Clase 37:

Teorema de Fubbini

CDIVV - 2023 - 2sem

Eugenio Ellis

eellis@fing.edu.uy

Teorema de Fubbini

$D = [a,b] \times [c,d]$

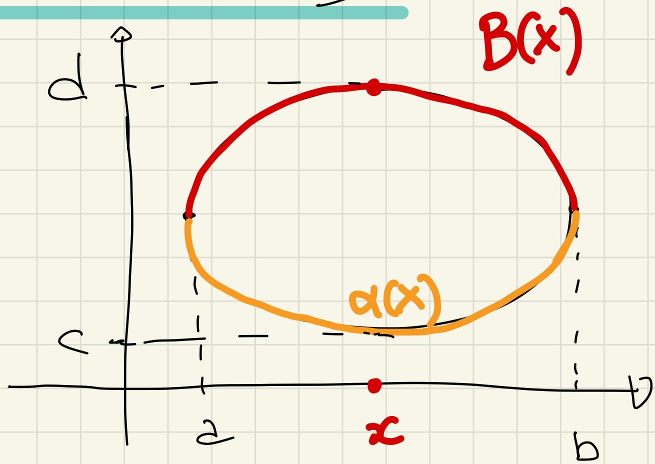
$f : D \longrightarrow \mathbb{R}$ continua entonces

$$\iint_D f = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

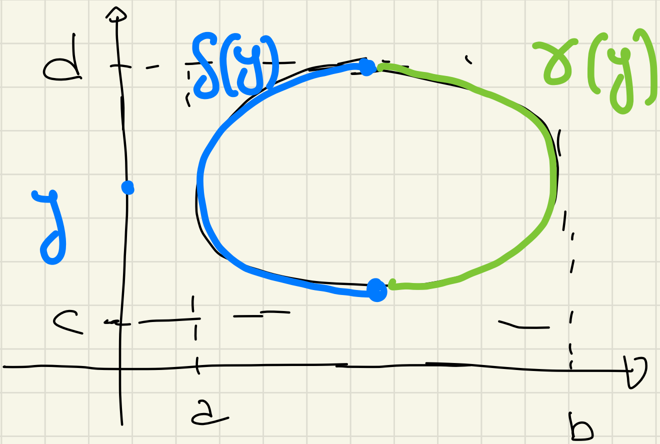
$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

Teorema de Fubbini II

D =



D =



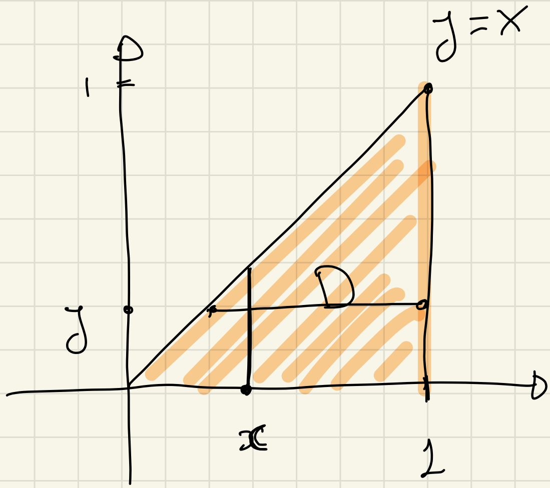
$$\boxed{\iint_D f = \int_a^b \int_{\alpha(x)}^{B(x)} f(x,y)\,dy\,dx = \int_c^d \int_{\delta(y)}^{\gamma(y)} f(x,y)\,dx\,dy}$$

Ejercicio

$f: D \to \mathbb{R}$

$f(x,y) = e^{-x^2}$

¿ $\iint_D f$?



Por TF

$$\iint_D f = \int_0^1 \left(\int_0^x e^{-x^2}\, dy \right) dx$$

$$= \int_0^1 \left(\int_y^1 e^{-x^2}\, dx \right) dy$$

se nos complica calcularlo porque e^{-x^2} no tiene primitiva elemental

$$\int_0^1 \left(\int_0^x e^{-x^2} \, dy \right) dx = \int_0^1 \left(y e^{-x^2} \Big|_0^x \right) dx$$

$$= \int_0^1 x e^{-x^2} - 0 e^{-x^2} \, dx$$

$$= \int_0^1 x e^{-x^2} \, dx = \int_{u(0)=0}^{u(1)=-1} e^u \cdot \frac{du}{-2}$$

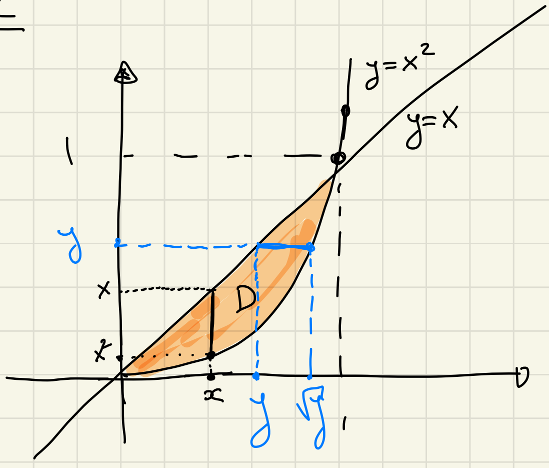
$$u = -x^2$$

$$du = -2x \, dx$$

$$= \int_{-1}^{0} \frac{e^u}{2} \, du = \frac{1}{2} e^u \Big|_{-1}^{0}$$

$$= \frac{1}{2}\left(e^0 - \frac{1}{e} \right) = \frac{1}{2}\left(1 - \frac{1}{e} \right)$$

Ejemplo 2



Hallar el area de D

$$\underset{\text{área de D}}{m(D)} = \iint_D 1 \overset{TF}{=} \int_0^1 \left(\int_{x^2}^{x} 1 \, dy \right) dx$$

$$= \int_0^1 \left(\int_y^{\sqrt{y}} 1 \, dx \right) dy$$

Ejemplo:

$$\underbrace{\int_0^1 \left(\int_0^{1-x} \frac{1}{(1+x+y)^2}\, dy \right) dx}$$

Tomamos $u = 1 + x + y$

parámetro (x) variable (y)

$$du = dy$$

$$\int_0^{1-x} \frac{1}{(1+x+y)^2}\, dy = \int_{1+x\, =\, u(0)}^{2\, =\, u(1-x)} \frac{1}{u^2}\, du.$$

$$= \int_{1+x}^{2} \frac{1}{u^2}\, du = \frac{-1}{u}\Big|_{1+x}^{2} = -\left(\frac{1}{2} \quad \frac{-1}{1+x} \right)$$

$$-\int_0^1 \frac{1}{2} - \frac{1}{1+x}\, dx = -\int_0^1 \frac{1}{2} dx + \int_0^1 \frac{1}{1+x} dx$$

$$= -\frac{1}{2} + L(1+x)\Big|_0^1 = -\frac{1}{2} + L(2).$$

Cambio de variables

Defi: $U, V \subseteq \mathbb{R}^2$ conjuntos abiertos

$g: U \longrightarrow V$ es un cambio de variable

si
- g es biyectiva
- g es diferenciable $\forall u \in U$
- $\det J_g(u) \neq 0 \quad \forall u \in U.$

Ejemplo coordenadas polares

$$g(r, \theta) = (r\cos\theta,\ r\,\text{sen}\,\theta)$$

$U = (0, +\infty) \times (0, 2\pi)$

$V = \mathbb{R}^2 \setminus \overrightarrow{O_x}$

$U = (0, +\infty) \times [0, 2\pi)$

$V = \mathbb{R}^2 - \{(0,0)\}$

Ejemplo Cambio de variable lineal

$g: \mathbb{R}^2 \to \mathbb{R}^2$

$$g(u,v) = (au + bv,\ cu + dv) = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{A} \begin{pmatrix} u \\ v \end{pmatrix}$$

- g es invertible $\iff \boxed{ad - bc \neq 0}$.

- ¿g es diferenciable en (u,v)?

$$g(u+h, v+k) = g(u,v) + J_g(u,v)\begin{pmatrix} h \\ k \end{pmatrix} + r(h,k)$$

una matriz

$$A \cdot \begin{pmatrix} u+h \\ v+k \end{pmatrix} = A \cdot \begin{pmatrix} u \\ v \end{pmatrix} + A\begin{pmatrix} h \\ k \end{pmatrix} + 0 \quad ✓$$

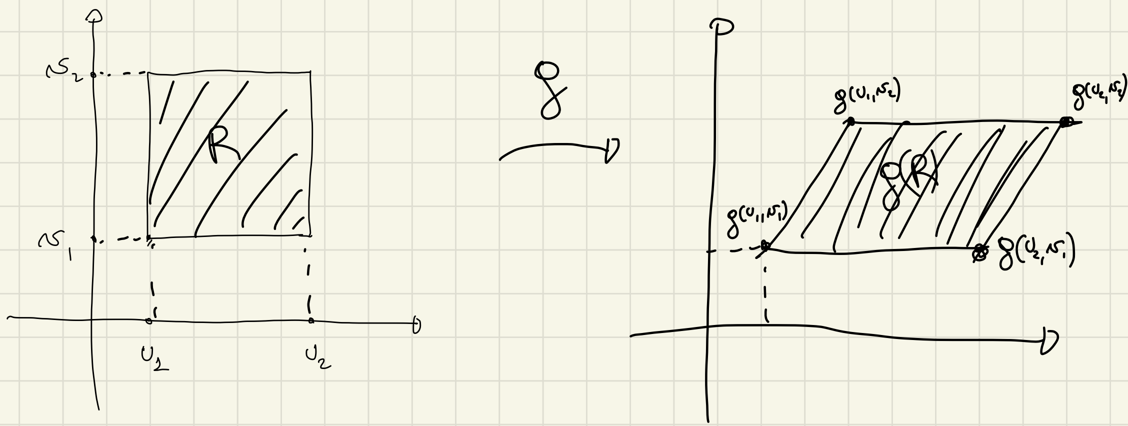
- $\det(J_g(u,v)) = \det A \neq 0$ ✓

Sea $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad g\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

c.v lineal

$\det(A) \neq 0$



¿Cuál es la relación entre

$$\iint_R 1 \quad \text{y} \quad \iint_{g(R)} 1 \quad ?$$

$m(R)$ $\qquad$ $m(g(R))$

$$m(R) = (u_2 - u_1)(v_2 - v_1).$$

área de R.

$$\underline{m(g(R))} = \left\| \big(g(u_1,v_1) - g(u_1,v_2)\big) \wedge \big(g(u_2,v_1) - g(u_1,v_2)\big) \right\|$$

área de $g(R)$.

$$= \left\| \left(A\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} - A\begin{pmatrix} u_1 \\ v_2 \end{pmatrix}\right) \wedge \left(A\begin{pmatrix} u_2 \\ v_1 \end{pmatrix} - A\begin{pmatrix} u_1 \\ v_2 \end{pmatrix}\right) \right\|$$

$$= \left\| \begin{pmatrix} b v_1 - b v_2 \\ d v_1 - d v_2 \end{pmatrix} \wedge \begin{pmatrix} a u_2 - a u_1 \\ c u_2 - c u_1 \end{pmatrix} \right\|$$

$$= \underbrace{(u_2 - u_1)(v_2 - v_1)}_{m(R)} \cdot \underbrace{(ad - cd)}_{\det(A)}.$$

ejercicio

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} a u_1 + b v_1 \\ c u_1 + d v_1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a u_1 + b v_2 \\ c u_1 + d v_2 \end{pmatrix}$$

Teorema de cambio de variable

Sea $f: \underset{\underset{\mathbb{R}^2}{\cap}}{D} \longrightarrow \mathbb{R}$ continua y

$g: U \longrightarrow V$ un cambio de variable

con $D \subseteq U$

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{g^{-1}(D)} f(g(u,v)) \cdot \left|\det J_g(u,v)\right| du\,dv}$$