1. Indicar si las siguientes series son convergentes o no, hallando sus suma en caso de serlo.

$$a)\ \sum_{n=0}^{+\infty}\left(\frac{1}{3}\right)^n \qquad b)\ \sum_{n=1}^{+\infty}\left(\frac{1}{\sqrt{3}}\right)^{n+3} \qquad c)\ \sum_{n=1}^{+\infty}5^{n+1} \qquad d)\ \sum_{n=1}^{+\infty}\frac{3}{n(n+3)}$$

$$e)\ \sum_{n=1}^{+\infty}\log\left(\frac{n^2+2n+1}{n^2}\right) \qquad f)\ \sum_{n=1}^{+\infty}\frac{n}{(n+1)(n+2)(n+3)} \qquad g)\ \sum_{n=1}^{\infty}\frac{n\,arctg(n+1)-(n+1)\,arctg(n)}{n(n+1)}$$

a) $\displaystyle\sum_{n=0}^{K}\left(\frac{1}{3}\right)^n = b_K \qquad \lim_{n\to\infty} b_K$

$$1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots+\left(\frac{1}{3}\right)^K = \frac{1-\left(\frac{1}{3}\right)^{K+1}}{\left(1-\left(\frac{1}{3}\right)\right)}$$

$$\left(1+\frac{1}{3}+\cdots+\left(\frac{1}{3}\right)^K\right)\left(1-\frac{1}{3}\right) =$$

$$1+\cancel{\frac{1}{3}}+\cancel{\cdots}\ \cancel{\left(\frac{1}{3}\right)^K}-\cancel{\frac{1}{3}}-\cancel{\left(\frac{1}{3}\right)^2}-\cancel{\cdots}-\left(\frac{1}{3}\right)^{K+1} = 1-\left(\frac{1}{3}\right)^{K+1}$$

Podemos afirmar que $b_K = \dfrac{1-\left(\frac{1}{3}\right)^{K+1}}{1-\frac{1}{3}}$

$$\lim_{K\to\infty} b_K = \lim_{K\to\infty}\frac{1-\left(\frac{1}{3}\right)^{K+1}}{1-\frac{1}{3}}$$

$$= \frac{1}{1-\frac{1}{3}} + \lim_{K\to\infty}\frac{-\left(\frac{1}{3}\right)^{K+1}}{1-\frac{1}{3}} = 0$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-1/3} = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}$$

b) $$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^{n+3} = x$$

lo puedo hacer porque $\left|\left(\frac{1}{\sqrt{3}}\right)\right| < 1$

$$\sum^{k} \left(\frac{1}{\sqrt{3}}\right)^n = \frac{1-\left(\frac{1}{\sqrt{3}}\right)^{k+1}}{1-\frac{1}{\sqrt{3}}}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{3}}} = \frac{1}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{3}-1}$$

$$x = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{3}}\right)^{n+3} = \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^5 + \cdots$$

Sabemos calcular $\sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{3}}\right) = 1 + \frac{1}{\sqrt{3}} + \cdots$

$$\left(\frac{1}{\sqrt{3}}\right)^4 \left(\sum_{n=0}^{\infty} \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^5 + \cdots = x$$

$$\left(\frac{1}{9}\right) \sum^{\infty} \frac{1}{\sqrt{3}} = x$$

$$\boxed{\frac{1}{9}\left(\frac{1}{1-1/\sqrt{3}}\right)}$$

c) $$\sum_{n=1}^{\infty} 5^{n+1} = \sum_{n=0}^{\infty} 5^{n+2}$$

$$= 5^2 + 5^3 + \cdots = \left(5^2\right)\left(\sum_{n=0}^{\infty} 5^n\right)$$

$$\sum_{n=0}^{K} 5^n = \frac{1-5^{K+1}}{1-5}$$

$$\lim_{K\to\infty} \frac{1-5^{K+1}}{1-5} = \frac{1}{1-5} - \lim \frac{5^{K+1}}{1-5} \to +\infty$$

$$= +\infty$$

$\sum_{n=0}^{\infty} a_n$ una condicion necesaria para que la serie converga es $\lim a_n = 0$

En este caso $\lim 5^{n+2} = +\infty$

e) $$\sum_{n=1}^{\infty} \log\left(\frac{n^2+2n+1}{n^2}\right)$$

Deberíamos verificar que $\lim_{n\to\infty} \log\left(\frac{n^2+2n+1}{n^2}\right) = 0$

de lo contrario, la serie diverge

$$\lim_{n\to+\infty} \log\left(\frac{n^2+2n+1}{n^2}\right) = \lim_{n\to+\infty} \log\left(\frac{(n+1)^2}{n^2}\right) =$$

$$\lim_{K\to\infty} 2\log\left(\frac{n+1}{n}\right) = \lim 2\left(\log(n+1) - \log n\right)$$

$$= 0$$

$$\sum_{n=1}^{\infty} 2\log\left(\frac{n+1}{n}\right) = 2\left(\sum_{n=1}^{\infty} \log(n+1) - \log n\right)$$

$$= 2\Big(\cancel{\log(1+1)} - \log(1) + \cancel{\log(2+1)} - \cancel{\log(2)}$$

$$+ \log(3+1) + \cancel{\log(3)} + \cdots +$$

$$\cancel{\log(n+1)} - \log(n) + \log(n+2) - \cancel{\log(n+1)}$$

$$+ \cdots$$

Serie telescopica

$$\sum_{n=1}^{K} \log\left(\frac{n^2+2n+1}{n^2}\right) = 2\left(\sum_{n=1}^{K} (\log(n+1) - \log(n))\right)$$

$$= 2\left(\underline{\sum_{n=2}^{K+1} \log(n)} - \underline{\sum_{n=1}^{K} \log(n)}\right)$$

$$= 2\Big(\underline{\cancel{\log(2)} + \cancel{\log(3)} + \cdots + \cancel{\log(K)} + \log(K+1)}$$

$$\underline{- \log(1) - \cancel{\log(2)} - \cancel{\log(3)} - \cdots - \cancel{\log(K)}}\Big)$$

$$= 2\left(-\log(1) + \log(K+1)\right)$$

$$\sum_{n=1}^{K} \log\left(\frac{n^2+2n+1}{n^2}\right) = 2\left(-\log(1) + \log(K+1)\right)$$

$$\sum_{n=1}^{\infty} \log\left(\frac{n^2+2n+1}{n^2}\right) = \lim_{K\to\infty} 2\left(-\log(1) + \log(K+1)\right)$$

$$= +\infty$$

Esta serie diverge a $+\infty$

1. Indicar si las siguientes series son convergentes o no, hallando sus suma en caso de serlo.

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$e)\ \sum_{n=1}^{+\infty}\log\left(\frac{n^2+2n+1}{n^2}\right) \quad f)\ \sum_{n=1}^{+\infty}\frac{n}{(n+1)(n+2)(n+3)} \quad g)\ \sum_{n=1}^{\infty}\frac{n\,arctg(n+1)-(n+1)\,arctg(n)}{n(n+1)}$

f) $\sum^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverge a $+\infty$

$$\frac{n}{(n+1)(n+2)(n+3)} = \frac{a}{n+1} + \frac{b}{n+2} + \frac{c}{n+3}$$

$$= \frac{-1}{2(n+1)} + \frac{2}{n+2} - \frac{3}{2(n+3)}$$

$$= \frac{-1}{2(n+1)} + \frac{4}{2(n+2)} - \frac{3}{2(n+3)}$$

$$= \left(\frac{-1}{2(n+1)} + \frac{1}{2(n+2)}\right) + \left(\frac{3}{2(n+2)} - \frac{3}{2(n+3)}\right)$$

$$x_n = \left(\frac{1}{2}\right)\frac{1}{n}$$

$$\frac{n}{(n+1)(n+2)(n+3)} = (-x_{n+1} + x_{n+2}) + 3(x_{n+2} - x_{n+3})$$

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} = \sum_{n=1}^{\infty} (-x_{n+1} + x_{n+2}) + 3(x_{n+2} - x_{n+3})$$

$$= \sum_{n=2}^{\infty} -x_n + \sum_{n=3}^{\infty} x_n + 3\left(\sum_{n=3}^{\infty} x_n - \sum_{n=4}^{\infty} x_n\right)$$

$$= -x_2 - \cancel{\sum_{n=3}^{\infty} x_n} + \cancel{\sum_{n=3}^{\infty} x_n} + 3\left(x_3 + \cancel{\sum_{n=4}^{\infty} x_n} - \cancel{\sum_{n=4}^{\infty} x_n}\right)$$

$$= -x_2 + 3x_3 = -\frac{1}{2}\,\frac{1}{2} + 3\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$$

2. Determinar si las siguientes series son convergentes o divergentes aplicando el criterio de comparación.

a) $\sum_{n=1}^{+\infty} \frac{1}{n^n}$ *b)* $\sum_{n=1}^{+\infty} e^{-\sqrt{n+1}}$

$\left(\frac{1}{n}\right)^n$, sabemos que $\frac{1}{n} < 1$

$$\left(\frac{1}{n}\right)^2 < \frac{1}{n} < 1$$

Se puede probar que $\left(\frac{1}{n}\right)^n < \left(\frac{1}{n}\right)^2$

$$\sum^{\infty} \frac{1}{n^2} < \infty$$