1.7.7 Modeling of Complex Flow Phenomena

Many flows of practical interest are difficult to describe exactly mathematically, let alone solve exactly. These flows include turbulence, combustion, multiphase flow, and are very important. Since exact description is often impracticable, one usually uses semi-empirical models to represent these phenomena. Examples are turbulence models (which will be treated in some detail in Chap. 9), combustion models, multiphase models, etc. These models, as well as the above mentioned simplifications affect the accuracy of the solution. The errors introduced by the various approximations may either augment or cancel each other; therefore, care is needed when drawing conclusions from calculations in which models are used. Due to the importance of various kinds of errors in numerical solutions we shall devote a lot of attention to this topic. The error types will be defined and described as they are encountered.

1.8 Mathematical Classification of Flows

Quasi-linear second order partial differential equations in two independent variables can be divided into three types: hyperbolic, parabolic, and elliptic. This distinction is based on the nature of the characteristics, curves along which information about the solution is carried. Every equation of this type has two sets of characteristics.

In the hyperbolic case, the characteristics are real and distinct. This means that information propagates at finite speeds in two sets of directions. In general, the information propagation is in a particular direction so that one datum needs to be given at an initial point on each characteristic; the two sets of characteristics therefore demand two initial conditions. If there are lateral boundaries, usually only one condition is required at each point because one characteristic is carrying information out of the domain and one is carrying information in. There are, however, exceptions to this rule.

In parabolic equations the characteristics degenerate to a single real set. Consequently, only one initial condition is normally required. At lateral boundaries one condition is needed at each point.

Finally, in the elliptic case, the characteristics are imaginary or complex so there are no special directions of information propagation. Indeed, information travels essentially equally well in all directions. Generally, one boundary condition is required at each point on the boundary and the domain of solution is usually closed although part of the domain may extend to infinity. Unsteady problems are never elliptic.

These differences in the nature of the equations are reflected in the methods used to solve them. It is an important general rule that numerical methods should respect the properties of the equations they are solving.

The Navier-Stokes equations are a system of non-linear second order equations in four independent variables. Consequently the classification scheme does not apply directly to them. Nonetheless, the Navier-Stokes equations do possess many of the properties outlined above and the many of the ideas used in solving second order equations in two independent variables are applicable to them but care must be exercised.

1.8.1 Hyperbolic Flows

To begin, consider the case of unsteady inviscid compressible flow. A compressible fluid can support sound and shock waves and it is not surprising that these equations have essentially hyperbolic character. Most of the methods used to solve these equations are based on the idea that the equations are hyperbolic and, given sufficient care, they work quite well; these are the methods referred to briefly above.

For steady compressible flows, the character depends on the speed of the flow. If the flow is supersonic, the equations are hyperbolic while the equations for subsonic flow are essentially elliptic. This leads to a difficulty that we shall discuss further below.

It should be noted however, that the equations for a viscous compressible flow are still more complicated. Their character is a mixture of elements of all of the types mentioned above; they do not fit well into the classification scheme and numerical methods for them are difficult to construct.

1.8.2 Parabolic Flows

The boundary layer approximation described briefly above leads to a set of equations that have essentially parabolic character. Information travels only downstream in these equations and they may be solved using methods that are appropriate for parabolic equations.

Note, however, that the boundary layer equations require specification of a pressure that is usually obtained by solving a potential flow problem. Subsonic potential flows are governed by elliptic equations (in the incompressible limit the Laplace equation suffices) so the overall problem actually has a mixed parabolic-elliptic character.

1.8.3 Elliptic Flows

When a flow has a region of recirculation i.e. flow in a sense opposite to the principal direction of flow, information may travel upstream as well as downstream. As a result, one cannot apply conditions only at the upstream end of the flow. The problem then acquires elliptic character. This situation occurs in subsonic (including incompressible) flows and makes solution of the equations a very difficult task.

It should be noted that unsteady incompressible flows actually have a combination of elliptic and parabolic character. The former comes from the fact that information travels in both directions in space while the latter results from the fact that information can only flow forward in time. Problems of this kind are called incompletely parabolic.

1.8.4 Mixed Flow Types

As we have just seen, it is possible for a single flow to be described by equations that are not purely of one type. Another important example occurs in steady transonic flows, that is, steady compressible flows that contain both supersonic and subsonic regions. The supersonic regions are hyperbolic in character while the subsonic regions are elliptic. Consequently, it may be necessary to change the method of approximating the equations as a function of the nature of the local flow. To make matters even worse, the regions can not be determined prior to solving the equations.

1.9 Plan of This Book

This book contains twelve chapters. We now give a brief summary of the remaining eleven chapters.

In Chap. 2 an introduction to numerical solution methods is given. The advantages and disadvantages of numerical methods are discussed and the possibilities and limitations of the computational approach are outlined. This is followed by a description of the components of a numerical solution method and their properties. Finally, a brief description of basic computational methods (finite difference, finite volume and finite element) is given.

In Chap. 3 finite difference (FD) methods are described. Here we present methods of approximating first, second, and mixed derivatives, using Taylor series expansion and polynomial fitting. Derivation of higher-order methods, and treatment of non-linear terms and boundaries is discussed. Attention is also paid to the effects of grid non-uniformity on truncation error and to the estimation of discretization errors. Spectral methods are also briefly described here.

In Chap. 4 the finite volume (FV) method is described including the approximation of surface and volume integrals and the use of interpolation to obtain variable values and derivatives at locations other than cell centers. Development of higher-order schemes and simplification of the resulting algebraic equations using the deferred-correction approach is also described. Finally, implementation of the various boundary conditions is discussed.

Applications of basic FD and FV methods are described and their use is demonstrated in Chaps. 3 and 4 for structured Cartesian grids. This restriction allows us to separate the issues connected with geometric complexity