IV. DISCRETIZATION PROBLEMS: ADVECTION SCHEMES

A. Introduction

The problem of discretizing the advection equation, or the advection terms in other prognostic equations, is still one of the unsettled problems in numerical modeling of the atmosphere. A number of approaches, methods and techniques have been proposed and used for this problem, with terminology such as "Eulerian," "Lagrangian," "semi-Lagrangian,"

"advective form," "flux (divergence) form," "flux correction," "constancy," "conservation," "dispersion," "dissipation," "overshooting/ undershooting," "positive-definite," "monotonic," and "shape-preserving." It is beyond the scope of this chapter to review these subjects even briefly. Instead, toward the end of the section, I present the major concern I have with this problem.

In any case, discretization of the advection equation is based on one of the following three forms:

1. Eulerian advective form:

$$\frac{\partial q}{\partial t} + \mathbf{V} \cdot \nabla q = 0, \tag{12}$$

for which "constancy" is automatic

2. Eulerian flux (divergence) form:

$$\frac{\partial (mq)}{\partial t} + \nabla \cdot (m\mathbf{V}q) = 0, \tag{13}$$

for which "conservation" is automatic

3. Lagrangian form:

$$\frac{Dq}{Dt} = 0, \tag{14}$$

for which "stability" is (almost) automatic.

Here q is a quantity per unit mass to be advected, $\mathbf{V}$ is the advecting velocity, m is the pseudo-density (i.e., the mass per unit horizontal area per unit increment of the vertical coordinate) predicted by the continuity equation, and D/Dt is the material time derivative. "Constancy" means that if initially q = constant, it remains so in time. "Conservation" means that $\overline{mq}$ does not change in time, where the overbar denotes the area-average over a closed domain. [Note that "conservation" here is that of the first moment, $\overline{mq}$, not that of the second moment $\overline{mq^2}$ as in (potential) enstrophy conservation or energy conservation (e.g., Arakawa and Lamb, 1981).] Finally, "stability" here means the boundedness of predicted q.

"Constancy" is perhaps one of the minimum requirements for any advection scheme. It is not automatically satisfied, however, in a scheme based on the flux (divergence) form, Eq. (13), which is a combination of the advection equation, Eq. (12), and the continuity equation, unless the scheme becomes equivalent to the discrete continuity equation used in the model when q is identically 1. Any reasonable Eulerian scheme should be able to be rewritten from the advective form to the flux (divergence) form, or vice versa, although actual computations are done using one of the two. Lagrangian (or semi-Lagrangian) schemes satisfy "constancy" but usually not "conservation." The time step of any explicit Eulerian schemes are restricted by the Courant-Friedrich-Levy (CFL) stability condition while such a restriction does not exist in Lagrangian (or semi-Lagrangian) schemes.

B. Computational Mode in Discrete Advection Equations

Most of the difficulties in discretizing the advection equation are associated with multidimensionality, nonuniformity (of the current), and nonlinearity. Problems, however, can exist even without these features. The existence of a computational mode is an example. As in Section III.B, a "computational mode" refers to a mode in the solutions of a finite-difference equation that has no counterpart in the solutions of the original differential equation. Because there is no corresponding true solution to compare with, a computational mode cannot be made more "accurate" by increasing the resolution or using a higher order scheme.

The existence of a computational mode in time with the leapfrog time differencing is well known. When the frequency is given, however, the relevant computational mode is in space, which commonly exists in most finite-difference schemes for the advection equation. The mode is especially visible in solutions with space-centered nondissipative schemes. To see the existence of a computational mode following Matsuno (1966), let us consider Eq. (12) in its simplest case of one-dimensional advection equation with a constant current U given by

$$\frac{\partial q}{\partial t} + U\frac{\partial q}{\partial x} = 0. \tag{15}$$

When the space derivative is replaced by the usual second-order centered finite difference, the relation between ν and $k\Delta x$ becomes as shown by the heavy half-sine curve in Fig. 10. Here ν is the frequency, k is the wave number, and Δx is the grid size. Unlike the continuous case, there are two wave numbers for a given frequency. As the grid size approaches zero for a given frequency, only the smaller wave number approaches the true wave number. The other wave number then represents a spurious mode, the computational mode in space. The group velocity associated with this mode is negative when $U > 0$, i.e., against the current. When the order of

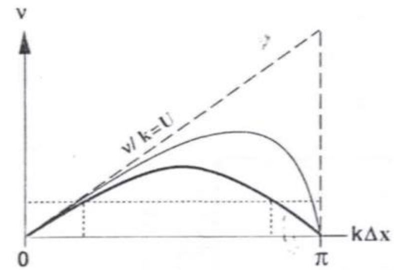

Figure 10 Dispersion relations for solutions of the advection equation, Eq. (15), with centered space finite differencing. See text for further explanation.

accuracy is raised to 4, for example, the relation becomes as shown by the thin solid line in Fig. 10, with a faster group velocity associated with the computational mode. When the order of accuracy is further increased to infinity, corresponding to the use of a spectral model, the relation becomes as shown by the dashed lines in Fig. 10. The computational mode still exists, now with the wave number equal to $\pi/\Delta x$ (i.e., wavelength equal to $2\Delta x$), with an infinite group velocity. This is a well-known problem with a spectral method applied to the Eulerian advection equation.

Note that, in the simple cases presented above with a uniform current and centered space finite differences, conservation of q^2 is automatic when time is continuous. Thus controlling a computational mode is a separate problem from the problem of conserving the second moment, such as the problem of enstrophy conservation in the nondivergent vorticity equation. In practice, the computational mode can only be handled by sacrificing the exact conservation of the second moment (while there is no justification for not conserving the first moment.) In the case of the nondivergent vorticity equation, however, enstrophy conservation helps the situation by preventing (or reducing) the spurious energy cascade to small scales, which may generate the computational mode. Also, the method of conserving the second moment for a nonlinear system can be applied in a modified way to guarantee that the deviation from conservation is dissipation, rather than generation, of the second moment (Takacs, 1985; Arakawa and Hsu, 1990; Hsu and Arakawa, 1990).

D. An Inherent Difficulty in Discretizing the Advection Equation

In my mind, an inherent difficulty in a discrete advection equation is in defining what we want in the solutions. To illustrate this point, let us consider solutions similar to those in Fig. 12. Figure 13 shows three hypothetical solutions: a solution with a perfect Lagrangian accuracy both in magnitude and phase (Fig. 13a), a solution satisfying conservation with no dispersion error (Fig. 13b), and a solution satisfying conservation with a perfect Lagrangian accuracy for the major peak but with dispersion error (Fig. 13c). All of these solutions are hypothetical and cannot be obtained in practice. Still they illustrate the problem of defining what we ultimately want in the solutions in view of their impact on the performance of the entire model.

If the quantity advected is the specific humidity, the solution of Fig. 13a does not conserve the total water content, while the solution of Fig. 13b gives excessive drying and, therefore, less cloudiness. Both of these cases are simple translations and, therefore, they are "shape preserving." The solution of Fig. 13c seems to be the optimum, but there is a question of whether we can tolerate such a large distortion of the shape. This kind of consideration makes me feel that the use of a grid fixed in space has inherent difficulties for both Eulerian and semi-Lagrangian schemes. This is one of the major reasons why I am in favor of the quasi-Lagrangian vertical coordinate, such as an isentropic coordinate, at least as one of the promising options.

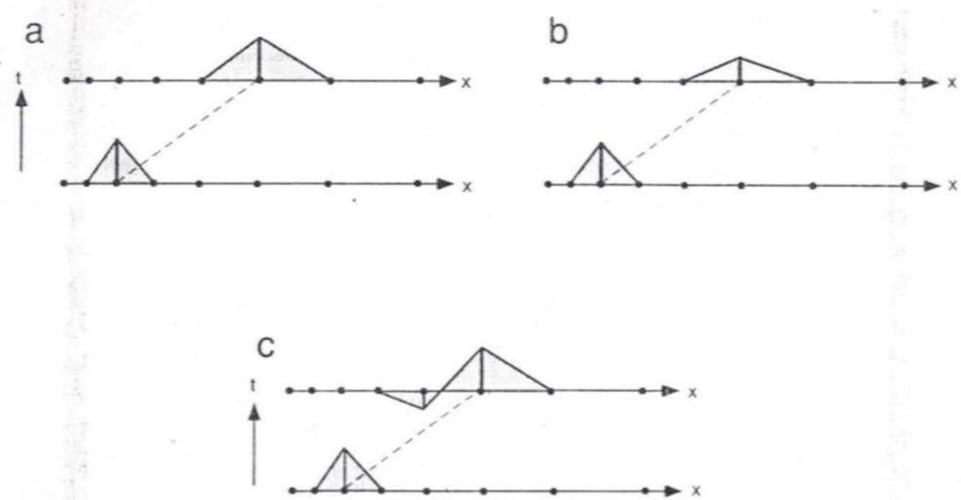

Figure 13 Three hypothetical solutions of the advection equation, Eq. (15), on a stretched grid: (a) a solution with a perfect Lagrangian accuracy both in magnitude and phase, (b) a solution satisfying conservation with no dispersion error, and (c) a solution satisfying conservation with a perfect Lagrangian accuracy for the major peak but with dispersion error.