Clase 32:

Regla de la cadena.

CDIVV - 2023 - 2sem

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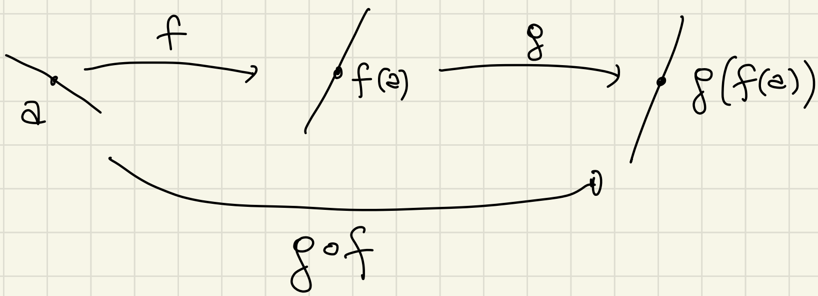
Regla de la cadena en funciones de una variable

$f: \mathbb{R} \longrightarrow \mathbb{R}$ derivable en a

$g: \mathbb{R} \longrightarrow \mathbb{R}$ derivable en $f(a)$

$\Rightarrow$ $g \circ f$ es derivable en a y

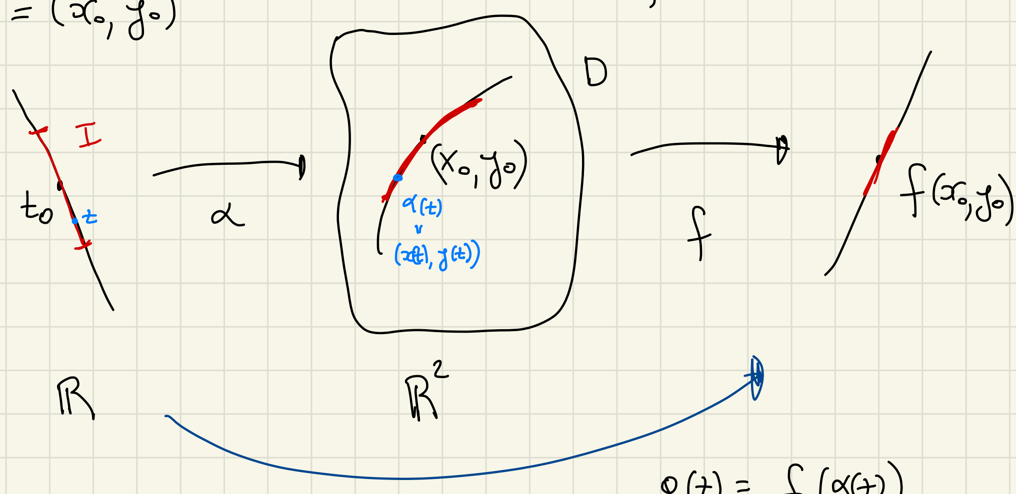
$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$



$\alpha(t) = (x(t), y(t))$

$\alpha(t_0) = (x_0, y_0)$

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



$g(t) = f(\alpha(t))$

Teorema (Regla de la cadena I)

$$\left.\begin{array}{l} f: \underset{\mathbb{R}^2}{\overset{D}{\cap}} \longrightarrow \mathbb{R} \quad \text{función diferenciable en } (x_0, y_0) \\ \alpha: \underset{\mathbb{R}}{\overset{I}{\cap}} \longrightarrow \mathbb{R}^2 \quad \alpha(t) = (x(t), y(t)) \\ \qquad x(t), y(t) \text{ son derivables en } t_0 \end{array}\right\} \Rightarrow$$

$\alpha(t_0) = (x_0, y_0)$, $\alpha(I) \subseteq D$

$\Rightarrow g(t) = f(x(t), y(t)) = f(\alpha(t))$ es derivable en t_0 y

$$g'(t_0) = \langle \nabla f(x_0, y_0), \alpha'(t_0) \rangle$$

$$= f_x(x_0, y_0) \cdot x'(t_0) + f_y(x_0, y_0)\, y'(t_0)$$

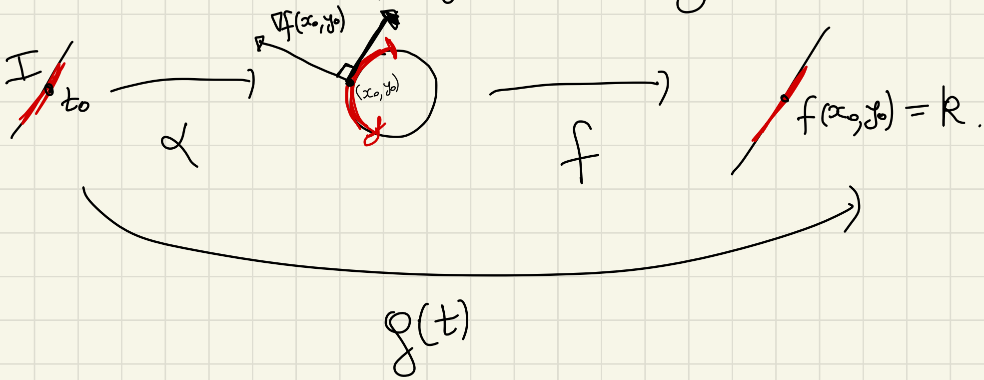
Aplicación: Si $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ diferenciable y $\alpha: I \longrightarrow \mathbb{R}^2$ es una curva en donde f es constante $f(\alpha(t)) = k \quad \forall\, t \in I$

$$\alpha(I) \subseteq C_k = \{ (x,y) \in \mathbb{R}^2 : f(x,y) = k \}$$

$g = f \circ \alpha$ es una función constante

$$g(t) = f(\alpha(t)) = k$$

$$\Rightarrow \quad g'(t) = 0$$



$$\left.\begin{array}{l} g'(t_0) = 0 \\ g'(t_0) \underset{\substack{\uparrow \\ \text{regla de} \\ \text{la cadena I}}}{=} \langle \nabla f(x_0,y_0), (x'(t_0), y'(t_0)) \rangle \end{array}\right\} \Rightarrow$$

$$\langle \nabla f(x_0,y_0), (x'(t_0), y'(t_0)) \rangle = 0$$

$$\Rightarrow \quad \nabla f(x_0,y_0) \perp (x'(t_0), y'(t_0))$$

Dem: Queremos probar que g es derivable en t_0 y queremos hallar la derivada

$$g'(t_0) \overset{\text{def}}{=} \lim_{h\to 0} \frac{g(t_0+h) - g(t_0)}{h} = ?$$

$$= \lim_{h\to 0} \frac{f(\alpha(t_0+h)) - f(\alpha(t_0))}{h}$$

$$= \lim_{h\to 0} \frac{f(x(t_0+h), y(t_0+h)) - f(x_0, y_0)}{h}$$

$x(t)$ es derivable en t_0 podemos escribir

$$x(t_0+h) = x(t_0) + x'(t_0)h + r_x(h) \quad \text{con} \quad \lim_{h\to 0} \frac{r_x(h)}{h} = 0$$

$y(t)$ es derivable en t_0 podemos escribir

$$y(t_0+h) = y(t_0) + y'(t_0)h + r_y(h) \quad \text{con} \quad \lim_{h\to 0} \frac{r_y(h)}{h} = 0$$

$$g'(t_0) = \lim_{h\to 0} \frac{f\left(x(t_0) + x'(t_0)h + r_x(h),\ y(t_0) + y'(t_0)h + r_y(h)\right) - f(x_0, y_0)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(x_0 + \overbrace{x'(t_0)h + r_x(h)}^{\Delta x},\ y_0 + \overbrace{y'(t_0)h + r_y(h)}^{\Delta y}\right) - f(x_0, y_0)}{h}$$

$$(\Delta x, \Delta y) = \left(x'(t_0)h + r_x(h),\ y'(t_0)h + r_y(h)\right) \xrightarrow[h\to 0]{} (0,0)$$

Como f es diferenciable en (x_0, y_0)

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x$$
$$+ f_y(x_0, y_0)\Delta y$$
$$+ r_f(\Delta x, \Delta y)$$

con $\displaystyle \lim_{(\Delta x, \Delta y)\to(0,0)} \frac{r_f(\Delta x, \Delta y)}{\|(\Delta x, \Delta y)\|} = 0$

$$g'(t_0) = \lim_{h\to 0} \frac{\cancel{f(x_0, y_0)} + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + r_f(\Delta x, \Delta y) - \cancel{f(x_0, y_0)}}{h}$$

$$g'(t_0) = \lim_{h\to 0} \frac{f_x(x_0,y_0)\left(x'(t_0)h + r_x(h)\right) + f_y(x_0,y_0)\left(y'(t_0)h + r_y(h)\right) + r_f(\Delta x, \Delta y)}{h}$$

$$= \lim_{h\to 0} \frac{f_x(x_0,y_0)x'(t_0)\cancel{h} + f_x(x_0,y_0)r_x(h) + f_y(x_0,y_0)y'(t_0)\cancel{h} + f_y(x_0,y_0)r_y(h) + r_f(\Delta x, \Delta y)}{\cancel{h}}$$

$$f_x(x_0,y_0)x'(t_0) + f_y(x_0,y_0)y'(t_0) + \lim_{h\to 0} \frac{f_x(x_0,y_0)r_x(h) + f_y(x_0,y_0)r_y(h) + r_f(\Delta x, \Delta y)}{h}$$

Teremos que probar que ese límite es 0

$$\lim_{h\to 0} \underbrace{f_x(x_0,y_0)\overbrace{\frac{r_x(h)}{h}}^{\to 0} + f_y(x_0,y_0)\overbrace{\frac{r_y(h)}{h}}^{\to 0}}_{h\to 0 \ \to 0} + \frac{r_f(\Delta x, \Delta y)}{h}$$

$$\lim_{h\to 0} \frac{r_f(\Delta x, \Delta y)}{h} = \lim_{h\to 0} \overbrace{\frac{r_f(\Delta x, \Delta y)}{\|(\Delta x, \Delta y)\|}}^{h\to 0 \ \to 0} \frac{\|(\Delta x, \Delta y)\|}{h}$$

¿está acotado?

Probemos que efectivamente $\dfrac{\|(\Delta x, \Delta y)\|}{h}$ está acotado

$$\left|\frac{\|(\Delta x, \Delta y)\|}{h}\right| = \left|\frac{\|(x'(t_0)h + r_x(t),\ y'(t_0)h + r_y(t))\|}{h}\right|$$

$$= \left\|\left(x'(t_0) + \underbrace{\frac{r_x(t)}{h}}_{\to 0},\ y'(t_0) + \underbrace{\frac{r_y(t)}{h}}_{\to 0}\right)\right\| \xrightarrow[h\to 0]{} \|(x'(t_0), y'(t_0))\|$$

Como existe $\displaystyle\lim_{h\to 0}\left|\frac{\|(\Delta x, \Delta y)\|}{h}\right| = \|(x'(t_0), y'(t_0))\|$

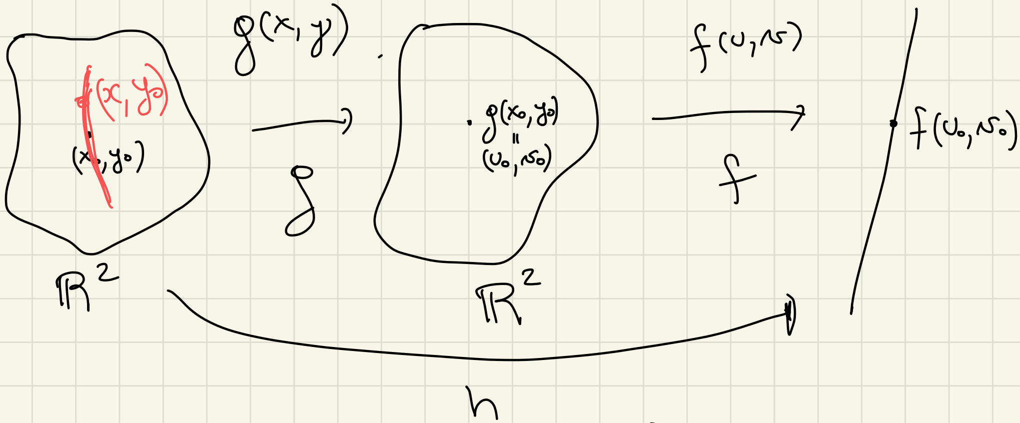
entonces $\dfrac{\|(\Delta x, \Delta y)\|}{h}$ está acotado.

$$\Rightarrow \quad g'(t_0) = f_x(x_0, y_0)\cdot x'(t_0) + f_y(x_0, y_0)\, y'(t_0)$$

$$= \langle \nabla f(x_0, y_0),\ \alpha'(t_0) \rangle$$

$\square$

Teorema (Regla de la Cadena II)



$$g(x,y) = \left(g_1(x,y),\ g_2(x,y)\right)$$

$$\left.\begin{array}{l} f \text{ es diferenciable en } (u_0, v_0) \\ g_1 \text{ y } g_2 \text{ son diferenciables en } (x_0, y_0). \end{array}\right\} \Rightarrow$$

$h = f \circ g$ es diferenciable en (x_0, y_0) y

$$\frac{\partial h}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial u}(u_0,v_0) \cdot \frac{\partial g_1}{\partial x}(x_0,y_0) + \frac{\partial f}{\partial v}(u_0,v_0) \cdot \frac{\partial g_2}{\partial x}(x_0,y_0)$$

$$\frac{\partial h}{\partial y}(x_0,y_0) = \frac{\partial f}{\partial u}(u_0,v_0) \cdot \frac{\partial g_1}{\partial y}(x_0,y_0) + \frac{\partial f}{\partial v}(u_0,v_0) \cdot \frac{\partial g_2}{\partial y}(x_0,y_0)$$

$$k(x) = h(x, y_0) = (f \circ g)(x, y_0) = f(g(x, y_0))$$

$$k'(x_0) = \frac{\partial h}{\partial x}(x_0, y_0)$$

$$g(x, y_0) = (g_1(x, y_0), g_2(x, y_0))$$

$$\begin{array}{ccc} g(t, y_0) = (& g_1(t, y_0) & , \; g_2(t, y_0)) \\ \| & \| & \| \\ \alpha(t) & x(t) & y(t) \end{array}$$

$$k(t) = f(g(t, y_0))$$

$$\Longrightarrow \qquad k'(t_0) = \langle \nabla f(g(t_0, y_0)), g'(t_0, y_0) \rangle$$

Teorema Regla de la Cadena I

$$k'(x_0) = \langle \nabla f(g(x_0, y_0)), g'(x_0, y_0) \rangle$$

$$\|$$

$$\frac{\partial h}{\partial x}(x_0, y_0)$$

$$\frac{\partial h}{\partial x}(x_0, y_0)$$

$$= \left\langle \left(\frac{\partial f}{\partial u}(u_0, v_0), \frac{\partial f}{\partial v}(u_0, v_0) \right) \left(\frac{\partial g_1}{\partial x}(x_0, y_0), \frac{\partial g_2}{\partial x}(x_0, y_0) \right) \right\rangle$$

$$= \frac{\partial f}{\partial u}(u_0, v_0) \cdot \frac{\partial g_1}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial v}(u_0, v_0) \cdot \frac{\partial g_2}{\partial x}(x_0, y_0)$$