Clase 3:

Raiz compleja.

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Un orden total en un conjunto A es un orden parcial $\leq$ tal que para todo $a, b \in A$

$$a \leq b \quad ó \quad b \leq a$$

- $a \leq a \quad \forall a \in A$ (reflexiva)
- $a \leq b$ y $b \leq a \Rightarrow a = b$ (antisimétrica)
- $\left.\begin{array}{l} a \leq b \\ b \leq c \end{array}\right\} \Rightarrow a \leq c$ (transitiva).

$$P(x) = ax^2 + bx + c \qquad a, b, c \in \mathbb{C}$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow P(z) = 0$$

hay que probar.

¿que es esto?

Raíces complejas

Sea $w \in \mathbb{C}$ queremos hallar $\sqrt{w}$

¿que significa esto? queremos encontrar

un $z \in \mathbb{C}$ tal que $z^2 = w$

Si vemos a -1 como número complejo

¿existe $z \in \mathbb{C}$ tal que $z^2 = -1$?

$z = a + bi$ $\qquad z^2 = a^2 + b^2 i^2 + 2abi$

$$\boxed{\begin{aligned} &= a^2 - b^2 + 2abi \\ &= -1 \end{aligned}}$$

$$\Rightarrow \begin{cases} a^2 - b^2 = -1 \\ 2ab = 0 \end{cases}$$

$a = 0 \Rightarrow -b^2 = -1$

$\Rightarrow b^2 = 1 \Rightarrow b = \pm 1$

$b = 0 \qquad a^2 = -1$ no existe

$z = a + bi \qquad a = 0 \qquad b = 1$ o
$b = -1$.

$\boxed{z = i}$ o $\boxed{z = -i}$

son las soluciones complejas de la ecuación $z^2 = -1$

$z = re^{i\theta}$ $\qquad z^2 = -1$

$$\left.\begin{array}{l} -1 = e^{\pi i} \\ z^2 = \left(re^{i\theta}\right)^2 = r^2 e^{i2\theta} \end{array}\right\} \Rightarrow$$

$$\Rightarrow e^{\pi i} = r^2 e^{2\theta i} \Leftrightarrow \begin{cases} 1 = r^2 \\ \pi = 2\theta + 2k\pi \\ \qquad k \in \mathbb{Z} \end{cases}$$

$$\Leftrightarrow \begin{cases} r = 1 \\ \theta = \frac{\pi}{2} - k\pi \quad k \in \mathbb{Z} \end{cases}$$

$\Leftrightarrow \begin{cases} r = 1 \\ \theta = \pi/2 \end{cases}$ y $\begin{cases} r = 1 \\ \theta = \frac{3\pi}{2} \end{cases}$

$\Leftrightarrow z = e^{\pi/2\, i} = i$ $\quad z = e^{\frac{3\pi}{2} i} = -i$

Raices complejas de la unidad

Sea $n \in \mathbb{N}$ hallar todos los $z \in \mathbb{C}$ tales que $z^n = 1$.

$\cancel{z = a + bi}$ $\quad z = re^{\theta i}$ ✓

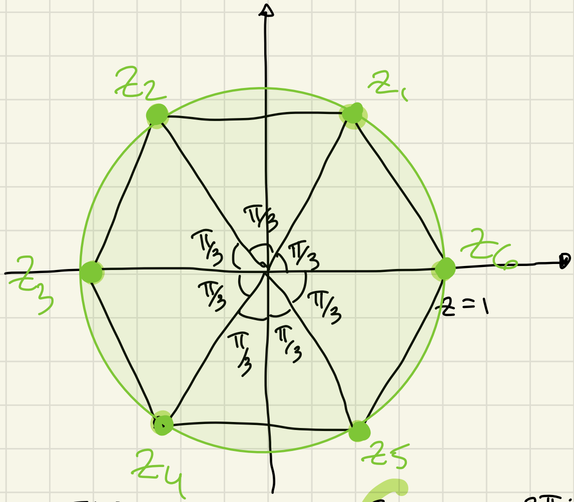
$$z^n = \left(re^{\theta i}\right)^n = r^n e^{n\theta i}$$

$$\|$$

$$1 = 1 . e^{0\pi i}$$

$$z^n = 1 \Leftrightarrow \begin{cases} r^n = 1 \\ n\theta = 0 + 2\pi k \end{cases} \quad k \in \mathbb{Z}$$

$$\Leftrightarrow \begin{cases} r = 1 \\ \theta = \frac{2\pi k}{n} \end{cases} \quad k \in \mathbb{Z}$$



$n=6$

$$\frac{2\pi}{6} = \frac{\pi}{3}$$

$$\frac{4\pi}{6} = \frac{2\pi}{3}$$

$$\boxed{z^n = 1} \iff z = e^{\frac{2\pi k i}{n}} \iff \begin{cases} z = e^{\frac{2\pi i}{n}} \\ z = e^{\frac{4\pi i}{n}} \\ \vdots \\ z = e^{\frac{(n-1)\cdot 2\pi i}{n}} \\ \vdots \\ z = e^{\frac{n\, 2\pi i}{n}} = e^{2\pi i} = 1 \end{cases}$$

$k = 1, \dots, n$

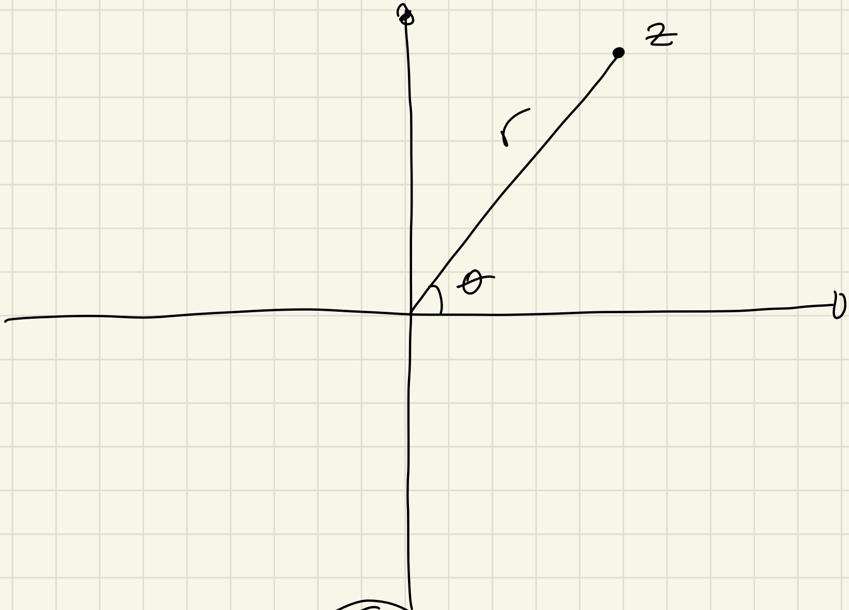
raíces n-ésimas de la unidad $\rightarrow$ $\boxed{z_k = e^{\frac{2\pi k i}{n}}}$

Sea $z \in \mathbb{C}$ $\quad \sqrt[n]{z} \quad z = r e^{\theta i}$

Queremos hallar $w^n = z$

$w = s e^{\varphi i}$

$$w^n = s^n e^{n\varphi i}$$
$$\parallel$$
$$z = r e^{\theta i}$$



Podemos considerar $\frac{z}{r}$ y es un complejo de módulo 1.

$$\begin{aligned} w^n &= s^n e^{n\varphi i} \\ \parallel \\ z &= r\, e^{\theta i} \end{aligned} \iff \begin{aligned} & s^n = r \Rightarrow s = \sqrt[n]{r} \\ & n\varphi = \theta + 2k\pi \\ & \varphi = \frac{\theta + 2k\pi}{n} \quad k \in \mathbb{Z} \end{aligned}$$

Si $z = re^{\theta i} \Rightarrow \boxed{w_k = \sqrt[n]{r}\, e^{\frac{\theta + 2k\pi}{n}} \qquad k = 1, \dots, n}$

Ejemplo:

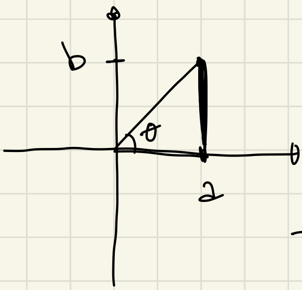
$$z = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{2021} \Rightarrow \operatorname{Re}(z) = -1/\sqrt{2} = \operatorname{Im}(z)$$

V o F

$$w = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \rightsquigarrow w = re^{\theta i}$$

$$r = \frac{1}{2} + \frac{1}{2} = \boxed{1 = r}$$

$$\theta = \operatorname{arctg}\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right) = \pi/4$$



$$\theta = \operatorname{arctg} b/a$$

$$w = e^{\pi/4\, i}$$

$$w^{2021} = e^{2021 \cdot \frac{\pi}{4} i} = e^{(2020+1)\frac{\pi i}{4}}$$

$2021 = 2020 + 1$

$$= e^{505\pi i} \cdot e^{\frac{\pi i}{4}}$$

$$= e^{\pi i} \cdot \underbrace{e^{504\pi i}}_{=1} \cdot e^{\frac{\pi i}{4}}$$

$$= e^{(\pi + \pi/4)i} = z = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$