & -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\cos\theta\cos\varphi \\ \operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ \cos\varphi & 0 & -\rho\operatorname{sen}\varphi \end{pmatrix}$$

$$|J_g(\rho,\theta,\varphi)| = |\det(J_g(\rho,\theta,\varphi))|$$

$$\det\begin{pmatrix} \operatorname{sen}\varphi\cos\theta & -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\cos\theta\cos\varphi \\ \operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \\ \cos\varphi & 0 & -\rho\operatorname{sen}\varphi \end{pmatrix}$$
$$\begin{matrix} \operatorname{sen}\varphi\cos\theta & -\rho\operatorname{sen}\varphi\operatorname{sen}\theta & \rho\cos\theta\cos\varphi \\ \operatorname{sen}\varphi\operatorname{sen}\theta & \rho\operatorname{sen}\varphi\cos\theta & \rho\cos\varphi\operatorname{sen}\theta \end{matrix}$$

$$= \Big| -\rho^2\operatorname{sen}^3\varphi\cos^2\theta - \rho^2\cos^2\varphi\operatorname{sen}\varphi\operatorname{sen}^2\theta$$
$$- \rho^2\cos^2\varphi\cos^2\theta\operatorname{sen}\varphi - \rho^2\operatorname{sen}^3\varphi\operatorname{sen}^2\theta \Big|$$

$$= \left| -\rho^2 \left(\operatorname{sen}^3\varphi \cos^2\theta + \cos^2\varphi \operatorname{sen}\varphi \operatorname{sen}^2\theta + \cos^2\varphi \cos^2\theta \operatorname{sen}\varphi + \operatorname{sen}^3\varphi \operatorname{sen}^2\theta \right) \right|$$

$$= \left| -\rho^2 \left(\operatorname{sen}^3\varphi \overbrace{\left(\cos^2\theta + \operatorname{sen}^2\theta\right)}^{=1} + \operatorname{sen}\varphi \cdot \cos^2\varphi \underbrace{\left(\operatorname{sen}^2\theta + \cos^2\theta\right)}_{=1} \right) \right|$$

$$= \left| -\rho^2 \left(\operatorname{sen}^3\varphi + \operatorname{sen}\varphi \cos^2\varphi \right) \right|$$

$$= \left| -\rho^2 \operatorname{sen}\varphi \left(\underbrace{\operatorname{sen}^2\varphi + \cos^2\varphi}_{=1} \right) \right|$$

$$= \left| -\rho^2 \operatorname{sen}\varphi \right| = \rho^2 \operatorname{sen}\varphi.$$

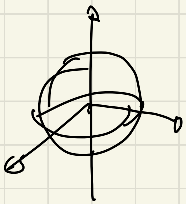
$\operatorname{sen}\varphi \geq 0 \quad \varphi \in [0, \pi]$

$\rho^2 > 0$

$$|J_g(\rho,\theta,\varphi)| = \rho^2 \operatorname{sen}\varphi.$$

Ejemplo:

$$f(x,y,z) = z$$



$$D = \{(x,y,z): \quad x^2+y^2+z^2 \le 1\}.$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz$$

$$\iiint_D z \; dx\,dy\,dz \underset{\text{CV}}{=} \iiint_{g^{-1}(D)} \rho\cos\varphi \cdot \rho^2 \operatorname{sen}\varphi \, d\rho\, d\theta\, d\varphi$$

CV coordenadas esféricas

$$g^{-1}(D) = \{(\rho,\varphi,\theta) \in \mathbb{R}^3 : \rho \in (0,1],\ \varphi \in [0,\pi),\ \theta \in [0,2\pi)\}$$

$x = \rho \operatorname{sen}\varphi \cos\theta$ $\qquad \rho > 0$

$y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta$. $\qquad \theta \in [0, 2\pi)$

$z = \rho \cos\varphi$. $\qquad \varphi \in [0, \pi)$

$$g(\rho, \varphi, \theta) = (x, y, z)$$

$$x^2 + y^2 + z^2 = \rho^2 \leq 1$$

$$g^{-1}(D) = \{ (\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \in (0, 1], \varphi \in [0, \pi), \theta \in [0, 2\pi) \}$$

$$\iiint_{g^{-1}(D)} \rho \cos\varphi \cdot \rho^2 \operatorname{sen}\varphi \, d\rho \, d\theta \, d\varphi =$$

Teorema de Fubbini

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^3 \cos\varphi \operatorname{sen}\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi} \int_0^{2\pi} \frac{\rho^4}{4} \cos\varphi \,\text{sen}\,\varphi \Big|_1^1 \, d\theta \, d\varphi$$

$$= \int_0^{\pi} \int_0^{2\pi} \frac{1}{4} \cos\varphi \,\text{sen}\,\varphi \, d\theta \, d\varphi$$

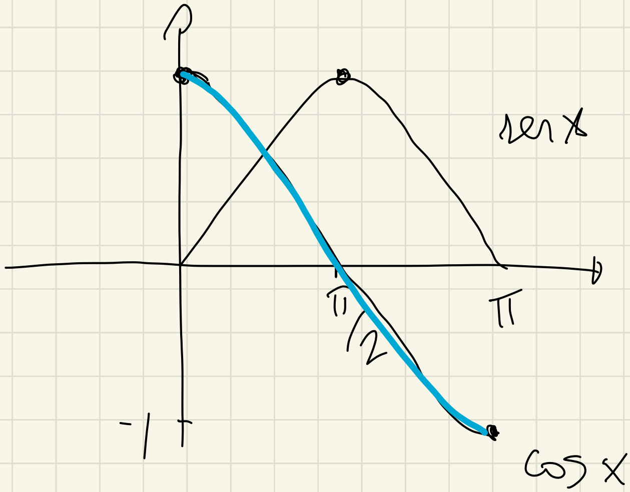
$$= \frac{1}{4} \int_0^{\pi} . \, 2\pi \cos\varphi \,\text{sen}\,\varphi \, d\varphi$$

$$= \frac{\pi}{2} \int_0^{\pi} \cos\varphi \, \underbrace{\text{sen}\,\varphi \, d\varphi}_{=u} . = \frac{\pi}{2} \int_1^{-1} u.(-du) .$$

$u = \cos\varphi$

$du = -\,\text{sen}\,\varphi \, d\varphi .$

$$= \frac{\pi}{2} \int_{-1}^{1} u \, du$$



$$= \frac{\pi}{2} \left. \frac{s^2}{2} \right|_{-1}^{1}$$

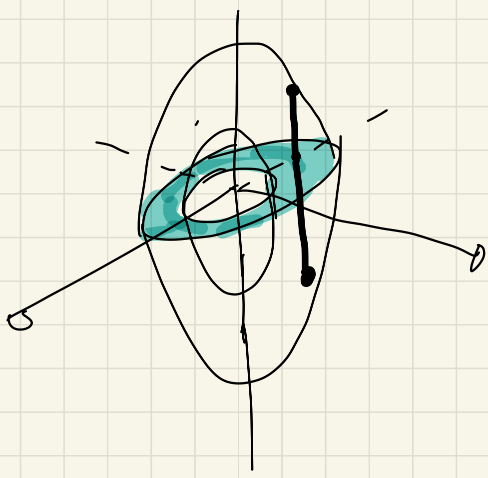
$$= \frac{\pi}{2} \left(\frac{1}{2} - \frac{1}{2} \right)$$

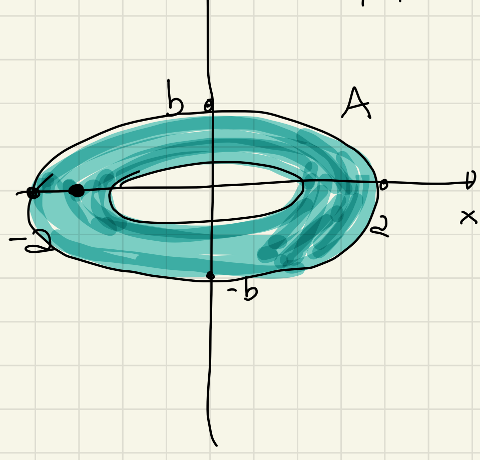
$$= 0$$

Ejemplo: Calcular el volumen del conjunto

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{1}{4} \leq \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \right\}$$

$a, b, c > 0$





$$\iiint_D 1 \underset{CV}{=} \iiint_{g^{-1}(D)} \overbrace{f(g(u,v,w))}^{1} \cdot \underbrace{|J_g(u,v,w)|}_{|abc|}$$

$$u = \frac{x}{a} \qquad v = \frac{y}{b} \qquad w = \frac{z}{c}$$

$$g(u,v,w) = (\underset{x}{\underset{\shortparallel}{au}}, \underset{y}{\underset{\shortparallel}{bv}}, \underset{z}{\underset{\shortparallel}{cw}})$$

$$g^{-1}(D) = \left\{ (u,v,w) : \frac{1}{4} \leq u^2 + v^2 + w^2 \leq 1 \right\}$$

$$J_g(u,v,w) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$|J_g(u,v,w)| = |a \cdot b \cdot c|$$

$$\shortparallel$$

$$|\det J_g(u,v,w)|$$

$$= \iiint_{g^{-1}(D) = \tilde{D}} abc \, du\,dv\,dw = \iiint_{g^{-1}(\tilde{D})} abc \, |J_g(\rho,\varphi,\theta)| =$$

CV esféricas

$$u = \rho \operatorname{sen}\varphi \cos\theta \qquad \rho > 0$$
$$v = \rho \operatorname{sen}\varphi \operatorname{sen}\theta . \qquad \theta \in [0, 2\pi)$$
$$w = \rho \cos\varphi . \qquad \varphi \in [0, \pi)$$

$$g(\rho,\varphi,\theta) = (\rho \operatorname{sen}\varphi \cos\theta, \rho \operatorname{sen}\varphi \operatorname{sen}\theta, \rho\cos\varphi) .$$

$$= \iiint_{g^{-1}(\tilde{D})} abc \cdot \rho^2 \operatorname{sen}\varphi \, d\rho \, d\theta \, d\varphi$$

$$g^{-1}(\tilde{D}) = \{ (\rho,\varphi,\theta) : \tfrac{1}{4} \le \rho^2 \operatorname{sen}^2\varphi \cos^2\theta + \rho^2 \operatorname{sen}^2\varphi \operatorname{sen}^2\theta$$

$$+ \ \rho^2 \cos^2\varphi \leq 1 \ \}$$

$$= \{(\rho, \varphi, \theta) : \ \frac{1}{4} \leq \rho^2 \leq 1, \ \varphi \in [0, \pi), \ \theta \in [0, 2\pi) \}$$

$$= \{(\rho, \varphi, \theta) : \ \frac{1}{2} \leq \rho \leq 1, \ \varphi \in [0, \pi), \ \theta \in [0, 2\pi) \}$$

$$\iiint_{g^{-1}(\tilde{D})} abc \cdot \rho^2 \operatorname{sen}\varphi \, d\rho \, d\theta \, d\varphi \ \underset{\text{Teorema de Fubbini}}{=}$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_{1/2}^{1} abc \, \rho^2 \operatorname{sen}\varphi \, d\rho \, d\varphi \, d\theta = V$$

$$V = abc \cdot \frac{4}{3} \cdot \frac{7}{8} \pi$$

$$\tilde{D} = \left\{ (x,y,z) \in \mathbb{R}^3 : \frac{1}{4} \leq \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \right\}.$$

$$\left(\frac{z}{c}\right)^2 = r^2 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$$

$$z^2 = c^2 r^2 - \left(\frac{cx}{a}\right)^2 - \left(\frac{cy}{b}\right)^2$$

$$V(\tilde{D}) = V(D_1) - V(D_{1/2})$$

$$D_r = \left\{ (x,y,z) \in \mathbb{R}^3 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq r^2 \right\}.$$

$$\iiint_{D_r} 1 \, dx\,dy\,dz = \iint_R 2 \sqrt{c^2 r^2 - \left(\frac{cx}{a}\right)^2 - \left(\frac{cy}{b}\right)^2} \, dx\,dy$$