$$f(x,y) = \begin{cases} \dfrac{xy}{x+y} & x \neq -y \\ 0 & x = -y \end{cases}$$

Existe $\lim_{(x,y)\to(0,0)} f(x,y)$

límite dirección (x, ax)

$$\lim_{x\to 0} f(x,ax) = \lim_{x\to 0} \frac{xax}{x+ax}$$

$$= \lim_{x\to 0} \frac{x^2 a}{x(1+a)} = \lim_{x\to 0} \frac{xa}{1+a}$$

¿ $g(a) = \dfrac{a}{1+a}$ es acotada? No se puede concluir

Si fijamos $a \neq 0$ $\lim x\left(\dfrac{a}{1+a}\right) = 0$

$x = r\cos\theta$
$y = r\,\text{sen}\,\theta$

$$g(r,\theta) = f(r\cos\theta, r\,\text{sen}\,\theta) = \frac{r^2\cos\theta\,\text{sen}\,\theta}{r(\cos\theta + \text{sen}\,\theta)} = k(r)\cdot h(\theta)$$

$$h(\theta) = \frac{\cos\theta\,\text{sen}\,\theta}{\cos\theta + \text{sen}\,\theta}$$

$h(\theta)$ no es acotada
$\Rightarrow$ No se puede usar polares.

(x_n, y_n) con $x_n \xrightarrow{n} 0$

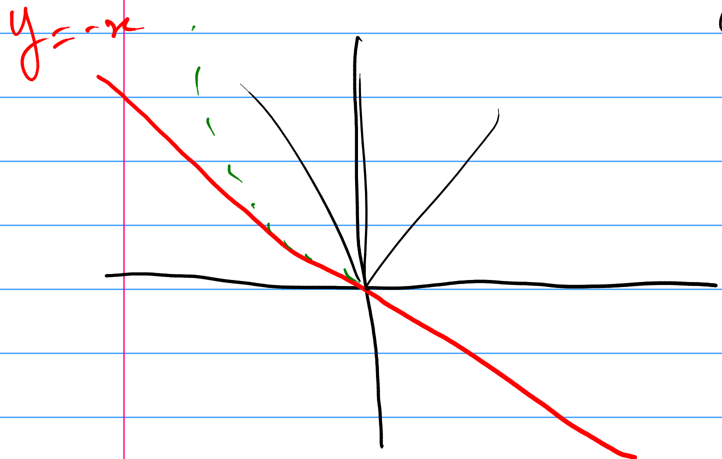
$y_n \xrightarrow{n} 0$

Calculamos $f(x_n, y_n) = \dfrac{x_n y_n}{x_n + y_n}$ si $x_n \neq -y_n$

Como $\lim\limits_{x \to 0} f(x, ax) = 0 \quad \forall a \neq 0$

Para que el $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exista

$f(x_n, y_n) = 0 \quad \forall x_n, y_n$ sucesiones convergentes a 0.



$x_n = \dfrac{1}{n}$

$y_n = -\dfrac{1}{n} + \dfrac{1}{n^2}$

$$\lim_{n\to\infty} f(x_n, y_n) = \lim_{n\to+\infty} \frac{\frac{1}{n}\left(-\frac{1}{n} + \frac{1}{n^2}\right)}{\cancel{\frac{1}{n}} + \left(\cancel{-\frac{1}{n}} + \frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{-\frac{1}{n^2} + \frac{1}{n^3}}{\left(\frac{1}{n^2}\right)} = \lim_{n\to+\infty} -1 + \frac{1}{n}$$

$$= -1$$

No existe $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

7. Decidir si los límites siguientes existen y en caso afirmativo calcularlos.

(a) $\lim_{(x,y,z)\to(2,5,3)} \frac{x-y}{x^2+y-z}$ (b) $\lim_{(x,y,z)\to(1,0,1)} \frac{\operatorname{sen}(x^2+e^y-z)}{x^2+\tan(\frac{1}{\cos(xyz)})}$ (c) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2yz-z^4}{x^4+y^4+z^4}$

8. Calcular:

(a) $\lim_{(x,y)\to(1,2)} \frac{x^2+xy+1}{x^2-x-y}$ (b) $\lim_{(x,y)\to(0,0)} x\,y\log|y|$ (c) $\lim_{(x,y)\to(1,1)} \frac{x^2+xy-2y^2}{x^2-y^2}$

(d) $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ (e) $\lim_{(x,y)\to(0,0)} \frac{e^{x-y}-1}{x^2-y^2}$ (f) $\lim_{(x,y)\to(0,0)} \frac{\log(1+x^2+y^2)}{x^2+y^2+x^3y}$

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x-y}-1}{x^2-y^2}$$

$$\lim_{x\to 0} e^x-1 = \lim_{x\to 0} x$$

$x = r\cos\theta$

$y = r\operatorname{sen}\theta$

$$\frac{e^{-r\cos\theta - r\operatorname{sen}\theta}-1}{r^2\cos^2\theta - r^2\operatorname{sen}^2\theta} = f(r)\cdot h(\theta)$$

$$\frac{e^{-r(\cos\theta+\operatorname{sen}\theta)}-1}{r^2(\cos^2\theta-\operatorname{sen}^2\theta)} = \frac{e^{-r}}{r^2}\left(\frac{e^{\cos\theta+\operatorname{sen}\theta}}{\quad}\right.$$

$$e^{ab} = e^a\cdot e^b$$

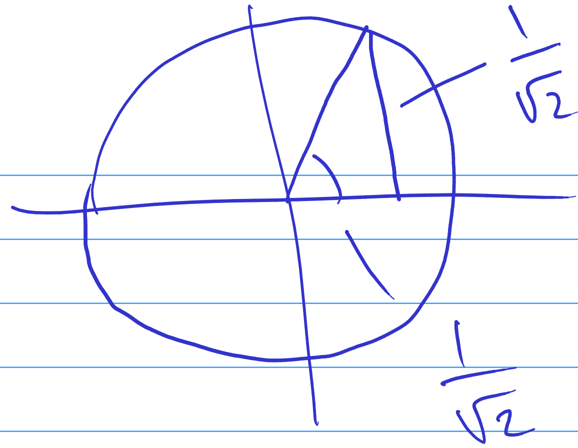
$$1 = \cos^2\theta + \operatorname{sen}^2\theta$$

$$-\operatorname{sen}^2\theta = \cos^2\theta - 1$$

$$\frac{e^{-r}}{r^2}\left(\frac{e^{\cos\theta+\operatorname{sen}\theta}}{2\cos^2\theta-1}\right)$$

no es acotada

$\cos^2\theta = \frac{1}{2}$, $\cos\theta = \frac{1}{\sqrt{2}}$



$\theta = \pi/4$

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

$$\lim_{x \to 0} f(x,x) = \lim_{x \to 0} \frac{e^{-2x} - 1}{x^2 - x^2}$$

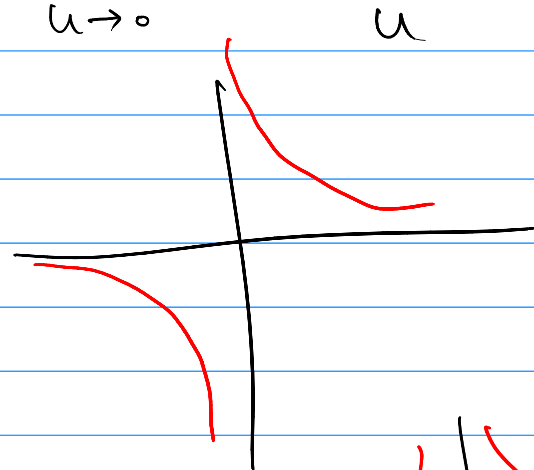
$$= \lim_{x \to 0} \frac{-2x}{x^2 - x^2}$$

$$= \lim_{x \to 0} \frac{-2}{x - x} \nexists$$

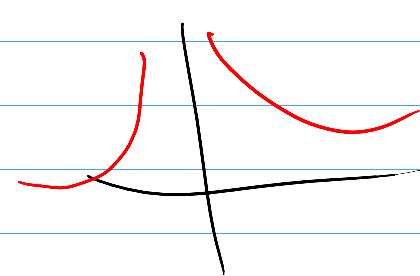
$u = x - x \to 0$

$$\lim_{u \to 0} \frac{-2}{u} \nexists$$

$$\lim_{x \to 0} \frac{1}{x} \nexists$$



$$\lim_{x \to 0} \left|\frac{1}{x}\right| = +\infty$$



9. Se considera la función

$$f(x,y) = \frac{ax + y + by^2}{\operatorname{sen} y + \log(1+x)} \qquad a, b \in \mathbb{R}.$$

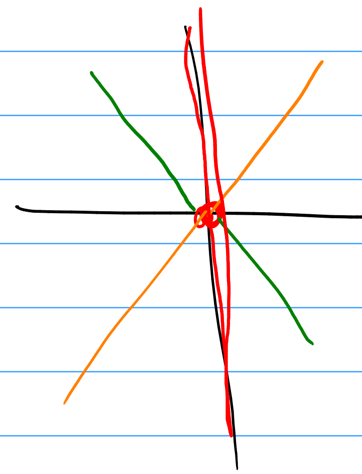
a) Determinar a y b para que todos los límites direccionales de f en $(0,0)$ sean iguales.

b) Para los a y b determinados en la parte anterior, probar que f carece de límite.

10. Discutir según $\alpha, \beta \in \mathbb{R}$ la existencia del límite:

$$\lim_{(x,y)\to(0,0)} \frac{x^\alpha y^\beta}{x^2 + xy + y^2}$$

a) $\boxed{\begin{array}{l} \lim\limits_{x\to 0} f(x, mx) \\ \\ \lim\limits_{y\to 0} f(0,y) \end{array}}$ $\quad m \in \mathbb{R}$



$$\lim_{x\to 0} \frac{ax + mx + bm^2x^2}{\operatorname{sen} mx + \log(1+x)}$$

L'Hopital $\quad \lim\limits_{x\to 0} ax + mx + bm^2x^2 = 0$

$\quad \lim\limits_{x\to 0} \operatorname{sen}(mx) + \log(1+x) = 0$

$$\lim_{x\to 0} \frac{a + m + 2bm^2x}{\underbrace{m\cos(mx) +}_{1} \frac{1}{1+x}} = \frac{a+m}{m+1} = \frac{m+a}{m+1}$$

Para que $\frac{m+a}{m+1} = 1$ $\forall m \to a = 1$

¿qué pasa si $m = -1$?

L'Hôpital

Queremos que sea

$$\lim_{x\to 0} \frac{2bm^2}{-m^2 \operatorname{sen}(mx) + \frac{1}{(1+x)^2}} = 1$$

Si $m=-1$

$$\lim_{x\to 0} \frac{2b}{-\operatorname{sen}(-x) - \frac{1}{(1+x)^2}} = -2b = 1$$

$$b = -\frac{1}{2}$$

$$\frac{x+y+y^2/2}{\operatorname{sen}(y)+\log(1+x)}$$

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{y + y^2/2}{\operatorname{sen}(y)}$$

$$= \lim_{y\to 0} \frac{y(y/2+1)}{\operatorname{sen}(y)} = 1$$

$$\lim_{y\to 0} \frac{y}{\operatorname{sen}(y)} = 1$$

La dirección "problemática" es $m=-1$

$y=-x$

$$\frac{x+y-y^2/2}{\operatorname{sen}(y)+\log(1+x)}$$

$$(x, -x+x^2)$$

$$\lim_{x\to 0} f(x, -x+x^2) =$$

$$= \lim_{x\to 0} \frac{x + (-x+x^2) - (-x+x^2)^2/2}{\operatorname{sen}(-x+x^2) + \log(1+x)}$$

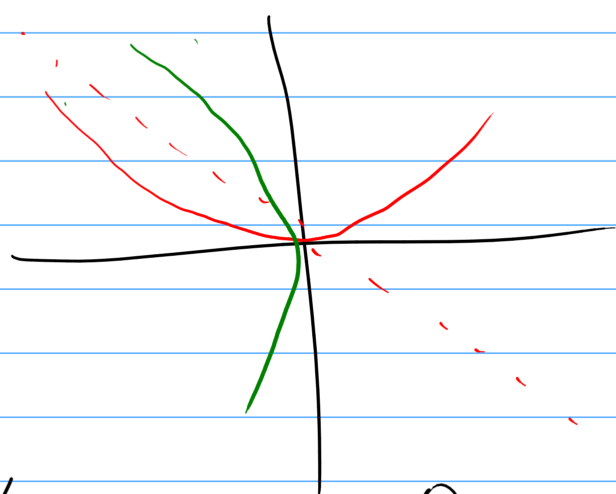
$$= \lim_{x\to 0} \frac{x^2 - (x^2 - 2x^3 + x^4)/2}{\operatorname{sen}(-x+x^2) + \log(1+x)}$$

Se complican las cuentas

$x = y - y^2/2$

$$\frac{x + y - y^2/2}{\operatorname{sen}(y) + \log(1+x)}$$

$$\lim_{y\to 0} f(y+y^2/2, y)$$



$$\lim_{y\to 0} \frac{-y + y^2/2 + y - y^2/2}{\quad} = 0$$

L'Hopital prueban que $\lim = 0$

$$\lim_{\to(1,1)} \frac{x^2 + xy - 2y^2}{x^2 - y^2}$$

$$\lim_{y\to 1} \cancel{f(0,y)} = \cancel{\lim \frac{-2y^2}{-y^2} = 2}$$

$$f(1,y) = \lim_{y\to 1} \frac{1+y-2y^2}{1-y^2}$$