Clase 39:

Integrales triples

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Eugenia Ellis

eellis@fing.edu.uy

Teorema de cambio de variable

Sea $f: \underset{\mathbb{R}^3}{\underset{\cap}{D}} \longrightarrow \mathbb{R}$ continua y

$g: U \longrightarrow V$ un cambio de variable

con $D \subseteq V$

$$\boxed{\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{g^{-1}(D)} f(g(u,v,w))\,|\det J_g(u,v,w)|\,du\,dv\,dw}$$

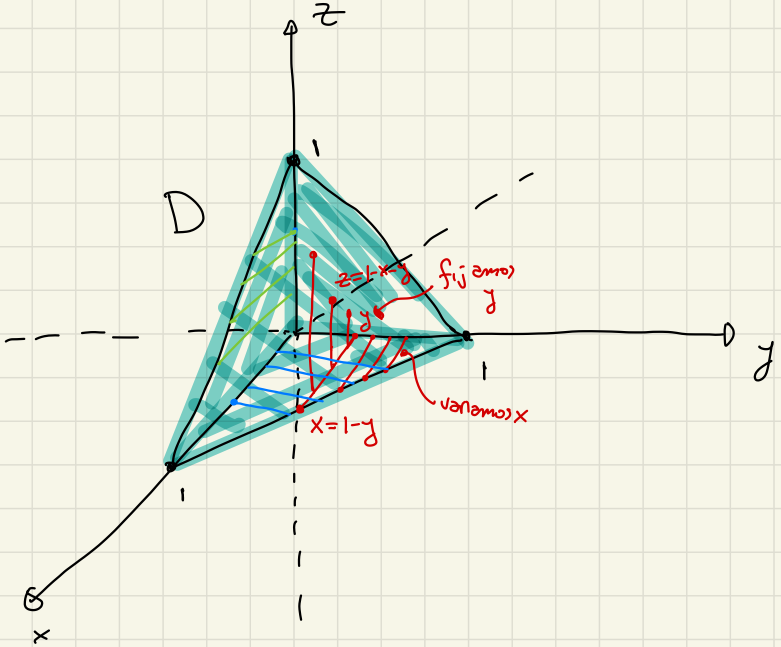
Ejemplo:

$D = \{(x,y,z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$

$f(x,y,z) = xyz$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz$$

TF CV



$$D = \{(x,y,z)\in\mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$$

$$D = \{(x,y,z)\in\mathbb{R}^3 : 0 \leq y \leq 1,\ 0 \leq x \leq 1-y,\ 0 \leq z \leq 1-x-y\}$$

$$\iiint_D f = \int_0^1 \int_0^{1-y} \int_0^{1-x-y} f(x,y,z)\, dz\, dx\, dy$$

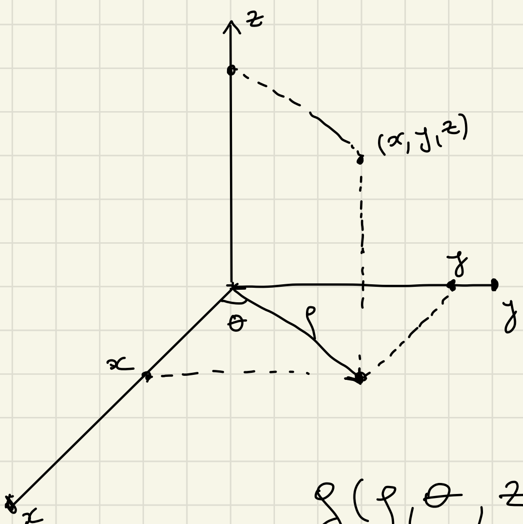
$$= \int_0^1 \int_0^{1-y} \int_0^{1-x-y} xyz \, dz\, dx\, dy.$$

$$= \ldots\ldots$$

ejercicio

Cambios de variables en $\mathbb{R}^3$

Coordenadas cilíndricas



$$x = \rho \cos\theta$$

$$y = \rho \operatorname{sen}\theta$$

$$z = z$$

$$g(\rho, \theta, z) = (\rho\cos\theta, \rho \operatorname{sen}\theta, z)$$

$$J_g(\rho,\theta,z) = \left(\begin{array}{cc|c} \cos\theta & -\rho\,\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & \rho\cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$|J_g(\rho,\theta,z)| = \begin{vmatrix} \cos\theta & -\rho\,\text{sen}\,\theta \\ \text{sen}\,\theta & \rho\cos\theta \end{vmatrix}$$

$$= |\rho\cos^2\theta - (-\rho\,\text{sen}^2\theta)|$$

$$= |\rho\,(\underbrace{\cos^2\theta + \text{sen}^2\theta}_{=1})|$$

$$= \rho$$

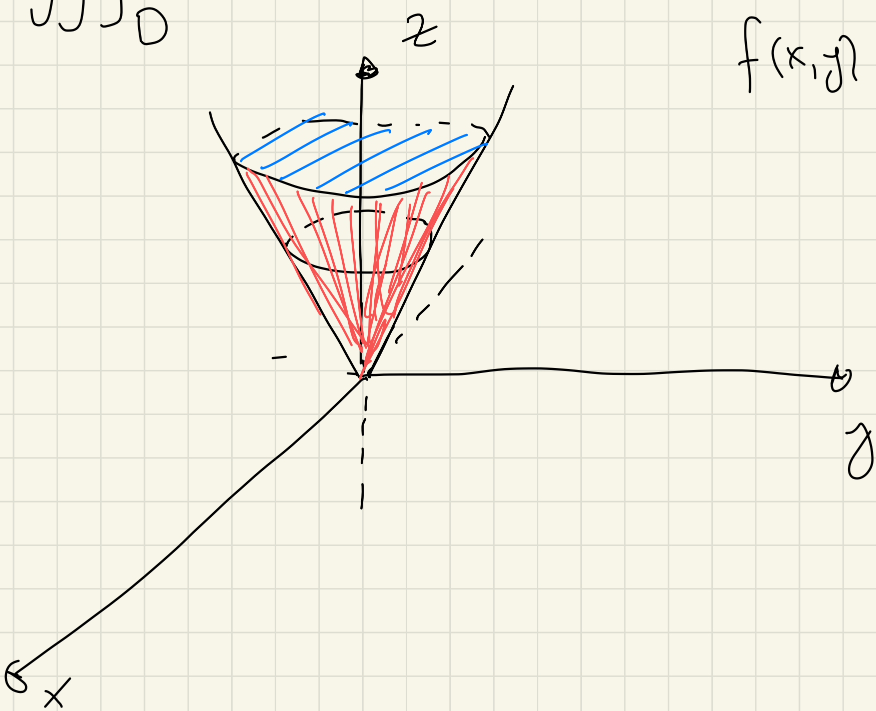
Ejemplo:

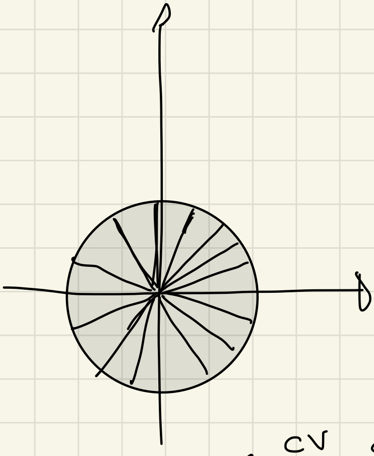
$$D = \{ (x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \leq z \leq 1 \}.$$

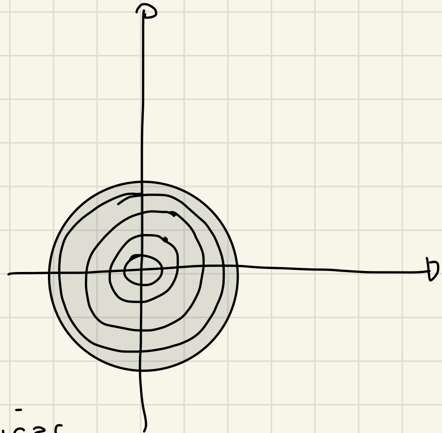
$$f(x,y,z) = 1$$

$$\iiint_D 1 \, dx\,dy\,dz = \text{volumen de } D.$$

$$f(x,y) = \sqrt{x^2+y^2}$$







cv cilindricas

$$D = g\left(g^{-1}(D)\right)$$

$$g^{-1}(D) = \left\{ (\rho, \theta, z) : 0 \le \rho \le 1 \,,\ \theta \in [0, 2\pi) \,,\ \rho \le z \le 1 \right\}.$$

$$\sqrt{x^2+y^2} \le z \le 1$$

$$\underbrace{\sqrt{\rho^2\cos^2\theta + \rho^2 \operatorname{sen}^2\theta}}_{=} \le z \le 1$$

$$\sqrt{\rho^2} = \rho \le z \le 1$$

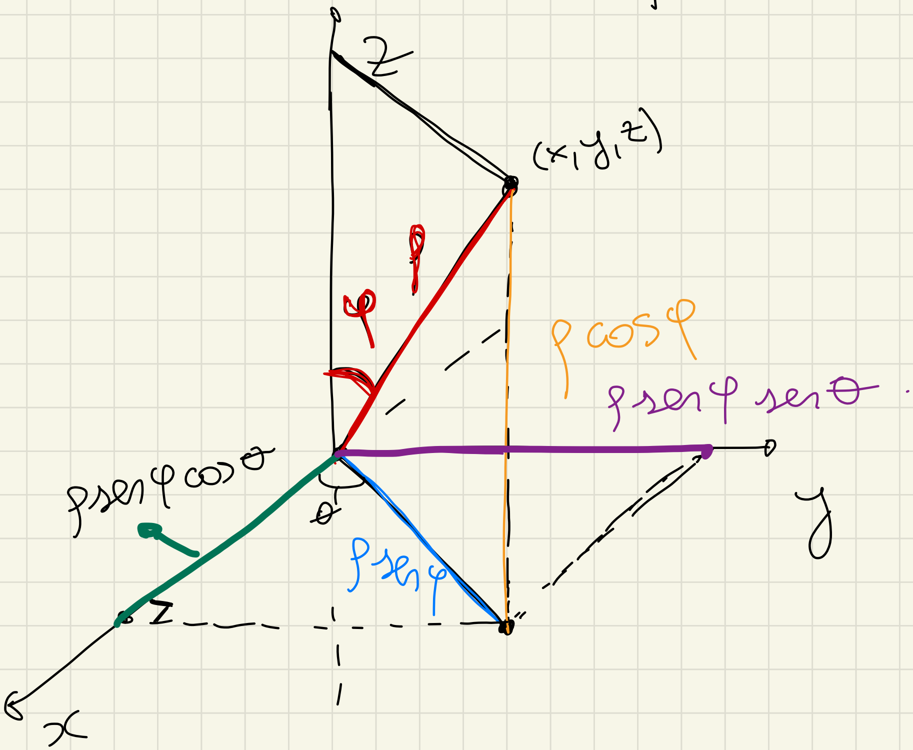
$$\iiint_D 1\,dx\,dy\,dz \underset{\substack{\uparrow \\ \text{CV} \\ \text{cilindrica}}}{=} \iiint_{g^{-1}(D)} 1\cdot\rho\, d\theta\, d\rho\, dz$$

$$\underset{\substack{\uparrow \\ \text{TF}}}{=} \int_0^1 \int_0^{2\pi} \int_\rho^1 \rho\, dz\, d\theta\, d\rho$$

$$\underset{\substack{\uparrow \\ \text{wentc}}}{=} \int_0^1 \int_0^{2\pi} \left(\rho z \Big|_\rho^1 \right) d\theta\, d\rho$$

$$= \int_0^1 \int_0^{2\pi} (\rho - \rho^2)\, d\theta\, d\rho = \cdots$$

Coordenadas esféricas



$$x = \rho \operatorname{sen}\varphi \cos\theta$$

$$y = \rho \operatorname{sen}\varphi \operatorname{sen}\theta$$

$$z = \rho \cos\varphi$$

$$\rho > 0$$

$$\theta \in [0, 2\pi)$$

$$\varphi \in [0, \pi)$$

$$J_g(\rho,\theta,\varphi) = \begin{pmatrix} \operatorname{sen}\varphi \cos\theta & -\rho \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \cos\theta \cos\varphi \\ \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \operatorname{sen}\varphi \cos\theta & \rho \cos\varphi \operatorname{sen}\theta \\ \cos\varphi & 0 & -\rho \operatorname{sen}\varphi \end{pmatrix}$$

$$|J_g(\rho,\theta,\varphi)| = |\det(J_g(\rho,\theta,\varphi))|$$

$$\det \begin{pmatrix} \operatorname{sen}\varphi \cos\theta & -\rho \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \cos\theta \cos\varphi \\ \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \operatorname{sen}\varphi \cos\theta & \rho \cos\varphi \operatorname{sen}\theta \\ \cos\varphi & 0 & -\rho \operatorname{sen}\varphi \end{pmatrix}$$
$$\begin{matrix} \operatorname{sen}\varphi \cos\theta & -\rho \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \cos\theta \cos\varphi \\ \operatorname{sen}\varphi \operatorname{sen}\theta & \rho \operatorname{sen}\varphi \cos\theta & \rho \cos\varphi \operatorname{sen}\theta \end{matrix}$$

$$= \left| -\rho^2 \operatorname{sen}^3\varphi \cdot \cos^2\theta - \rho^2 \cos^2\varphi \operatorname{sen}\varphi \operatorname{sen}^2\theta - \rho^2 \cos^2\varphi \cos^2\theta \operatorname{sen}\varphi - \rho^2 \operatorname{sen}^3\varphi \operatorname{sen}^2\theta \right|$$

$$= \left| -\rho^2 \left(\operatorname{sen}^3\varphi \cos^2\theta + \cos^2\varphi \operatorname{sen}\varphi \operatorname{sen}^2\theta + \cos^2\varphi \cos^2\theta \operatorname{sen}\varphi + \operatorname{sen}^3\varphi \operatorname{sen}^2\theta \right) \right|$$

$$= \left| -\rho^2 \left(\operatorname{sen}^3\varphi \overbrace{\left(\cos^2\theta + \operatorname{sen}^2\theta \right)}^{=1} + \operatorname{sen}\varphi \cdot \cos^2\varphi \underbrace{\left(\operatorname{sen}^2\theta + \cos^2\theta \right)}_{=1} \right) \right|$$

$$= \left| -\rho^2 \left(\operatorname{sen}^3\varphi + \operatorname{sen}\varphi \cos^2\varphi \right) \right|$$

$$= \left| -\rho^2 \operatorname{sen}\varphi \left(\underbrace{\operatorname{sen}^2\varphi + \cos^2\varphi}_{=1} \right) \right|$$

$$= \left| -\rho^2 \operatorname{sen}\varphi \right| = \rho^2 \operatorname{sen}\varphi.$$

$\operatorname{sen}\varphi \geq 0 \quad \varphi \in [0, \pi]$

$\rho^2 > 0$

$$|J_g(\rho,\theta,\varphi)| = \rho^2 \operatorname{sen}\varphi.$$

Ejemplo:

$$f(x,y,z) = z$$

$$D = \{(x,y,z) : x^2+y^2+z^2 \leq 1\}.$$

$$¿\iiint_D f(x,y,z)\,dx\,dy\,dz?$$