Clase 34: Regla de la Cadena III

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Eugenia Ellis

eellis@fing.edu.uy

$f: \mathbb{R}^n \to \mathbb{R}^m$ es diferenciable en a

$\Longleftrightarrow$

Existe una transformación lineal $df_a : \mathbb{R}^n \to \mathbb{R}^m$ tal que

$$f(a+h) = f(a) + \boxed{df_a(h)} + r(h)$$

en donde $\lim\limits_{h \to 0} \dfrac{r(h)}{\|h\|} = 0 \in \mathbb{R}^m$

$0 \in \mathbb{R}^n$

$df_a : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $\qquad df_a(h) = Jf_a \cdot h$

transformación lineal $\qquad$ matriz

$f = (f_1, \dots, f_m) \qquad f_i : \mathbb{R}^n \to \mathbb{R}$

$$Jf_a = \begin{pmatrix} \nabla f_1(a) \\ \vdots \\ \nabla f_m(a) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_m}(a) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_m}(a) \end{pmatrix}$$

Teorema (Regla de la cadena II)

$$\left.\begin{array}{l} g: \mathbb{R}^k \longrightarrow \mathbb{R}^n \text{ diferenciable en } a \in \mathbb{R}^k \\ f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ diferenciable en } g(a) \in \mathbb{R}^n \end{array}\right\}$$

$\Rightarrow (f \circ g): \mathbb{R}^k \longrightarrow \mathbb{R}^m$ es diferenciable en $a \in \mathbb{R}^k$ y

$$J_{f\circ g}(a) = J_f(g(a)) \cdot J_g(a)$$

$$d(f\circ g)(a) = df_{g(a)} \circ dg_a$$

Dem: $f \circ g$ es diferenciable en $a \overset{\text{def}}{\iff}$

Existe una transformación lineal $d_{f\circ g}: \mathbb{R}^k \longrightarrow \mathbb{R}^m$ tal que

$$(f\circ g)(a+h) = (f\circ g)(a) + d_{f\circ g}(h) + R_{f\circ g}(h)$$

en donde $\dfrac{R_{f\circ g}(h)}{\|h\|} \xrightarrow[h\to 0]{} 0$

Queremos probar lo descrito anteriormente.

Sabemos que

$$g(a+h) = g(a) + dg_a(h) + r_g(h)$$

g es diferenciable en a

con $\frac{r_g(h)}{\|h\|} \xrightarrow[h\to 0]{} 0$

$\Rightarrow$

$$f(g(a+h)) = f\Big(g(a) + \underbrace{dg_a(h) + r_g(h)}_{v \in \mathbb{R}^n}\Big)$$

$$= f(g(a) + v)$$

$$= f(g(a)) + df_{g(a)}(v) + r_f(v)$$

f es diferenciable en g(a)

con

$$\frac{r_f(v)}{\|v\|} \xrightarrow[v\to 0]{} 0$$

$\Rightarrow$

$$f(g(a+h)) = f(g(a)) + df_{g(a)}\big(dg_a(h) + r_g(h)\big) + r_f\big(dg_a(h) + r_g(h)\big)$$

✓ $df_{g(a)}$ es una trasformación lineal

$\Rightarrow$

$$(f\circ g)(a+h) = (f\circ g)(a) + \overbrace{df_{g(a)}\left(dg_a(h)\right)}^{\left(df_{g(a)}\circ dg_a\right)(h)} +$$

$$\underbrace{+\, df_{g(a)}\left(r_g(h)\right) + r_f\left(dg_a(h) + r_g(h)\right)}_{R_{f\circ g}(h)}$$

Si probamos que $\lim\limits_{h\to 0} \dfrac{R_{f\circ g}(h)}{\|h\|} = 0$

tenemos que $f\circ g$ es diferenciable en a

y $\quad d(f\circ g)_a = df_{g(a)} \circ dg_a$

<u>Ejercicio:</u> Probar que $\lim\limits_{h\to 0} \dfrac{R_{f\circ g}(h)}{\|h\|} = 0$

$\square$

Ejemplo: $f:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$f(x,y,z) = (x^2y + e^z,\ \operatorname{sen} x + yz)$$

$$g:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$g(u,v) = (uv,\ e^u,\ \cos(v))$$

$h = f \circ g$

$$\boxed{¿\, J_h(1,\pi)\, ?}$$

$$h(u,v) = (f\circ g)(u,v) = f(g(u,v)) = f(\overset{x}{\overline{uv}},\ \overset{y}{\overline{e^u}},\ \overset{z}{\overline{\cos v}})$$

$$= (\underbrace{u^2v^2e^u + e^{\cos v}}_{h_1(u,v)},\ \underbrace{\operatorname{sen}(uv) + e^u\cos v}_{h_2(u,v)})$$

$$J_h(u,v) = \begin{pmatrix} \dfrac{\partial h_1}{\partial u}(u,v) & \dfrac{\partial h_1}{\partial v}(u,v) \\[2ex] \dfrac{\partial h_2}{\partial u}(u,v) & \dfrac{\partial h_2}{\partial v}(u,v) \end{pmatrix}$$

$$\frac{\partial h_1}{\partial u}(u,v) = v^2(2u e^u + u^2 e^u) = v^2 e^u(2u + u^2)$$

$$\frac{\partial h_1}{\partial v}(u,v) = 2v u^2 e^u - e^{\cos v} \cdot \operatorname{sen} v$$

$$\frac{\partial h_2}{\partial u}(u,v) = v \cdot \cos(uv) + e^{u} \cos v$$

$$\frac{\partial h_2}{\partial v}(u,v) = u \cos(uv) - e^{u} \operatorname{sen} v$$

$$J_h(u,v) = \begin{pmatrix} v^2 e^{u}(2u+u^2) & 2vu^2e^{u} - e^{\cos v} \operatorname{sen} v \\ v \cdot \cos uv + e^{u}\cos v & u\cos(uv) - e^{u}\operatorname{sen} v \end{pmatrix}$$

$$J_h(1,\pi) = \begin{pmatrix} \pi^2 \cdot e \cdot 3 & 2\pi e - e^{-1}\overbrace{\operatorname{sen}\pi}^{=0} \\ \pi \cos\pi + \underline{e\cos\pi} & \underbrace{\cos\pi}_{=-1} - \overbrace{e \operatorname{sen}\pi}^{=0} \end{pmatrix}$$

$$\boxed{J_h(1,\pi) = \begin{pmatrix} \pi^2 e 3 & 2\pi e \\ -\pi - e & -1 \end{pmatrix}}$$

¿Qué pasa si usamos la regla de la cadena?

$$J_h(1,\pi) = J_{(f\circ g)}(1,\pi) \underset{\substack{\uparrow \\ \text{regla de} \\ \text{la cadena}}}{=} J_f(g(1,\pi)) \cdot J_g(1,\pi)$$

$g(u,v) = (uv, e^u, \cos(v))$

$f(x,y,z) = (x^2y + e^z, \operatorname{sen} x + yz)$

$$g(1,\pi) = (\pi, e, -1)$$

$$J_f(x,y,z) = \begin{pmatrix} 2xy & x^2 & e^z \\ \cos x & z & y \end{pmatrix}$$

$$J_f(\pi,e,-1) = \begin{pmatrix} 2\pi e & \pi^2 & e^{-1} \\ -1 & -1 & e \end{pmatrix}$$

$$J_g(u,v) = \begin{pmatrix} v & u \\ e^u & 0 \\ 0 & -\operatorname{sen} v \end{pmatrix} \qquad J_g(1,\pi) = \begin{pmatrix} \pi & 1 \\ e & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \pi & 1 \\ e & 0 \\ 0 & 0 \end{pmatrix}$$

$$J_h(1,\pi) = \begin{pmatrix} 2\pi e & \pi^2 & e^{-1} \\ -1 & -1 & e \end{pmatrix} \begin{pmatrix} 2\pi^2 e + \pi^2 e & 2\pi e \\ -\pi - e & -1 \end{pmatrix}$$

$$J_h(1,\pi) = \begin{pmatrix} 3\pi^2 e & 2\pi e \\ -\pi - e & -1 \end{pmatrix}$$

Desarrollo de Taylor

$f: \mathbb{R} \longrightarrow \mathbb{R}$ de clase C^{k+1}

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2} h^2 + \frac{f^{(3)}}{3!}(x_0) h^3$$

$$+ \dots + \frac{f^{(k)}(x_0)}{k!} h^k + r(h)$$

con $\lim\limits_{h \to 0} \dfrac{r(h)}{h^k} = 0$

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ de clase C^{k+1} en $a \in \mathbb{R}^n$

$$f(a+h) = f(a) + df_a(h) + \frac{1}{2} d^2 f_a(h)$$

$$+ \frac{1}{3!} d^3 f_a(h) + \dots$$

$h = (h_1, \dots, h_n)$

$$+ \dots \frac{1}{k!} \boxed{d^k f_a(h)} + r_k(h)$$

$$\lim_{h \to (0,\dots,0)} \frac{r_k(h)}{\|h\|^k} = 0$$

$$\boxed{d^k f_a(h) = \sum_{i_1, i_2 \dots i_p = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_p}}(a)\, h_{i_1} \cdots h_{i_p}}$$

$\boxed{k=2}$

$$d^2 f_a(h) \overset{?}{=}$$

$\boxed{n=2}$

$$d^k f_a(h,k) \overset{?}{=}$$