11. Determinar en qué puntos de $\mathbb{R}^2$ las siguientes funciones $f : \mathbb{R}^2 \mapsto \mathbb{R}$ son continuas y discontinuas.

(a) $f(x,y) = \begin{cases} (4x^2y^3)/(4x^2+y^6) & \text{si } (x,y) \neq (0,0) \\ 0 & \text{si } (x,y) = (0,0) \end{cases}$

(b) $f(x,y) = \begin{cases} x/y & \text{si } y \neq 0 \\ 0 & \text{si } y = 0 \end{cases}$

(c) $f(x,y) = \begin{cases} x^2+2y-1 & \text{si } x \geq 0 \\ 3x+y^2 & \text{si } x < 0 \end{cases}$

$f : D \to \mathbb{R}^d$, $x \in D$

· f es continua en a si $\lim_{x \to a} f(x) = f(a)$

(a) que f sea continua equivale a que

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$$

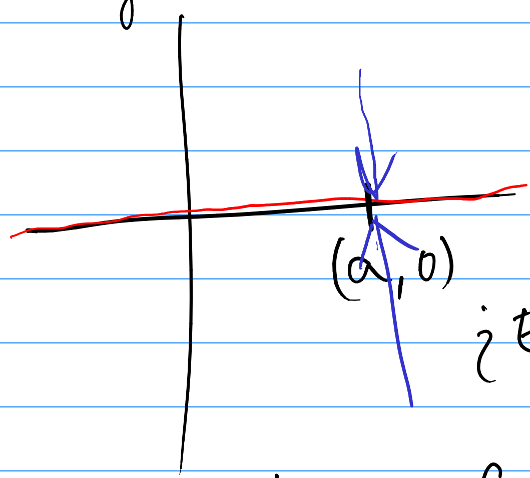
$$\lim_{(x,y)\to(0,0)} \frac{4x^2y^3}{4x^2+y^6} = \lim_{(x,y)\to(0,0)} \overbrace{y^3}^{\to 0} \overbrace{\left(\frac{4x^2}{4x^2+y^6}\right)}^{\text{acotado}} = 0$$

$$0 < \frac{4x^2}{4x^2+y^6} \leq 1$$

si $(x,y) \neq (0,0)$; $y^6 \geq 0$

f es continua en $\mathbb{R}^2$

(b) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases} x/y & y\neq 0 \\ 0 & y=0 \end{cases}$



$\{(a,0) : x\in\mathbb{R}\}$

¿Es continua f en estos puntos?

¿ $\lim\limits_{(x,y)\to(a,0)} f(x,y) = f(a,0)$?

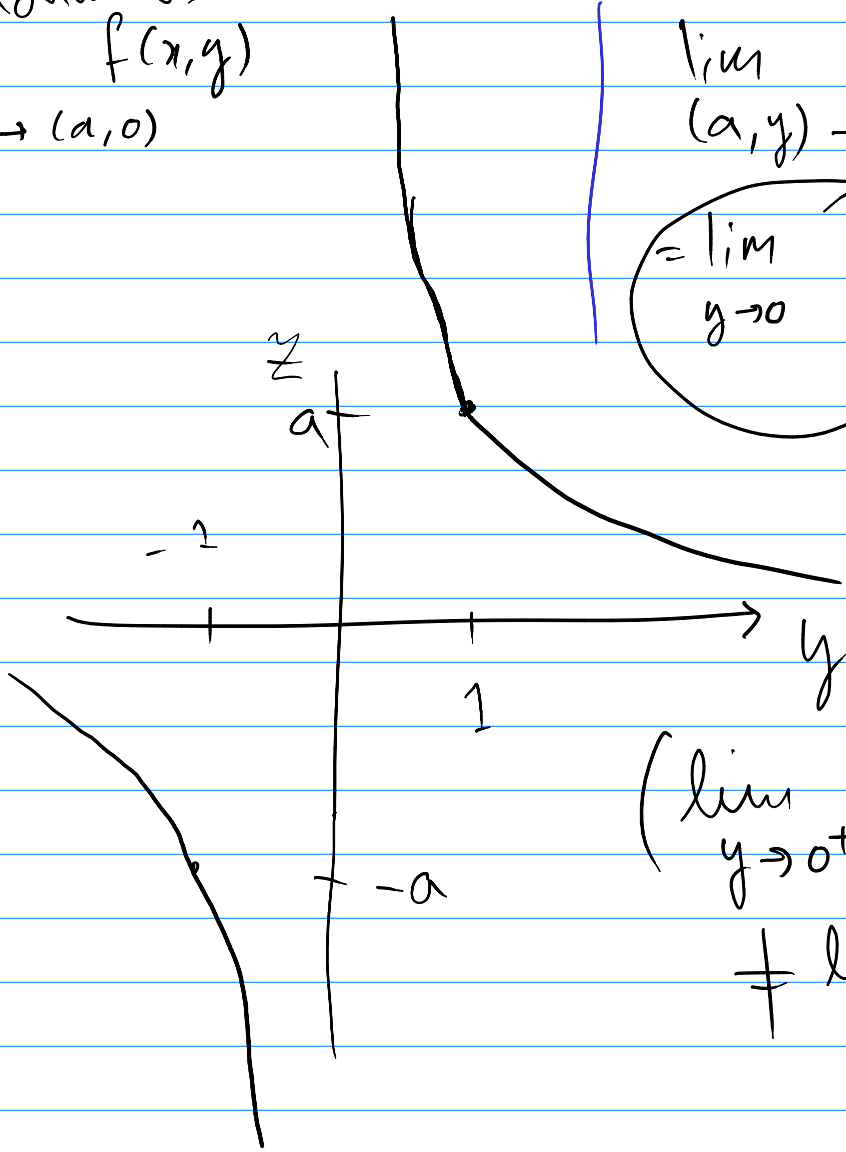
Estudiamos:

$\lim\limits_{(x,y)\to(a,0)} f(x,y)$

$\lim\limits_{(a,y)\to(a,0)} f(x,y) = \lim\limits_{y\to 0} \frac{a}{y}$

No existe el límite

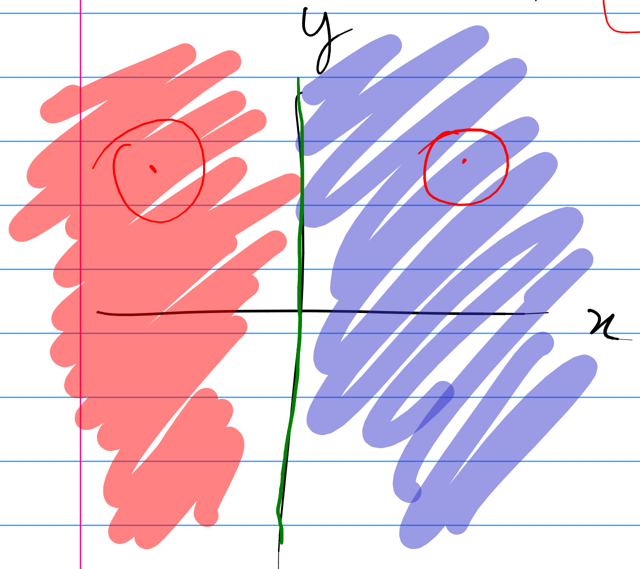
$\frac{a}{y}$



$\left(\lim\limits_{y\to 0^+} \frac{a}{y} = +\infty \neq \lim\limits_{y\to 0^-} \frac{a}{y} = -\infty\right)$

La función es continua en $\mathbb{R}^2 \setminus \{y=0\}$

(c) $f(x,y) = \begin{cases} x^2+2y-1 & x \geq 0 \\ 3x+y^2 & x<0 \end{cases}$



$\{x=0\} = \{(0,a),\ a\in\mathbb{R}\}$

$\lim_{(x,y)\to(0,a)} f(x,y)$

$x^2+2y-1 = 3x+y^2$

$x^2-3x-1 = y^2-2y$

$x=0 \downarrow$

$-1 = y^2-2y$

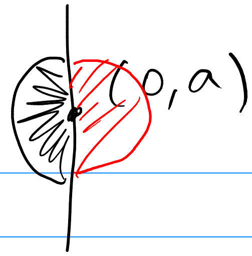
$y^2-2y+1 = 0$

$\frac{2\pm\sqrt{4-4}}{2} = 1$

$(0,1)$

$\lim_{(x,y)\to(0,1)} f(x,y) = \lim_{(x,y)\to(0,1)} x^2+2y-1 = \lim_{(x,y)\to(0,1)} 3x+y^2$

$= 1$



$\{(0,a),\ a\in\mathbb{R}\setminus\{1\}\}$

$$\lim_{(x,y)\to(0,a)} f(x,y)$$

$$\lim_{x\to 0^+} f(x,a) \qquad\qquad \lim_{x\to 0^-} f(x,a)$$

$$\parallel \qquad\qquad \parallel$$

$$\lim_{x\to 0^+} x^2+2a-1 \qquad\qquad \lim_{x\to 0^-} 3x+a^2$$

$$\parallel \qquad\qquad \parallel$$

$$2a-1 \quad \neq \quad a^2$$

Hay dos maneras de acercarme a $(0,a)$ cuyos límites no coinciden

→ No existe $\lim_{(x,y)\to(0,a)} f(x,a)$

$$\lim_{(x,y)\to(0,1)} f(x,y) = \underbrace{\lim_{(x,y)\to(0,1)} x^2+2y-1}_{} = \underbrace{\lim_{(x,y)\to(0,1)} 3x+y^2}_{}$$

$$= 1$$

$f(0,1) = 1$

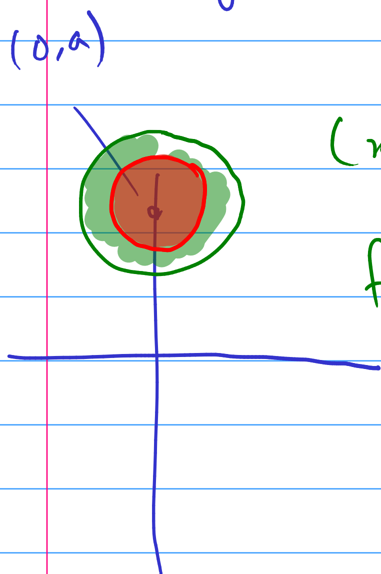
$$\lim_{(x,y)\to(0,1)} f(x,y) = 1$$

$$\forall \varepsilon>0 \quad \exists \delta>0 : \quad \forall (x',y') \in B((0,1),\delta)$$

$$f(x',y') \in B(1,\varepsilon)$$

fijo $\varepsilon>0$ Para la función $\overbrace{x^2+2y-1}^{f_1}$

$\exists \delta_1$ que cumple la def. de continuidad



$$(x',y') \in B((0,1),\delta_1)$$

$$f_1(x',y') \in B(1,\varepsilon)$$

Para la función $\overbrace{3x+y^2}^{f_2}$

$\exists \delta_2$

$$(x',y') \in B((0,1),\delta_2)$$

$$f_2(x',y') \in B(1,\varepsilon)$$

Si tomamos $\delta = \min\{\delta_1, \delta_2\}$

Tenemos que $f(x,y) \in B(1,\varepsilon)$

$$\begin{cases} x \geq 0 & f_1 \in B(1,\varepsilon) \\ x < 0 & f_2 \in B(1,\varepsilon) \end{cases}$$

(porque $\delta \leq \delta_1$, $\delta \leq \delta_2$)

12. ¿Cuáles de las siguientes funciones se pueden extender en forma continua a todo el plano?

(a) $\dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}$ (b) $x^2\log(x^2+y^2)$ (c) $\dfrac{\operatorname{sen}(x^4+y^4)}{x^2+y^2}$

12)

$$\frac{\operatorname{sen}(x^2+y^2)}{x^2+y^2} \leftarrow \quad x^2+y^2 \geq 0$$

$(0,0)$ no esta definido $\dfrac{\operatorname{sen}(0^2)}{0^2+0^2}$

Si $\exists \lim\limits_{(x,y)\to(0,0)} \dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2}$

$x = r\cos\theta$

$y = r\operatorname{sen}\theta$

$$\frac{\operatorname{sen}(r^2(\cos^2\theta+\operatorname{sen}^2\theta))}{r^2(\cos^2\theta+\operatorname{sen}^2\theta)}$$

$$= \frac{\operatorname{sen}(r^2)}{r^2}(1)$$

$$\lim_{(x,y)\to(0,0)} \frac{\operatorname{sen}(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0^+} \frac{\operatorname{sen}(r^2)}{r^2}$$

$$= \lim_{r\to 0^+} \frac{r^2}{r^2} = 1$$

$$f(x,y) = \begin{cases} \dfrac{\operatorname{sen}(x^2+y^2)}{x^2+y^2} & (x,y)\neq(0,0) \\ 1 & (x,y)=(0,0) \end{cases}$$

f es una extension continua de $\operatorname{sen}(x^2+y^2)/(x^2+y^2)$

(b) $x^2 \log(x^2+y^2)$

$\log : \{x > 0\} \to \mathbb{R}$

$x^2+y^2 \geq 0$

$$\lim_{(x,y)\to(0,0)} x^2 \log(x^2+y^2)$$

$x = r\cos\theta$ $\quad\to\quad$ $r^2 \cos\theta \log(r^2)$

$y = r\sin\theta$ $\qquad\qquad$ $r^2 \log(r^2) \cos\theta$

$$\lim_{r\to 0^+} r^2 \log(r^2)\cos\theta =$$

L'H $\displaystyle \lim_{r\to 0^+} \frac{\log(r^2)}{r^{-2}} = \lim_{r\to 0} \left(\frac{2r}{r^2}\right) \cdot r^3)$

$$\left(\frac{1}{r^2}\right)' = \frac{-1}{r^3} \qquad\qquad = \lim_{r\to 0} 2r^2 = 0$$