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Optics and optical instruments — Field procedures for testing geodetic and surveying instruments —

Part 1: Theory

Optique et instruments d'optique — Méthodes d'essai sur site pour les instruments géodésiques et d'observation —

Partie 1: Théorie

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Foreword

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The procedures used to develop this document and those intended for its further maintenance are described in the ISO/IEC Directives, Part 1. In particular the different approval criteria needed for the different types of ISO documents should be noted. This document was drafted in accordance with the editorial rules of the ISO/IEC Directives, Part 2 (see www.iso.org/directives).

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For an explanation on the meaning of ISO specific terms and expressions related to conformity assessment, as well as information about ISO's adherence to the WTO principles in the Technical Barriers to Trade (TBT) see the following URL: Foreword - Supplementary information

The committee responsible for this document is ISO/TC 172, *Optics and photonics*, Subcommittee SC 6, *Geodetic and surveying instruments*.

This third edition cancels and replaces the second edition (ISO 17123-1:2010).

ISO 17123 consists of the following parts, under the general title *Optics and optical instruments — Field procedures for testing geodetic and surveying instruments*:

— *Part 1: Theory*

— *Part 2: Levels*

— *Part 3: Theodolites*

— *Part 4: Electro-optical distance meters (EDM measurements to reflectors)*

— *Part 5: Total stations*

— *Part 6: Rotating lasers*

— *Part 7: Optical plumbing instruments*

— *Part 8: GNSS field measurement systems in real-time kinematic (RTK)*

Introduction

This part of ISO 17123 specifies field procedures for adoption when determining and evaluating the uncertainty of measurement results obtained by geodetic instruments and their ancillary equipment, when used in building and surveying measuring tasks. Primarily, these tests are intended to be field verifications of suitability of a particular instrument for the immediate task. They are not proposed as tests for acceptance or performance evaluations that are more comprehensive in nature.

The definition and concept of uncertainty as a quantitative attribute to the final result of measurement was developed mainly in the last two decades, even though error analysis has already long been a part of all measurement sciences. After several stages, the CIPM (Comité Internationale des Poids et Mesures) referred the task of developing a detailed guide to ISO. Under the responsibility of the ISO Technical Advisory Group on Metrology (TAG 4), and in conjunction with six worldwide metrology organizations, a guidance document on the expression of measurement uncertainty was compiled with the objective of providing rules for use within standardization, calibration, laboratory, accreditation and metrology services. ISO/IEC Guide 98-3 was first published as an International Standard (ISO document) in 1995.

With the introduction of uncertainty in measurement in ISO 17123 (all parts), it is intended to finally provide a uniform, quantitative expression of measurement uncertainty in geodetic metrology with the aim of meeting the requirements of customers.

ISO 17123 (all parts) provides not only a means of evaluating the precision (experimental standard deviation) of an instrument, but also a tool for defining an uncertainty budget, which allows for the summation of all uncertainty components, whether they are random or systematic, to a representative measure of accuracy, i.e. the combined standard uncertainty.

ISO 17123 (all parts) therefore provides, for defining for each instrument investigated by the procedures, a proposal for additional, typical influence quantities, which can be expected during practical use. The customer can estimate, for a specific application, the relevant standard uncertainty components in order to derive and state the uncertainty of the measuring result.

Optics and optical instruments — Field procedures for testing geodetic and surveying instruments —

Part 1: **Theory**

1 Scope

This part of ISO 17123 gives guidance to provide general rules for evaluating and expressing uncertainty in measurement for use in the specifications of the test procedures of ISO 17123-2, ISO 17123-3, ISO 17123-4, ISO 17123-5, ISO 17123-6, ISO 17123-7 and ISO 17123-8.

ISO 17123-2, ISO 17123-3, ISO 17123-4, ISO 17123-5, ISO 17123-6, ISO 17123-7 and ISO 17123-8 specify only field test procedures for geodetic instruments without ensuring traceability in accordance with ISO/IEC Guide 99. For the purpose of ensuring traceability, it is intended that the instrument be calibrated in the testing laboratory in advance.

This part of ISO 17123 is a simplified version based on ISO/IEC Guide 98-3 and deals with the problems related to the specific field of geodetic test measurements.

2 Normative references

The following documents, in whole or in part, are normatively referenced in this document and are indispensable for its application. For dated references, only the edition cited applies. For undated references, the latest edition of the referenced document (including any amendments) applies.

ISO/IEC Guide 99, *International vocabulary of metrology — Basic and general concepts and associated terms (VIM)*

3 Terms and definitions

For the purposes of this document, the terms and definitions given in ISO/IEC Guide 99 and the following apply.

3.1 General metrological terms

3.1.1
(measurable) quantity
property of a phenomenon, body or substance, where the property has a magnitude that can be expressed as a number and a reference

EXAMPLE 1 Quantities in a general sense: length, time, temperature.

EXAMPLE 2 Quantities in a particular sense: length of a rod.

3.1.2
value
value of a quantity
quantity value
number and reference together expressing the magnitude of a quantity

EXAMPLE Length of a rod: 3,24 m.

3.1.3
true value
true value of a quantity
true quantity value
value consistent with the definition of a given quantity

Note 1 to entry: This is a value that would be obtained by perfect measurement. However, this value is in principle and in practice unknowable.

3.1.4
reference value
reference quantity value
quantity value used as a basis for comparison with values of quantities of the same kind

Note 1 to entry: A reference quantity value can be a true quantity value of the measurand, in which case it is normally unknown. A reference quantity value with associated measurement uncertainty is usually provided by a reference measurement procedure.

3.1.5
measurement
process of experimentally obtaining one or more quantity values that can reasonably be attributed to a quantity

Note 1 to entry: Measurement implies comparison of quantities and includes counting of entities.

3.1.6
measurement principle
phenomenon serving as the basis of a measurement (scientific basis of measurement)

Note 1 to entry: The measurement principle can be a physical phenomenon like the Doppler effect applied for length measurements.

3.1.7
measurement method
generic description of a logical organization of operations used in a measurement

Note 1 to entry: Methods of measurement can be qualified in various ways, such as “differential method” and “direct measurement method”.

3.1.8
measurand
quantity intended to be measured

EXAMPLE Coordinate x determined by an electronic tacheometer.

3.1.9
indication
quantity value provided by a measuring instrument or measuring system

Note 1 to entry: An indication and a corresponding value of the quantity being measured are not necessarily values of quantities of the same kind.

3.1.10
measurement result
result of measurement
set of quantity values attributed to a measurand together with any other available relevant information

Note 1 to entry: A measuring result can refer to

— the indication,

— the uncorrected result, or

— the corrected result.

A measurement result is generally expressed as a single measured quantity value and a measurement uncertainty.

3.1.11
measured quantity value
quantity value representing a measurement result

3.1.12
error
error of measurement
measurement error
measured quantity value minus a reference quantity value

3.1.13
random measurement error
random error
component of measurement error that in replicate measurements varies in an unpredictable manner

Note 1 to entry: Random measurement errors of a set of replicate measurements form a distribution that can be summarized by its expectation, which is generally assumed to be zero, and its variance.

3.1.14
systematic error
systematic error of measurement
component of measurement error that in replicate measurements remains constant or varies in a predictable manner

Note 1 to entry: Systematic error, and its causes, can be known or unknown. A correction can be applied to compensate for a known systematic measurement error.

3.2 Terms specific to this part of ISO 17123

3.2.1
accuracy of measurement
closeness of agreement between a measured quantity value and the true value of the measurand

Note 1 to entry: “Accuracy” is a qualitative concept and cannot be expressed in a numerical value.

Note 2 to entry: “Accuracy” is inversely related to both systematic error and random error.

3.2.2
experimental standard deviation
estimate of the standard deviation of the relevant distribution of the measurements

Note 1 to entry: The experimental standard deviation is a measure of the uncertainty due to random effects.

Note 2 to entry: The exact value arising in these effects cannot be known. The value of the experimental standard deviation is normally estimated by statistical methods.

3.2.3
precision
measurement precision
closeness of agreement between measured quantity values obtained by replicate measurements on the same or similar objects under specified conditions

Note 1 to entry: Measurement precision is usually expressed by measures of imprecision, such as experimental standard deviation under specified conditions of measurement.

3.2.4
repeatability condition
repeatability condition of measurement
condition of measurement, out of a set of conditions

Note 1 to entry: Conditions of measurement include

— the same measurement procedure,

— the same observer(s),

— the same measuring system,

— the same meteorological conditions,

— the same location, and

— replicate measurements on the same or similar objects over a short period of time.

3.2.5
repeatability
measurement repeatability
measurement precision under a set of repeatability conditions of measurement

3.2.6
reproducibility conditions of measurement
condition of measurement, out of a set of conditions

Note 1 to entry: Conditions of measurement include

— different locations,

— different observers,

— different measuring systems, and

— replicate measurements on the same or similar objects.

3.2.7
reproducibility
measurement reproducibility
measurement precision under reproducibility conditions of measurement

3.2.8
influence quantity
quantity, which in a direct measurement does not affect the quantity that is actually measured, but affects the relation between the indication of a measuring system and the measurement result

EXAMPLE Temperature during the length measurement by an electronic tacheometer.

3.3 The term "uncertainty"

3.3.1
uncertainty
uncertainty of measurement
measurement uncertainty
non-negative parameter characterizing the dispersion of quantity values attributed to a measurand, based on the information used

Note 1 to entry: Measurement uncertainty comprises, in general, many components. Some of these components can be evaluated by a Type A evaluation of measurement uncertainty from the statistical distribution of the quantity values from series of measurements and can be characterized by an experimental standard deviation. The other components, which can be evaluated by a Type B evaluation of measurement uncertainty, can also be characterized by an approximation to the corresponding standard deviations, evaluated from assumed probability distributions based on experience or other information.

3.3.2
Type A evaluation
Type A evaluation of measurement uncertainty
evaluation of a component of measurement uncertainty (standard uncertainty) by a statistical analysis of quantity values obtained by measurements under defined measurement conditions

Note 1 to entry: For information about statistical analysis, see 4.1 and ISO/IEC Guide 98-3.

3.3.3
Type B evaluation of measurement uncertainty
evaluation of a component of measurement uncertainty (standard uncertainty) determined by means other than a Type A evaluation of measurement uncertainty

EXAMPLE The component of measurement uncertainty can be based on

— previous measurement data,

— experience with, or general knowledge of, the behaviour and property of relevant instruments or materials,

— manufacturer's specifications,

— data provided in calibration and other reports,

— uncertainties assigned to reference data taken from handbooks, and

— limits deduced through personal experiences.

Note 1 to entry: For more information see 4.3 and ISO/IEC Guide 98-3.

3.3.4
standard uncertainty
standard uncertainty of measurement
standard measurement uncertainty
measurement uncertainty expressed as a standard deviation

Note 1 to entry: Standard uncertainty can be estimated either by a Type A evaluation or by a Type B evaluation.

3.3.5
combined standard uncertainty
combined standard measurement uncertainty
standard (measurement) uncertainty, obtained by using the individual standard uncertainties (and covariances as appropriate), associated with the input quantities in a measurement model

Note 1 to entry: The procedure for combining standard uncertainties is often called the "law of propagation of uncertainties" and in common parlance the "root-sum-of-squares" (RSS) method.

3.3.6
coverage factor
numerical factor larger than one, used as a multiplier of the (combined) standard uncertainty in order to obtain the expanded uncertainty

Note 1 to entry: The coverage factor, which is typically in the range of 2 to 3, is based on the coverage probability or level of confidence required of the interval.

3.3.7
expanded uncertainty
expanded measurement uncertainty
half-width of a symmetric coverage interval, centred around the estimate of a quantity with a specific coverage probability

Note 1 to entry: A fraction can be viewed as the coverage probability or level of confidence of the interval.

3.3.8
coverage interval
interval containing the set of true quantity values of a measurand with a stated probability, based on the information available

Note 1 to entry: It is intended that a coverage interval not be termed "confidence interval" in order to avoid confusion with the statistical concept. To associate an interval with a specific level of confidence requires explicit or implicit assumptions regarding the probability distribution, characterized by the measurement result.

3.3.9
coverage probability
probability that the set of true quantity values of a measurand is contained within a specific coverage interval

Note 1 to entry: The probability is sometimes termed "level of confidence" (see ISO/IEC Guide 98-3).

3.3.10
uncertainty budget
statement of a measurement uncertainty, of the components of that measurement uncertainty, and of their calculation and combination

Note 1 to entry: It is intended that an uncertainty budget include the measurement model, estimates, measurement uncertainties associated with the quantities in the measurement model, type of applied probability density functions and type of evaluation of measurement uncertainty.

3.3.11
measurement model
mathematical relation among all quantities known to be involved in a measurement

3.4 Symbols

Table 1 — Symbols and definitions

a	Half-width of a rectangular distribution of possible values of input quantity X_i:$a = (a_+ - a_-)/2$
a_+	Upper bound or upper limit of input quantity X_i
a_-	Lower bound or lower limit of input quantity X_i
A	Design or Jacobian matrix ($N \times n$)
c_i	Partial derivates or sensitive coefficient: $c_i = \frac{\partial f}{\partial x_i}$ $(i = 1, 2, ..., N)$
c	Vector of sensitive coefficients c_i $(i = 1, 2, ..., N)$
e	Unit vector

Table 1 *(continued)*

f_k	Functional relationship between a measurand, Y_k, and the input quantity, X_j, and between output estimate, y_k, and input estimates, x_j
f	Vector with elements $f_k(x^T)$ ($k = 1, 2, ..., n$)
$F_{1-\alpha/2}(v, v)$	Fisher's F (or Fisher-Snedecor) distribution with degrees of freedom (v, v) and confidence level of $(1 - \alpha)$ %
g_j	Functional relationship between the estimate of input quantity, x_j, and the observables, l_i
k	Coverage factor used to calculate expanded uncertainty $U = k \times u_c(y)$ of the output estimate y from its combined uncertainty $u_c(y)$
l_i	Observables, random variables ($i = 1, 2, ..., m$)
m	Number of observations, l_i
M	Number of input quantities, whose uncertainties can be estimated by a Type A evaluation
n	Number of output quantities, measurands
N	Number of input quantities
$N - M$	Number of input quantities, whose uncertainties can be estimated by a Type B evaluation
$\mathbf{N}$	Normal equation matrix ($n \times n$)
p_j	Weight of the input estimates x_j ($j = 1, 2, ..., N$)
P	Weight matrix of p_j ($N \times N$)
Q_{ykyk}	Cofactor of the output estimate, y_k
Q_y	Cofactor matrix of the output estimates, y_k ($n \times n$)
r_j	Residual of input estimates, x_j ($j = 1, 2, ..., N$)
r	Vector of residuals, r_j
$r(x_i, x_j)$	Correlation coefficient between the input estimates, x_i and x_j
s	Experimental standard deviation (general notation)
$s(y_k)$	Experimental standard deviation of the output estimate y_k
$t_\alpha(v)$	Student's t-distribution with the degree of freedom, v, and a confidence level of $(1 - \alpha)$ %
u	Standard uncertainty (general notation)
$u(y_k)$	Standard uncertainty of the output estimate y_k
$u(x_j)$	Standard uncertainty of the input estimate x_j
$u_c(y_k)$	Combined standard uncertainty of the output estimate y_k
U	Expanded uncertainty (general notation)
x_j	Estimate of input quantity, input estimate ($j = 1, 2, ..., N$)
x	Vector of the estimates of input quantities x_j
X_j	jth input quantity on which the measurand Y_k depends
X	Vector of input quantities X_j
y_k	Estimate of measurand Y_k, output estimate; ($k = 1, 2, ..., n$)
y	Vector of output estimates of measurands y_k
Y_k	kth measurand ($k = 1, 2, ..., n$)
Y	Vector of measurands Y_k
α	Probability of error, as a percentage
$(1 - \alpha)$	Confidence level
v	Degrees of freedom

Table 1 *(continued)*

σ	Standard deviation of the normal distribution
$\chi^2_{1-\alpha}(\nu)$	Chi-squared distribution with the degree of freedom, ν, and a confidence level of $(1 - \alpha)$ %

4 Evaluating uncertainty of measurement

4.1 General

The general concept is documented in ISO/IEC Guide 98-3, which represents the international view of how to express uncertainty in measurement. It is just a rigorous application of the variance-covariance law, which is very common in geodetic and surveying data analysis. However, the philosophy behind it has been extended in order to consider not only random effects in measurements, but also systematic errors in the quantification of an overall measurement uncertainty.

In principle, the result of a measurement is only an approximation or estimate of the value of the specific quantity subject to a measurement; that is the measurand. Thus, the result is complete only when accompanied by a quantitative statement of its quality, the uncertainty.

The uncertainty of the measurement result generally consists of several components, which may be grouped into two categories according to the method used to estimate their numerical values:

a) those which are evaluated by statistical methods;

b) those which are evaluated by other means.

Basic to this approach is that each uncertainty component, which contributes to the uncertainty of a measuring result by an estimated standard deviation, is termed standard uncertainty with the suggested symbol u.

The uncertainty component in category A is represented by a statistically estimated experimental standard deviation, s_i, and the associated number of degrees of freedom, ν_i. For such a component, the standard uncertainty $u_i = s_i$. The evaluation of uncertainty components by the statistical analysis of observations is termed a Type A evaluation of measurement uncertainty (see 4.2).

In a similar manner, an uncertainty component in category B is represented by a quantity, u_j, which may be considered an approximation of the corresponding standard deviation and which may be attributed an assumed probability distribution based on all available information. Since the quantity u_j is treated as a standard deviation, the standard uncertainty of category B is simply u_j. The evaluation of uncertainty by means other than statistical analysis of series of observations is termed a Type B evaluation of measurement uncertainty (see 4.3).

Correlation between components of either category are characterized by estimated covariances or estimated correlation coefficients.

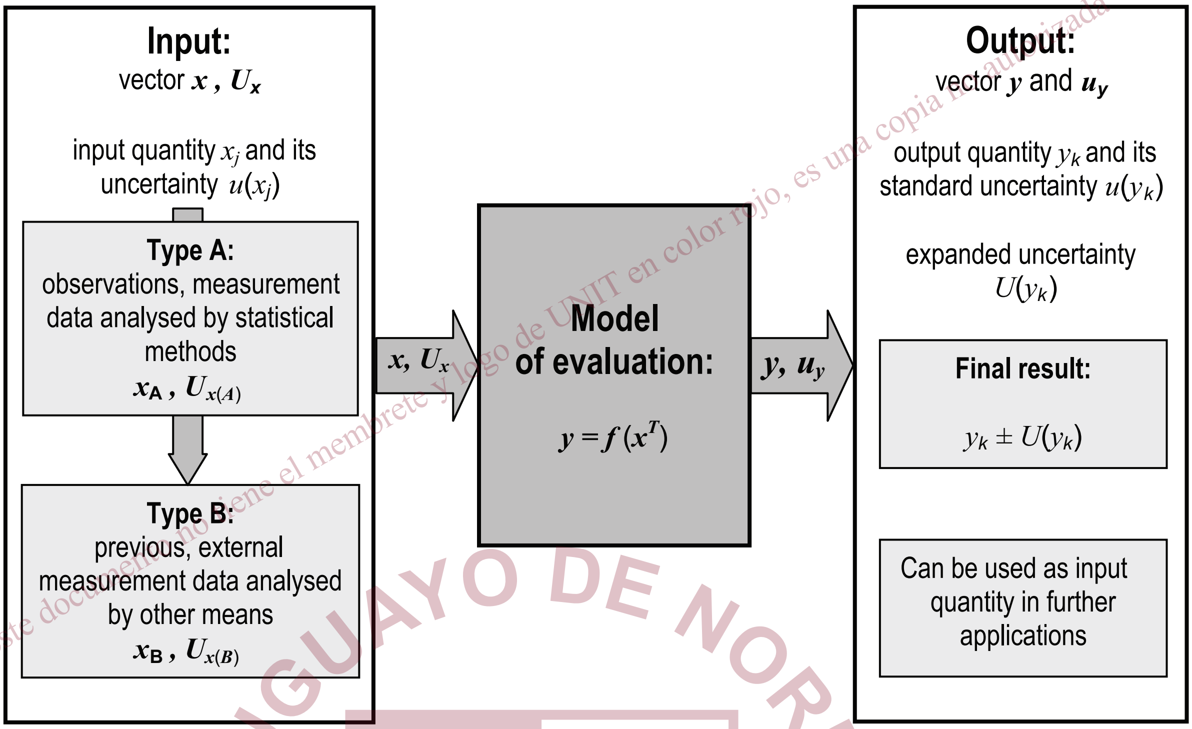


Figure 1 — Universal mathematical model and uncertainty evaluation

4.2 Type A evaluation of standard uncertainty

4.2.1 General mathematical model

In most cases, a measurand, Y, is not measured directly, but is determined by N other quantities $X_1, X_2, ..., X_N$ through the functional relationship given as Formula (1):

$$Y = f(X_1, X_2, ..., X_N) \tag{1}$$

An estimate of the measurand, Y, the output estimate, y, is obtained from Formula (1) by using the input estimates, $x_1, x_2, ..., x_N$, thus the output estimate, y, which is the result of measurements, is given by Formula (2):

$$y = f(x_1, x_2, ..., x_N) \tag{2}$$

In most cases, the measurement result (output estimate, y) is obtained by this functional relationship.

But in some cases, especially in geodetic and surveying applications, the measurement result is composed of several output estimates, $y_1, y_2, ..., y_n$ which are obtained by multiple, e.g. N, measurements (input estimates).

From this follows the general model function (see Figure 1) given as Formula (3):

$$y = f(x^T) \tag{3}$$

Assuming that

x is a vector ($N \times 1$) of input quantities x_j ($j = 1, 2, ..., N$);

y is a vector ($n \times 1$) of output quantities y_k ($k = 1, 2, ..., n$);

f is a vector ($n \times 1$) with the elements $f_k(x^T)$ ($k = 1, 2, ..., n$);

f can be understood as a suitable algorithm to determine the output quantities y (see Annex C).

4.2.2 General law of Type A uncertainty propagation

Often in geodetic measuring processes, the input quantity, x_j, is a function of several observables, the random variables:

$$l^T = (l_1, l_2, l_3, ..., l_m) \tag{4}$$

The reason for this can be, for example, internal measuring processes of the instrument, correction parameters obtained by calibration or even multiple measurements of the same observable.

The associated uncertainty matrix may be given by Formula (5):

$$U_l = \begin{pmatrix} u_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_m^2 \end{pmatrix} \tag{5}$$

Assuming the general function

$$x_j = g_j(l) \ (j = 1, 2, ..., N) \tag{6}$$

the linearized model

$$x_j = g_0 + g_j^T l \tag{7}$$

with

$$g_j^T = (g_{j1}, g_{j2}, \ldots, g_{jm}) = (\frac{\partial g_j}{\partial l_1}, \frac{\partial g_j}{\partial l_2}, \ldots, \frac{\partial g_j}{\partial l_m}) \tag{8}$$

yields the standard uncertainty of the input quantity, x_j, as given by Formula (9):

$$u(x_j) = \sqrt{g_j^T U_l g_j} \tag{9}$$

Under the assumption that the observables are random,

$$u(x_j) = s(x_j) \tag{10}$$

which is called the experimental standard deviation of x_j.

Of course, u_{jk} can also be introduced in Formula (5) covariances such that U_l becomes a fully occupied matrix.

The numerical example in C.1 illustrates this approach of a Type A evaluation for calculating the standard uncertainty.

If there are N functions of X, all dependent on the observables l, they are treated according to Formula (7):

$$x = g_0 + Gl \tag{11}$$

With the Jacobian matrix:

$$G = \begin{pmatrix} g_{11} & \cdots & g_{lm} \\ \vdots & \ddots & \vdots \\ g_{N1} & \cdots & g_{Nm} \end{pmatrix} \tag{12}$$

Finally, Formula (9) can be written in the general form of the known law of error propagation:

$$U_x = GU_l G^T = \begin{pmatrix} u^2(x_1) & u(x_1,x_2) & \cdots & u(x_1,x_N) \\ u(x_2,x_1) & u^2(x_2) & \cdots & u(x_2,x_N) \\ \vdots & \vdots & \ddots & \vdots \\ u(x_M,x_1) & u(x_M,x_2) & \cdots & u^2(x_N) \end{pmatrix} \tag{13}$$

From the diagonal elements, the standard uncertainties can be derived as given by Formula (14):

$$u_x = \left[u(x_1), u(x_2), ..., u(x_N)\right]^T \tag{14}$$

Respectively, the empirical standard deviations are

$$s_x = \left[s(x_1), s(x_2), ..., s(s_N)\right]^T \tag{15}$$

Following the flowchart of Figure 1 in which the output quantities are obtained from the input estimates x by a linear transformation, then

$$y = f(x^T) = h_0 + H(x) \tag{16}$$

Taking Formula (11) into account,

$$y = h_0 + H(g_0 + Gl) = \bar{h}_0 + HGl \tag{17}$$

and, according to Formula (13), the uncertainty matrix becomes:

$$U_y = HU_xH^T = HGU_lG^TH^T \tag{18}$$

The diagonal elements of the matrix U_y incorporate the standard uncertainty vector given as Formula (19):

$$u_y = \left[u(y_1), u(y_2), \ ..., \ u(y_N)\right]^T \tag{19}$$

of the output estimates $y_1, y_2, ..., y_N$.

Again, if the input quantities vary randomly, the standard uncertainties in Formula (19) match the empirical standard deviations of the output estimate y.

$$u_y = s_y \text{ or } u(y_k) = s(y_k)\ (k = 1, 2, ..., n) \tag{20}$$

The nesting in Formula (18) can be arbitrarily enhanced for further applications (see Figure 1), e.g. $z = M(y)$.

The numerical example in C.2 illustrates this approach of a Type A evaluation for calculating the standard uncertainty.

4.2.3 Least squares approach

Often, more model equations according to Formula (3) are given than output quantities, y_k, have to be determined. In such a case ($N > n$), it is suitable to solve the equation system by the known method of a least-squares adjustment. For this, it is necessary to restate the model function of Formula (3) in a system of (nonlinear) observation equations:

$$x + r = F(y) \tag{21}$$

or in a linearized notation (neglecting higher-order terms):

$$x + r = F(y_0) + \frac{\partial F}{\partial y}(y - y_0) \tag{22}$$

where

x is the vector ($N \times 1$) of the observations or measurable input quantities;

r is the vector ($N \times 1$) of the residuals;

y is the vector ($n \times 1$) of unknowns, output estimates;

y_0 is the vector ($n \times 1$) of the approximate values of y.

Substituting in Formula (22):

$$y - y_0 = \tilde{y},$$

$$x - F(y_0) = l \tag{23}$$

and

$$\frac{\partial F}{\partial y} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_N}{\partial y_1} & \cdots & \frac{\partial F_N}{\partial y_n} \end{pmatrix} = A \tag{24}$$

yields Formula (25):

$$r = A\tilde{y} - l \tag{25}$$

Often, it is necessary to introduce a stochastic model by the weight matrix of the measurable input quantities:

$$P = \begin{pmatrix} p_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_N \end{pmatrix} \text{ with } p_j = \frac{s_0^2}{s_j^2} \tag{26}$$

The weights, p_j, can be determined under consideration of Formula (13), respectively Formula (15).

Following the Gauß-Markov model, the solution vector is:

$$\tilde{y} = (A^T PA)^{-1} A^T Pl = N^{-1} n \tag{27}$$

With the results of Formula (27), the residuals can be calculated from Formula (25). Thus, the a posteriori variance factor can be derived from Formula (28):

$$s_0^2 = \frac{r^T Pr}{v} \tag{28}$$

where

$v = N-n$ (degree of freedom).

From this, the experimental standard deviation of the output estimates, y, can be calculated by the known relationships

$$s(y_k) = s_0 \sqrt{Q_{y_k y_k}} \quad k = 1, 2, ..., n \tag{29}$$

with

$$Q_{ykyk} = diag Q_y \text{ and } Q_y = N^{-1} \tag{30}$$

Finally, the standard uncertainties, Type A evaluation, of all output estimates y_k can be stated as Formula (31):

$$u_y = s_y \text{ or } u(y_k) = s(y_k) \quad k = 1, 2, ..., n \tag{31}$$

But, the adjusted input values can also be quoted by Formula (32):

$$\tilde{\boldsymbol{x}} = \boldsymbol{l} + \boldsymbol{r} \tag{32}$$

and the estimated variance covariance matrix of $\tilde{\boldsymbol{x}}$ by Formula (33):

$$S_{\tilde{x}} = s_0^2 A N^{-1} A^T \tag{33}$$

Finally, from its diagonal elements, the experimental standard deviation is given by Formula (34):

$$s_{\tilde{x}} = (s_{\tilde{x}_1}, s_{\tilde{x}_2}, ..., s_{\tilde{x}_N}) = \sqrt{diag\, S_{\tilde{x}}} \tag{34}$$

Thus, the standard uncertainty of the adjusted input estimates, , yields Formula (35):

$$u_{\tilde{x}} = s_{\tilde{x}} \text{ or } u(\tilde{x}_j) = s(\tilde{x}_j) \quad (j = 1, 2, ..., N) \tag{35}$$

The numerical example in C.3 illustrates this approach of a Type A evaluation for calculating the standard uncertainty.

4.2.4 Special cases

4.2.4.1 Calculation of the standard uncertainty, $u(\bar{x}_i)$, of the arithmetic mean or average $\bar{x}_i$ for the ith series of measurements

Often, the input quantity X_i is estimated from j = 1, 2, ..., n independent repeated observations $x_{i,j}$. Following Formula (27), the best available estimate is Formula (36):

$$\bar{x}_i = (e^T Pe)^{-1} \quad e^T P x_i \tag{36}$$

With its experimental standard deviation, given as Formula (37):

$$s(\bar{x}_i) = \frac{s_0}{\sqrt{e^T Pe}} = \frac{s_0}{\sqrt{\sum p_{i\,j}}} \tag{37}$$

For uncorrelated equal accurate input estimates, $x_{i,j}$, the average yields Formula (38):

$$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij} \tag{38}$$

and the experimental standard deviation yields Formula (39):

$$s(\bar{x}_i) = \frac{s_0}{\sqrt{n}} = \sqrt{\frac{r^T r}{n(n-1)}}\text{, with } r = e\bar{x}_i - x_i \tag{39}$$

Then, the standard uncertainty is given by Formula (40):

$$u(\bar{x}_i) = s(\bar{x}_i) \tag{40}$$

4.2.4.2 Calculation of the standard uncertainty, $u(\bar{y}_i)$, of the arithmetic mean or average $\bar{y}_i$ for the ith series of double measurements

Often the output quantities, Y_i, are estimated by the mean $\bar{y}_i (i = 1, 2, \ldots, n)$ of pairs of measurements (two measurements with the same measurand):

$$(l_1, l_2) \text{ with } l_j = (l_{j1}, l_{j2}, \ldots, l_{jn})^T \text{ and } j = 1, 2. \tag{41}$$

The vector of the output estimates reads as Formula (42):

$$\bar{y} = \frac{1}{2}(l_1 + l_2) \tag{42}$$

The following evaluation implies that the measurement procedure eliminates systematic errors; this means that, for the expectation of the difference vector, it follows that:

$$E(d) = E(l_2 - l_1) = 0 \tag{43}$$

Furthermore, it is assumed that the same standard uncertainty $u_{l,j}$, with $j = 1, 2$, can be attributed to all pairs of measurements. Therefore

$$P_{l_1} = P_{l_2} = P \tag{44}$$

and

$$s_0^2 = \frac{d^T P d}{2n}$$

where

$$d = (l_2 - l_1) \tag{45}$$

If the same weight can be allocated to all observations, the experimental standard deviation reads as given in Formulae (46), (47) and (48):

for the measurements $l_{j,i}$:

$$s_l = \sqrt{\frac{d^T d}{2n}} \tag{46}$$

for the differences d_i:

$$s_d = \sqrt{\frac{d^T d}{n}} \tag{47}$$

and

for the output estimates $\bar{y}_i$:

$$s(\bar{y}_i) = \sqrt{\frac{d^T d}{4n}} \qquad (48)$$

To check if the assumption in Formula (43) is fulfilled, the following rule should be applied.

If Formula (49)

$$(e^T d)^2 < d^T d \qquad (49)$$

is true, it can be expected that $E(\boldsymbol{d}) = 0$. In this case, the standard uncertainty is given as Formula (50):

$$u(\bar{y}_i) = s(\bar{y}_i) \qquad (50)$$

4.2.4.3 Calculation of the overall standard uncertainty, *u*, for *m* series of measurements

The experimental standard deviation obtained for each of the *m* series of measurements is considered to be a separate estimate of the overall experimental standard deviation of the measurements. It is assumed that each of these estimates is of the same order of reliability, $\nu_i = \nu_1 = \nu_2 = ... = \nu_m$. Formulae (51) and (52) indicate how the individual experimental standard deviations are combined to give one overall experimental standard deviation which takes equal account of the experimental standard deviations calculated for each series of measurements.

$$\sum s^2 = \sum_{i=l}^{m} s_i^2 = s_1^2 + s_2^2 + ... + s_m^2 \qquad (51)$$

where

m is the number of series of measurements;

s_i is the experimental standard deviation of a single measured value within the *i*th series of measurements;

$\sum s^2$ is the sum of squares of all standard deviations, s_i, of the *m* series of measurements.

The overall experimental standard deviation, s, of m series of measurements yields Formula (52):

$$s = \sqrt{\frac{\sum s^2}{m}} \tag{52}$$

The number of degrees of freedom of all m series of measurements is obtained by Formula (53):

$$\nu = \sum_{i=1}^{m} \nu_i = m \times \nu_i \tag{53}$$

Finally, the overall standard uncertainty can be written as Formula (54):

$$u = s \tag{54}$$

Numerical examples in C.4 and C.5 illustrate these approaches of a Type A evaluation for calculating standard uncertainties.

4.3 Type B evaluation of standard uncertainty

4.3.1 General

Often, not all uncertainties of the N input quantities can be estimated by a Type A evaluation; this number of uncertainties, obtained by the Type A evaluation, is therefore assumed, M, so that the uncertainties of $N - M$ input quantities have to be determined by other means, namely by a Type B evaluation.

For an estimate x_j, $M < j \leqslant N$ of an input quantity, which has not been obtained from repeated observations or was derived from small samples, the evaluation of the standard uncertainty $u(x_j)$ is usually based on scientific judgment using all available information, which may include

— previous measurement data,

— experience with, or general knowledge of, the behaviour and properties of relevant materials and instruments,

— manufacturer's specifications,

— data provided in calibration reports,

— uncertainties assigned to reference data taken from handbooks.

Examples of such a Type B evaluation, which can be very helpful for practical use, are given in the following subclauses.

4.3.2 Quantity in question modelled by a normal distribution (see Annex A)

— Lower and upper limits are estimated by a_- and a_+.

— Estimated value of the quantity: $(a_+ + a_-)/2$.

— 50 % probability that the value lies in the interval a_- to a_+.

Then, the standard uncertainty yields Formula (55):

$$u_j \approx 1{,}48\,a \tag{55}$$

where $a = (a_+ - a_-)/2$

4.3.3 Quantity in question modelled by a normal distribution (see Annex A)

— Lower and upper limits are estimated by a_- and a_+.

— Estimated value of the quantity: $(a_+ + a_-)/2$.

— 67 % probability that the value lies in the interval a_- to a_+.

Then, the standard uncertainty yields Formula (56):

$$u_j \approx a \tag{56}$$

where $a = (a_+ - a_-)/2$

4.3.4 Quantity in question modelled by a uniform or rectangular probability distribution (see Annex A)

— Lower and upper limits are estimated by a_- and a_+.

— Estimated value of the quantity: $(a_+ + a_-)/2$.

— 100 % probability that the values lies in the interval a_- to a_+.

Then, the standard uncertainty yields Formula (57):

$$u_j = \frac{a}{\sqrt{3}} \approx 0{,}58\,a \tag{57}$$

where $a = (a_+ - a_-)/2$

4.3.5 Quantity in question modelled by a triangular probability distribution (see Annex A)

— Lower and upper limits are estimated by a_- and a_+.

— Estimated value of the quantity: $(a_+ + a_-)/2$.

— 100 % probability that the values lies in the interval a_- to a_+.

Then, the standard uncertainty yields Formula (58):

$$u_j = \frac{a}{\sqrt{6}} \approx 0{,}41\,a \tag{58}$$

where $a = (a_+ - a_-)/2$

The numerical Examples in C.6 illustrate these approaches of a Type B evaluation for calculating standard uncertainties.

4.4 Law of propagation of uncertainty and combined standard uncertainty

The combined standard uncertainty, $u_c(y_k)$, of a measurement result y_k is taken to represent the estimated standard deviation of the final result. It is obtained by combining the individual standard uncertainties, $u(x_i)$, and, if available, the covariances $u(x_i, x_j)$ of the input estimates $x_1, x_2, ..., x_M, x_{M+1}, x_{M+2}, ..., x_N$, whether arising from a Type A evaluation or a Type B evaluation. This method is called the law of propagation of uncertainty or in the parlance of geodetic metrology the root-sum-squares method of combining standard deviations.

It is assumed that for the input estimates

$$(x_1, x_2, ..., x_M) = \boldsymbol{x}_A^T \tag{59}$$

the standard uncertainties are from a Type A evaluation and given by Formula (60):

$$U_{x(A)} = \begin{pmatrix} u(x_1)^2 & 0 & \cdots & 0 \\ 0 & u(x_2)^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & u(x_M)^2 \end{pmatrix} \tag{60}$$

and for the input estimates

$$(x_{M+1}, x_{M+2}, ..., x_N) = \boldsymbol{x}_B^T \tag{61}$$

the standard uncertainties are from a Type B evaluation and given by Formula (62):

$$U_{x(B)} = \begin{pmatrix} u(x_{M+1})^2 & 0 & \cdots & 0 \\ 0 & u(x_{M+2})^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & u(x_N)^2 \end{pmatrix} \tag{62}$$

Hence

$$U_x = \begin{pmatrix} U_{x(A)} & 0 \\ 0 & U_{x(B)} \end{pmatrix} \tag{63}$$

and, according to Formulae (7) to (9),

$$y_k = c_0 + c_k^T \begin{pmatrix} x_A \\ x_B \end{pmatrix} \tag{64}$$

with

$$\boldsymbol{c}_k^T = \left(\frac{df_k}{dx_1}, \frac{df_k}{dx_2}, ..., \frac{df_k}{dx_N} \right) = (c_{k1}, c_{k2}, ..., c_{kN}) \tag{65}$$

The values c_{ki}, with $i = 1, ..., N$, are often called sensitivity coefficients and are determined either by the derivatives of the function f_k or, sometimes measured experimentally by an empirical first-order Taylor series expansion.

Finally, the combined standard uncertainty for the output estimate, y_k [see Formula (3)] yields Formula (66):

$$u_c(y_k) = \sqrt{c_k^T U_x c_k} \tag{66}$$

If the estimated covariance between x_i and x_j the $u(x_i, x_j) = u(x_j, x_i)$ are known, they can be regarded easily in Formulae (60), (62) and (63).

In this case, the degree of correlation is characterized by the estimated correlation coefficient

$$r(x_i, x_j) = \frac{u(x_i, x_j)}{u(x_i) \cdot u(x_j)} \tag{67}$$

where $-1 \le r(x_i, x_j) \le +1$. If $u(x_i)$ and $u(x_j)$ are independent, $r(x_i, x_j) = 0$.

The numerical examples in C.6 illustrate these approaches of calculating the combined standard uncertainties.

4.5 Expanded uncertainty

Although the combined standard uncertainty, $u_c(y)$, can be universally used, in some commercial, industrial applications, it is often necessary to give a measure of uncertainty that defines an interval about the measurement result, y, within which the value of the measurand, Y, is confidently believed to lie. The measure of uncertainty that meets the requirements of providing an interval is termed expanded uncertainty with the suggested symbol U and is obtained by multiplying the combined standard uncertainty by the coverage factor k as given by Formula (68):

$$U = k \times u_c(y) \tag{68}$$

It is confidently believed that

$$y - U \le Y \le y + U \tag{69}$$

which is conveniently expressed as Formula (70):

$$Y = y \pm U \tag{70}$$

In general, the value of the coverage factor, k, is chosen on the basis of the desired level of confidence intended to be associated with the interval defined by $\pm U$ and is typically in the range of 2 to 3.

If

$$U = 2 \times u_c(y) \tag{71}$$

the interval corresponds to a particular level of confidence of approximately $P = 95$ %, which is used typically in this series of standards and assumes for the output estimate a normal distribution.

Under the same precondition,

$$U = 3 \times u_c(y) \tag{72}$$

defines an interval having a level greater than $P = 99$ %.

However, for specific applications, k may be outside the stated range. Extensive experiences with full knowledge of the use to which the measurement result is intended to be put can facilitate the proper selection of the value k. For more information, see ISO/IEC Guide 98-3:2008, 6.3, and Annex G.

5 Reporting uncertainty

When reporting a measurement result and its uncertainty, the following information should be given:

— a clear description of the mathematical models and methods used to calculate the measurement result and its uncertainty (Type A and Type B evaluations) from the experimental observations and input data;

— a list of all uncertainty components together with their degrees of freedom and the resulting u_c;

— a detailed description of how each component of standard uncertainty was evaluated;

— a description of how k was chosen, if k is not taken equal to 2.

When the measure of uncertainty is $u_c(y)$, the numerical result of measurement should be stated in the following way:

D = 12 345,678 m $\qquad$ u_c = 9,1 mm

If the expanded uncertainty, U, is reported, the following notation is recommended:

D = 12 345,678 m $\qquad$ U = ±18 mm (k = 2)

or

D = (12 345,678 ± 0,018) m (k = 2)

6 Summarized concept of uncertainty evaluation

The following summary can be understood as a stepwise instruction for calculating the uncertainty in practice.

a) Clear description of measurands and measuring method: the relationship between the input quantities and output quantities, and the evaluation model shall be correctly described mathematically.

b) All corrections should be ascertained and, as far as possible, applied.

c) Detection of all causes (influence quantities) for evaluating uncertainty.

d) Calculation of the standard uncertainties applying the statistical procedures of a Type A evaluation.

e) Determination of the standard uncertainties of a Type B evaluation. For this,

   1) the knowledge of the probability distribution of the input quantity,

   2) information to estimate the distribution of the input quantity,

   3) upper and lower bounds of the variability of the limits of the input quantity, and

4) any other information, knowledge to quote the required standard uncertainty should be considered.

f) For each input quantity, the quantitative contribution of the standard uncertainty shall be calculated. Thus all sensitivity coefficients shall be determined according to the measuring model (mathematical model to calculate the output estimate).

g) Hereinafter, the law of propagation of the uncertainty can be applied; the result is the combined standard uncertainty of the output estimate.

h) Multiplication of the combined standard uncertainty by the coverage factor yields after all the expanded uncertainty.

i) Report of the final result by quoting the output estimate, the expanded uncertainty and the coverage factor.

7 Statistical tests

7.1 General

For the interpretation of the results, obtained from the full test procedure only, statistical tests shall be carried out using the experimental standard deviation, s, or the standard uncertainty, u, of a Type A evaluation. For tests, this Type A evaluation of standard uncertainty can be treated as an experimental standard deviation. For testing, the following questions shall be answered (see Table 2).

a) Is the calculated experimental standard deviation (standard uncertainty of a Type A evaluation), s, smaller than or equal to the manufacturer's or some other predetermined value of σ?

b) Do two experimental standard deviations (standard uncertainties of a Type A evaluation), s and $\tilde{s}$, as determined from two different samples of measurements belong to the same population, assuming that both samples have the same number of degrees of freedom, v (v being the number of degrees of freedom of all series of measurements)?

c) Respectively, d) is a parameter y_k obtained by adjustment equal to zero?

Table 2 — Statistical tests

Question	Null hypothesis	Alternative hypothesis
a)	$s \le \sigma$	$s > \sigma$
b)	$\sigma = \tilde{\sigma}$	$\sigma \ne \tilde{\sigma}$
c) respectively d)	$y_k = 0$	$y_k \uparrow 0$
NOTE σ is used instead of s because the null hypothesis checks if the two experimental standard deviations belong to the same population.		

7.2 Question a): is the experimental standard deviation, s, smaller than or equal to a given value σ?

Formulae (1) to (54) allow only the determination of the (experimental) standard deviation, s, or the standard uncertainty of a Type A evaluation, u, of the measurements. Because of the small size of the sample, this value can differ more or less from the theoretical standard deviation, σ, of the whole population as stated by the manufacturer of the instrument or predetermined in any other way.

The methods of mathematical statistics permit the decision whether an experimental standard deviation, s, is smaller than or equal to a given theoretical standard deviation, σ, on the confidence level $1 - \alpha$.

The null hypothesis $s \leq \sigma$ is not rejected if the following condition is fulfilled:

$$s \leq \sigma \times \sqrt{\frac{\chi^2_{1-\alpha}(\nu)}{\nu}} \tag{73}$$

Otherwise, the null hypothesis is rejected. $\chi^2_{1-\alpha}(\nu)$ may be taken from Table B.1.

The theoretical standard deviation, σ, is a predetermined value.

7.3 Question b): Do two samples belong to the same population?

The methods of mathematical statistics permit the decision as to whether two experimental standard deviations, s and $\tilde{s}$, or the standard uncertainties of a Type A evaluation, u and $\tilde{u}$, obtained from two different samples of measurements, belong to the same population on the confidence level $1 - \alpha$. The corresponding null hypothesis $\sigma = \tilde{\sigma}$ is not rejected if the following condition is fulfilled:

$$\frac{1}{F_{1-\alpha/2}(\nu,\nu)} \leq \frac{s^2}{\tilde{s}^2} \leq F_{1-\alpha/2}(\nu,\nu) \tag{74}$$

Otherwise, the null hypothesis is rejected.

Two samples of measurements with the same number $n = \tilde{n}$ are taken to determine the experimental standard deviations, s and $\tilde{s}$. These experimental standard deviations, s and $\tilde{s}$, may be obtained from:

— two samples of measurements by the same equipment, but different observers;

— two samples of measurements by the same equipment, but at different times;

— two samples of measurements by different equipment.

$F_{1-\alpha/2}(\nu, \nu)$ may be taken from Table B.1.

7.4 Question c) [respectively question d)]:Testing the significance of a parameter y_k

Formulae (21) to (35), the equations of adjustment by least squares, allow the determination of parameters y_k and their experimental standard deviations, $s(y_k)$, or standard uncertainties of a Type A evaluation, $u(y_k)$. Moreover, the methods of mathematical statistics permit the decision as to whether a parameter y_k is not equal to zero on the confidence level $1 - \alpha$. The null hypothesis of $y_k = 0$ is not rejected, if the following condition is fulfilled:

$$|y_k| \leq s(y_k) \times t_{1-\alpha/2}(\nu) \tag{75}$$

Otherwise, the null hypothesis is rejected.

yk is the parameter to be tested valid for all series of measurements.

If $m > 1$, y_k is calculated by the corresponding values $y_{k,i}$ for the m series of measurements:

$$y_k = \frac{\sum_{i=1}^{m} y_{k,i}}{m} \tag{76}$$

$y_{k,i}$ has to be estimated according to the equations for the full test procedure.

In this case

$$s(y_k) = \frac{s}{\sqrt{v}} \tag{77}$$

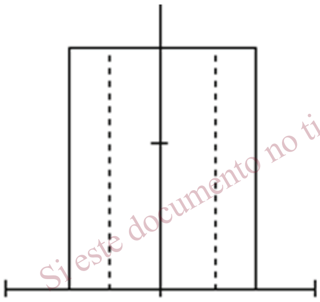
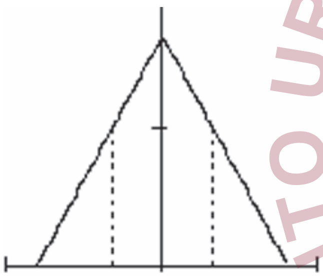
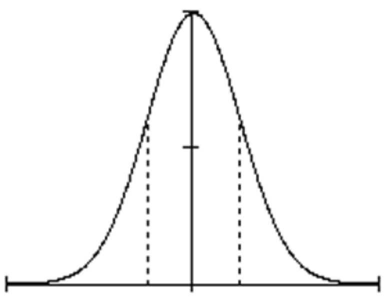
is the experimental standard deviation of the parameter y_k valid for all series of measurements, where v is a constant according to the equations for the full test procedure. If $m > 1$, $s(y_k)$ is calculated by the corresponding values $s(y_{k,i})$ for the m series of measurements:

$$s(y_k) = \sqrt{\frac{\sum_{i=1}^{m} s^2(y_{k,i})}{m}} = \frac{s}{\sqrt{v \times m}} \tag{78}$$

$t_{1-\alpha/2}(v)$ may be taken from Table B.1.

Annex A
(informative)

Probability distributions

Probability density distribution	Density function	Examples of application
Rectangular(uniform) distribution  $\mu - a$ $-\sigma$ μ $+\sigma$ $\mu + a$	Probability density function $f(x) = \frac{1}{2a}$ $(\mu - a \le x \le \mu + a)$ Standard deviation $\sigma = \frac{a}{\sqrt{3}}$	Tolerances, e.g. digital display resolutions, intervals, deviations.
Triangular distribution  $\mu - a$ $-\sigma$ μ $+\sigma$ $\mu + a$	Probability density function $f(x) = \frac{1}{a}\left[1 - \frac{1}{a}(\lvert x - \mu \rvert)\right]$ $(u - a \le x \le \mu + a)$ Standard deviation ..	Tolerances, the values of which show a high frequency in the middle and decrease linearly to both sides. Convolution of two rectangular distributions with the same half-width
Normal (Gaussian) distribution  $-\sigma$ μ $+\sigma$	Probability density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ $(-\infty < x < \infty, \sigma > 0)$ Standard deviation, σ, from statistical analysis	Standard deviation derived from a sample of uncorrelated measurements

Annex B
(normative)

χ^2 distribution, Fisher's distribution and Student's *t*-distribution

Table B.1 — χ^2 distribution, Fisher's distribution and Student's *t*-distribution

v	$\chi^2_{0,90}(v)$		$t_{0,95}(v)$	$\chi^2_{0,95}(v)$	$F_{0,975}(v,v)$	$t_{0,975}(v)$	$\chi^2_{0,99}(v)$	$F_{0,995}(v,v)$	$t_{0,995}(v)$
2	4,61	19,00	2,92	5,99	39,00	4,30	9,21	199,01	9,92
3	6,25	9,28	2,35	7,81	15,44	3,18	11,34	47,47	5,84
4	7,78	6,39	2,13	9,49	9,60	2,78	13,28	23,15	4,60
5	9,24	5,05	2,02	11,07	7,15	2,57	15,09	14,94	4,03
6	10,64	4,28	1,94	12,59	5,82	2,45	16,81	11,07	3,71
7	12,02	3,79	1,89	14,07	4,99	2,36	16,48	8,89	3,50
8	13,36	3,44	1,86	15,51	4,43	2,31	20,09	7,50	3,36
9	14,68	3,18	1,83	16,92	4,03	2,26	21,67	6,54	3,25
10	15,99	2,98	1,81	18,31	3,72	2,23	23,21	5,85	3,17
14	21,06	2,48	1,76	23,68	2,98	2,14	29,14	4,30	2,98
15	21,31	2,40	1,75	25,00	2,86	2,13	30,58	4,07	2,95
16	23,54	2,33	1,75	26,30	2,76	2,12	32,00	3,87	2,92
18	25,99	2,22	1,73	28,87	2,60	2,10	34,81	3,56	2,88
19	27,20	2,17	1,73	30,14	2,53	2,09	36,19	3,43	2,86
24	33,20	1,98	1,71	36,42	2,27	2,06	42,98	2,97	2,80
27	36,74	1,90	1,70	40,11	2,16	2,05	46,96	2,78	2,77
28	37,92	1,88	1,70	41,34	2,13	2,05	48,28	2,72	2,76
30	40,26	1,86	1,70	43,77	2,07	2,04	50,89	2,63	2,75
32	42,58	1,80	1,69	46,19	2,02	2,04	53,49	2,54	2,74
36	47,21	1,74	1,69	51,00	1,94	2,03	58,62	2,41	2,72
38	49,51	1,72	1,69	53,38	1,91	2,02	61,16	2,35	2,71
42	54,09	1,67	1,68	58,12	1,85	2,02	66,21	2,25	2,70
54	67,67	1,57	1,67	72,15	1,71	2,00	81,07	2,04	2,67
72	87,74	1,48	1,67	92,81	1,59	1,99	102,82	1,85	2,65
108	127,21	1,37	1,66	133,26	1,46	1,98	145,10	1,65	2,62

The test values $\chi^2_{1-\alpha}(v), F_{1-\alpha/2}(v,v)$ and t1 − α/2 (v) apply to the full test procedures of ISO 17123-2, ISO 17123-3, ISO 17123-4, ISO 17123-5, ISO 17123-6, ISO 17123-7 and ISO 17123-8, even if the number of series of measurements is less than provided there. If a different number of measurements is analysed, the number of degrees of freedom changes and the above-mentioned test values should be taken from a reference book on statistics.

Annex C
(informative)

Examples

NOTE Calculations are done with full precision from beginning to end, but intermediate and final results are shown as rounded values.

C.1 Example 1

Measurands:

Slope distance: l_1 = 142,432 m with u_1 = 12,0 mm

Zenith angle: l_2 = 78,412° u_2 = 0,055 mrad

$\boldsymbol{l}^T = (l_1 l_2) = (142{,}432\ 78{,}412)$

[m, °]

$$U_l = \begin{pmatrix} u_1^2 & 0 \\ 0 & u_2^2 \end{pmatrix} = \begin{pmatrix} 144 & 0 \\ 0 & 0{,}003\,0 \end{pmatrix} \begin{bmatrix} \mathrm{mm}^2 & \\ & \mathrm{mrad}^2 \end{bmatrix}$$

Wanted: Horizontal distance and its standard uncertainty

$x = g(l) = l_l \times \sin l_2 = 142{,}432 \times \sin 78{,}412°$

$x = 142{,}432 \times 0{,}97\,962 = 139{,}529$ m

$g^T = (g_1, g_2)$

$$g_1 = \frac{\partial g}{\partial l_1} = \sin l_2 = 0{,}979\,62$$

$$g_2 = \frac{\partial g}{\partial l_2} = l_1 \cos l_2 = 142{,}432 \times 0{,}200\,87 = 28{,}611 [\mathrm{m}]$$

$$u(x)^2 = \boldsymbol{g}^T \boldsymbol{U}_l \boldsymbol{g} = (0{,}98 \quad 28{,}61) \begin{pmatrix} 144 & 0 \\ 0 & 0{,}003 \end{pmatrix} \begin{pmatrix} 0{,}98 \\ 28{,}61 \end{pmatrix} = 140{,}646$$

$u(x) = s(x) = \mathbf{11{,}9\ mm}$

C.2 Example 2

By tacheometer measurements (measurands) the following input estimates were measured or manually entered:

s = 345,746 m slope distance;

z = 70,580 8° vertical angle;

c = 32,6 mm additive constant;

k_a = 12 ppm[1)] athmospheric correction.

As a result, it can be read from the display:

D = 326,111 6 m horizontal distance;

h = 114,964 9 m height.

According to Figure 1, the model of evaluation is given by

$x = g(l^T)$, respectively

$D = (s + c + s \times k_a)\sin z$

$h = (s + c + s \times k_a)\cos z$

For further evaluations, the standard uncertainties of the quantities D and h are needed.

1) The equivalent of 0,001 2% is 12 ppm; ppm is a deprecated unit.

For this, proceed according to 4.2.2. Following the notation in Formula (4), it is obtained:

$$l^T = (s\ c\ k_a\ z) = (345{,}746\ \ 32{,}6\ \ 12\ \ 70{,}580\ 8)\ [\text{m mm ppm}]$$

From calibration certificate uncertainties (Type A evaluation) of $\boldsymbol{l}$ were taken out, given by the vector,

$$u_l^T = \left[u(s) \quad u(c) \quad u(k_a) \quad u(z)\right] = (3 \quad 0{,}5 \quad 2 \quad 0{,}003)\left[\text{mm} \quad \text{mm} \quad \text{ppm} \quad \text{mrad}\right]$$

with

$$U_l = \begin{pmatrix} u(s)^2 & & & 0 \\ & u(c)^2 & & \\ & & u(k_a)^2 & \\ 0 & & & u(z)^2 \end{pmatrix} = \begin{pmatrix} 9 & & & 0 \\ & 0{,}25 & & \\ & & 4 & \\ 0 & & & 9\times10^{-6} \end{pmatrix}$$

$\boldsymbol{x} = \begin{pmatrix} D \\ h \end{pmatrix}$, $g_0 = 0$ and

$$G = \begin{pmatrix} \frac{\partial D}{\partial s} & \frac{\partial D}{\partial c} & \frac{\partial D}{\partial k_a} & \frac{\partial D}{\partial z} \\ \frac{\partial h}{\partial s} & \frac{\partial h}{\partial c} & \frac{\partial h}{\partial k_a} & \frac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} (1+k_a)\sin z & \sin z & s\times\sin z & \tilde{s}\times\cos z \\ (1+k_a)\cos z & \cos z & s\times\cos z & -\tilde{s}\times\sin z \end{pmatrix}$$

where

$$\tilde{s} = s + c + s\times k_a$$

It can be written (in order to obtain the result in square millimetres):

$$U_x = G U_l G^T$$

$$= \begin{pmatrix} 0{,}943 & 0{,}943 & 326 & 114{,}96 \\ 0{,}332 & 0{,}332 & 114{,}95 & -326 \end{pmatrix} \times \begin{pmatrix} 9 & & & 0 \\ & 0{,}25 & & \\ & & 4\times10^{-6} & \\ 0 & & & 9\times10^{-6} \end{pmatrix} \times \begin{pmatrix} 0{,}943 & 0{,}332 \\ 0{,}943 & 0{,}332 \\ 326 & 114{,}95 \\ 114{,}96 & -326 \end{pmatrix}$$

and finally yields:

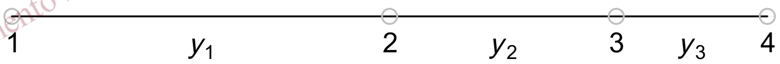
$$U_x = \begin{bmatrix} u(D)^2 \\ u(h)^2 \end{bmatrix} = \begin{pmatrix} 8{,}772 \\ 2{,}033 \end{pmatrix} \begin{bmatrix} \text{mm}^2 \\ \text{mm}^2 \end{bmatrix}$$

and

$u(D) = 3{,}0$ mm and $u(h) = 1{,}4$ mm.

C.3 Example 3

By EDM measurements (measurands) the following horizontal distances between four points located on a straight line were measured:



Observables: distances x

$1 - 2 = x_1 = 117{,}342$ m

$2 - 3 = x_2 = 68{,}454$ m

$3 - 4 = x_3 = 41{,}265$ m

$1 - 3 = x_4 = 185{,}811$ m

$2 - 4 = x_5 = 109{,}707$ m

$1 - 4 = x_6 = 227{,}058$ m

$$\boldsymbol{x}^T = (x_1 x_2 x_3 x_4 x_5 x_6)$$

Unknowns:

$$y^T = (y_1 y_2 y_3)$$

According to Formula (21), the system of observation equations yields

$$\begin{aligned} r_1 + 117{,}342 &= y_1 \\ r_2 + 68{,}454 &= y_2 \\ r_3 + 41{,}265 &= y_3 \\ r_4 + 185{,}811 &= y_1 + y_2 \\ r_5 + 109{,}707 &= y_2 + y_3 \\ r_6 + 227{,}058 &= y_1 + y_2 + y_3 \end{aligned}$$

As there already is a linear equation system, this can immediately be written using the matrix [see Formulae (24) and (23)]:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } x = l = \begin{pmatrix} 117{,}342 \\ 68{,}454 \\ 41{,}265 \\ 185{,}811 \\ 109{,}707 \\ 227{,}058 \end{pmatrix}$$

With $P = E$ the normal matrix is obtained [see Formula (27)]:

$$N = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \text{ and the vector } n = \begin{pmatrix} 530{,}211 \\ 591{,}030 \\ 378{,}030 \end{pmatrix}$$

The solution vector yields

$$y = N^{-1} n = \begin{pmatrix} 0{,}5 & -0{,}25 & 0 \\ -0{,}25 & 0{,}5 & -0{,}25 \\ 0 & -0{,}25 & 0{,}5 \end{pmatrix} \times \begin{pmatrix} 530{,}211 \\ 591{,}030 \\ 378{,}030 \end{pmatrix} = \begin{pmatrix} 117{,}348\,0 \\ 68{,}454\,7 \\ 41{,}257\,5 \end{pmatrix}$$

Finally, the residuals can be calculated according to Formula (25) by

$$A \times y - x = r$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} 117{,}348\,0 \\ 68{,}454\,7 \\ 41{,}257\,5 \end{pmatrix} - \begin{pmatrix} 117{,}342 \\ 68{,}454 \\ 41{,}265 \\ 185{,}811 \\ 109{,}707 \\ 227{,}058 \end{pmatrix} = \begin{pmatrix} +0{,}006\,0 \\ +0{,}000\,7 \\ -0{,}007\,5 \\ -0{,}008\,3 \\ +0{,}005\,2 \\ +0{,}002\,2 \end{pmatrix}$$

From this, the following can be derived [see Formula (28)]:

$$s_0 = \sqrt{\frac{r^T r}{v}} = \sqrt{\frac{192{,}01 \times 10^{-6}}{6-3}} = 0{,}008$$

According to Formula (29), the following can be quoted:

$$s_{y_k} = s_0 \sqrt{Q_{y_k y_k}} = 0{,}008 \times \sqrt{0{,}5}$$

$$s_{y_k} = 0{,}0057$$

Finally, the standard uncertainty (Type A evaluation) of the output estimates y_1, y_2, y_3 yields

$u_{yk} = s_{yk} = 5{,}7$ mm, $k = 1, 2, 3$

With

$$S_x = s_0 \begin{pmatrix} \mathbf{0{,}50} & -0{,}25 & 0{,}00 & 0{,}25 & -0{,}25 & 0{,}25 \\ -0{,}25 & \mathbf{0{,}50} & -0{,}25 & 0{,}25 & 0{,}25 & 0{,}00 \\ 0{,}00 & -0{,}25 & \mathbf{0{,}50} & -0{,}25 & 0{,}25 & 0{,}25 \\ 0{,}25 & 0{,}25 & -0{,}25 & \mathbf{0{,}50} & 0{,}00 & 0{,}25 \\ -0{,}25 & 0{,}25 & 0{,}25 & 0{,}00 & \mathbf{0{,}50} & 0{,}25 \\ 0{,}25 & 0{,}00 & 0{,}25 & 0{,}25 & 0{,}25 & \mathbf{0{,}50} \end{pmatrix}$$

and

$$s_{\tilde{x}}^T = (5{,}7 \quad 5{,}7 \quad 5{,}7 \quad 5{,}7 \quad 5{,}7 \quad 5{,}7) \ \text{[mm]}$$

the standard uncertainty of the adjusted input estimates $\tilde{x}$

$u_x = s_{\tilde{x}}$, respectively

$u(\tilde{x}_j) = s(\tilde{x}_j) = 5{,}7$ mm, j = 1, 2, ..., 6

C.4 Example 4

As a measurand (input quantity), an angle was observed several times with two different instruments:

Instrument I: $x_1 = 124°\ 39'\ 16''$	Instrument II: $x_4 = 124°\ 39'\ 13''$
$x_2 = 124°\ 39'\ 04''$	$x_5 = 124°\ 39'\ 09''$
$x_3 = 124°\ 39'\ 06''$	$x_6 = 124°\ 39'\ 08''$

The standard uncertainty of a single angle measurement was specified for instrument I with u_I = 5″ and for instrument II with u_{II} = 2″. With x_0 = 124° 39′ 00″

$$\bar{x} = x_0 + \Delta\bar{x}$$

and

$$\Delta x^T = x - e \times x_0 = (16 \quad 4 \quad 6 \quad 13 \quad 9 \quad 8) \ [''] $$

$$\Delta\bar{x} = (e^T P\, e)^{-1} e^T P\, \Delta x \ ,$$

with

$$p_1 = p_2 = p_3 = \frac{s_0^2}{u_I^2}, \quad p_4 = p_5 = p_6 = \frac{s_0^2}{u_{II}^2}$$

where s_0^2 is chosen as 100.

$$P = \begin{pmatrix} 4 & & & & & 0 \\ & 4 & & & & \\ & & 4 & & & \\ & & & 25 & & \\ & & & & 25 & \\ 0 & & & & & 25 \end{pmatrix}$$

Finally,

$$(e^T P\, e)^{-1} = 1/87 \ \text{ and } \ e^T P\, \Delta x = 854$$

From this result

$$\Delta\bar{x} = \frac{854}{87} = 9{,}8 \ [''] \text{ respectively}$$

$\bar{x}$ = 124° 39′ 00″ + 9,8″ = 124° 39′ 10″

The experimental standard deviation yields

$$s(\bar{x}) = \frac{s_0}{\sqrt{e^T P\, e}} \ \text{ with } \ s_0^2 = \frac{r^T P\, r}{v}$$

with

$$\boldsymbol{r}^T = (-6{,}2 \quad 5{,}8 \quad 3{,}8 \quad -3{,}2 \quad 0{,}8 \quad 1{,}8) \ [''] \text{ and } v = 5$$

$$s_0 = \sqrt{\frac{699{,}1}{5}} = 11{,}8'' \ \text{ and } \ s(\bar{x}) = \frac{11{,}8}{\sqrt{87}} = 1{,}3''$$

For the standard uncertainty of the input quantity, the arithmetic mean $\bar{x}$, the following is finally obtained:

$$u(\bar{x}) = s(\bar{x}) \ = \mathbf{1{,}3''}$$

C.5 Example 5

From different levelling lines, the measurands are known for the forward and backward readings of levelling staffs. To calculate the uncertainty, Formulae (41) to (50) can be applied.

The given heights are

$$\boldsymbol{l}_1^T = (10{,}473 \quad -15{,}213 \quad 28{,}775 \quad 12{,}742 \quad 13{,}155 \quad -6{,}989) \ \ [\mathrm{m}]$$

and

$$\boldsymbol{l}_2^T = (10{,}466 \quad -15{,}211 \quad 28{,}780 \quad 12{,}732 \quad 13{,}155 \quad -6{,}986) \ \ [\mathrm{m}]$$

Thus, the arithmetic mean $\bar{y}_i$, respectively the vector, is obtained:

$$\bar{\boldsymbol{y}}^T = (10{,}469\,5 \quad -15{,}212\,0 \quad 28{,}777\,5 \quad 12{,}737\,0 \quad 13{,}155\,0 \quad -6{,}987\,5) \ \ [\mathrm{m}]$$

and the differences

$$\boldsymbol{d}^T = (-7 \quad +2 \quad +5 \quad -10 \quad 0 \quad +3) \ \ [\mathrm{mm}]$$

As all observations l_j, with $j = 1, 2$, are from the same uncertainty level, the experimental standard deviation for the heights

$$s_l = \sqrt{\frac{187}{12}} = 3{,}9 \ \ [\mathrm{mm}]$$

and

for the averages $\bar{y}_i, i = 1, 2, ..., 6$

$$s(\bar{y}_i) = \sqrt{\frac{187}{24}} = 2{,}8 \ \ [\mathrm{mm}]$$

To check the condition $E(d) = 0$ the following is obtained from Formula (49):

$$(7)^2 < 187$$

This means that the condition is true and that the standard uncertainties can be written:

$u(l_j) = s_l = 3{,}9$ mm and $u(\bar{y}_i) = s(\bar{y}_i) = 2{,}8$ mm

C.6 Example 6

From a given Point $P_0(x, y, H)$, the coordinates (measurands) of a new point P were determined by the polar method using only face I observations (see Figure C.1).

Given:

Coordinates of P_0:

x_0 = 12 345,678 m y_0 = 87 654,321 m

$s(x_0)$ = 1,8 cm $s(y_0)$ = 1,6 cm

Bearing: t_A = 309,090 9

$s(t_A)$ = 1,3″

Measured:

Angle: α = 89,999 9° s(α) = 1,7″

Horizontal distance: D = 326,111 6 m

(taken from Example 2)

$u(D)$ = 3,0 mm

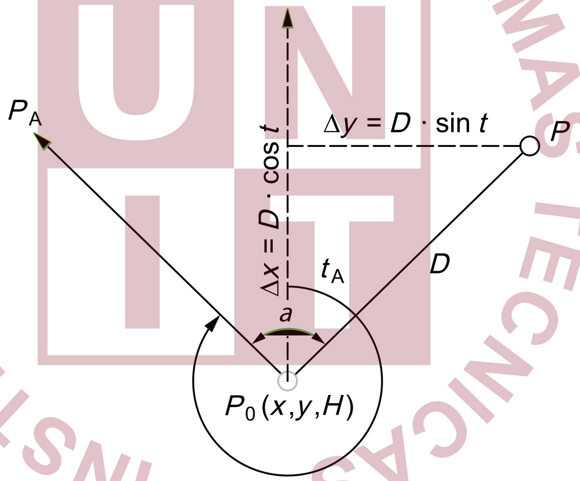


Figure C.1 — Polar survey

For the uncertainty evaluation, the following mathematical model is used:

$$y = f(x^T)$$

or

$$\begin{pmatrix} x \\ y \\ H \end{pmatrix} = \begin{pmatrix} x_0 + \Delta x \\ y_0 + \Delta y \\ H_0 + \Delta h \end{pmatrix}$$

Here only the calculation for the *x*-coordinate is exemplarily pursued:

$$x(P) = x_0 + \Delta x = x_0 + D \times \cos(\alpha + t_A)$$

In consideration of the collimation error, *c*, and the tilting axis error, *i*, the model has to be extended by the equivalent correction k_c and k_i (here directly attributed to the horizontal angle α due to sightings under different zenith angles):

$$x(P) = x_0 + D \times \cos(\alpha + k_c + k_i + t_A)$$

$$x(P) = 12\,345{,}678 + 326{,}111\,6 \times \cos(89{,}999\,9° + 0{,}003\,2° + 0{,}004\,3° + 309{,}090\,9°)$$

$$x(P) = 12\,345{,}678 + 253{,}084 = 12\,598{,}762 \text{ m}$$

To calculate the uncertainty, it is convenient to use tabular form in analogy to 4.2.2 and 4.4.

Additional uncertainty influences can still be estimated using Type B evaluation according to 4.3.

Centring excentricity, *e*, of the instrument:

With *e* = ± 3 mm and assuming a probability for this interval of 100 %, a standard uncertainty [see Formula (57)], is yielded:

$$u(e) = 0{,}58 \times e = 1{,}7 \text{ mm}$$

Sensitivity coefficient: c7 = 1

Horizontal refraction:

With an estimated influence of *r* = ± 7″ and assuming for this estimation a probability of 50 %, a standard uncertainty [see Formula (55)] is yielded:

$$u(r) = 1{,}48 \times r = 10{,}4''$$

Sensitivity coefficient: $c_8 = D \times \sin(\alpha + k_c + k_i + t_A) = 206$

Applying the law of propagation of uncertainty according to 4.4, yields

$$U_{x(A)} = \begin{pmatrix} u(x_0)^2 & & 0 \\ & u(D)^2 & \\ 0 & & u(t_A)^2 \end{pmatrix}, \; U_{x(B)} = \begin{pmatrix} u(\alpha)^2 & & & & 0 \\ & u(k_c)^2 & & & \\ & & u(k_i)^2 & & \\ & & & u(e)^2 & \\ 0 & & & & u(r)^2 \end{pmatrix}$$

and

$$U_{x(P)} = \begin{pmatrix} U_{x(A)} & 0 \\ 0 & U_{x(B)} \end{pmatrix}$$

With

$$\boldsymbol{c}^T = (1 \quad 0{,}78 \quad 206 \quad 206 \quad 206 \quad 206 \quad 1 \quad 206)$$

according to Formula (66), the combined standard uncertainty of the output estimate, the x-coordinate, can finally be stated:

$u[x(P)] = 21{,}1$ mm

The final result including the expanded uncertainty $\pm U$ ($k = 2$) is given by

$x(P) = (12\,598{,}762 \pm 0{,}042)$ m

$u_c[x(P)] = 21{,}1$ mm

$U[x(P)] = 2 \times u_c[x(P)] = \pm 42$ mm

NOTE Calculation is done always with full accuracy but intermediate results are shown as rounded numbers.

Table C.1 — Uncertainty budget

Input quantity X_i	Input estimates x_i [dim]	Standard uncertainty $u(x_j)$ [dim]	Distribution	Sensitivity coefficients[a] $c_i \equiv \partial f / \partial x_i$ [dim]	$u(x_i) \equiv c_i \times u(x_i)$ [mm]	Type of evaluation, source of uncertainty
x_0	12 345,678 m	18 mm	normal	1	18	A, estimation from previous least-squares adjustment
D	326,111 6 m	3,0 mm	normal	0,78	2,3	A, combined standard uncertainty
α	89,999 9° 1,570 795 rad	1,7″ 0,008 2 mrad	normal	206 m	1,7	B, random influences, experiences
k_c	0,003 2° 0,061 mrad	1″ 0,004 8 mrad	rectangular	206 m	1,0	B, general knowledge of the behaviour
k_i	0,004 3° 0,075 mrad	1″ 0,004 8 mrad	rectangular	206 m	1,0	B, general knowledge of the behaviour
t_A	309,090 9° 5,394 654 rad	1,3″ 0,006 3 mrad	normal	206 m	1,3	A, estimation from previous least-squares adjustment
e	0	1,7 mm	rectangular	1	1,7	B, centring eccentricity
r	0	10,4″ 0,050 2 mrad	normal	206 m	10,3	B, horizontal refraction
Output estimate, final result	**12 598,762 m**				**21,1 mm**	

a The partial derivates used in Formulae (12) or (17) are often called the sensitivity coefficients.

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