$X \sim \mathcal{N}(\mu, \sigma^2)$

$X_1, \dots, X_n$ MAS de $X$

intervalo de confianza para $\mu$ si no conocemos $\sigma^2$

$$\left[ \overline{X_n} - \frac{t_{\alpha/2}(n-1)}{\sqrt{n}} s_n , \overline{X_n} + \frac{t_{\alpha/2}(n-1)}{\sqrt{n}} s_n \right]$$

$$s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X_n})^2$$

$$\frac{(n-1)}{n} s_n^2 = \frac{\cancel{(n-1)}}{n} \frac{1}{\cancel{n-1}} \sum_{i=1}^{n} (X_i - \overline{X_n})^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - 2X_i \overline{X_n} + \overline{X_n}^2)$$

$$= \frac{1}{n} \left( \sum_{i=1}^{n} X_i^2 - 2\overline{X_n} \sum_{i=1}^{n} X_i + n\overline{X_n}^2 \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2\overline{X_n} \overbrace{\frac{1}{n} \sum_{i=1}^{n} X_i}^{\overline{X_n}} + \frac{1}{\cancel{n}} \cancel{n} \overline{X_n}^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} X_i^2 - 2\overline{X_n}^2 + \overline{X_n}^2$$

$$\frac{(n-1)}{n} s_n^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 - \overline{X_n}^2$$

$$\boxed{s_n^2 = \frac{1}{n-1} \left( \left( \sum_{i=1}^{n} X_i^2 \right) - n\overline{X_n}^2 \right)}$$

$$\sum_{i=1}^{n} \overline{X_n}^2 = \overline{X_n}^2 \underbrace{\sum_{i=1}^{n} 1}_{=n} = n\overline{X_n}^2$$

**Ejercicio 7**
Los contenidos de 7 recipientes similares de ácido sulfúrico son 9,8, 10,2, 10,4, 9,8, 10,0, 10,2 y 9,6 litros. Encontrar un intervalo de confianza 0,95 para la media de todos los recipientes, suponiendo una distribución normal.

$X$ = contenido de ácido sulfúrico de un recipiente

$X \sim \mathcal{N}(\mu, \sigma^2)$

buscamos intervalo de confianza 0,95 para $\mu$

el intervalo de confianza va a tener la forma

$$\left[ \overline{X_n} - \frac{t_{\alpha/2}(n-1)}{\sqrt{n}} s_n , \overline{X_n} + \frac{t_{\alpha/2}(n-1)}{\sqrt{n}} s_n \right]$$

| $X_i$ | $X_i^2$ |
|---|---|
| 9,8 | 96,04 |
| 10,2 | 104,04 |
| 10,4 | 108,16 |
| 9,8 | 96,04 |
| 10 | 100 |
| 10,2 | 104,04 |
| 9,6 | 92,16 |
| suma: 70 | 700,48 |

$$\overline{X_n} = \frac{70}{7} = 10$$

$$\boxed{\overline{X_n} = 10}$$

$$S_n^2 = \frac{1}{n-1}\left(\left(\sum_{i=1}^{n} X_i^2\right) - n\overline{X_n}^2\right)$$

$$= \frac{1}{6}\left(700{,}48 - 7 \cdot 100\right)$$

$$= \frac{0{,}48}{6}$$

$$= 0{,}08$$

$$\boxed{S_n = \sqrt{0{,}08}}$$

queremos intervalo de confianza $\underbrace{0{,}95}_{1-\alpha}$

entonces $\alpha = 0{,}05$, $n = 7$ $\Big\}$ $t_{\alpha/2}(n-1) = t_{0{,}025}(6) = 2{,}447$

$F_r^{-1}(1-\alpha/2)$

|  | 0.60 | 0.75 | 0.90 | 0.95 | $1-\alpha/2$ 0.975 | 0.99 |
|---|---|---|---|---|---|---|
| $r$ | $t_{0.40}(r)$ | $t_{0.25}(r)$ | $t_{0.10}(r)$ | $t_{0.05}(r)$ | $t_{0.025}(r)$ | $t_{0.01}(r)$ |
| 1 | 0.325 | 1.000 | 3.078 | 6.314 | 12.706 | 31.821 |
| 2 | 0.289 | 0.816 | 1.886 | 2.920 | 4.303 | 6.965 |
| 3 | 0.277 | 0.765 | 1.638 | 2.353 | 3.182 | 4.541 |
| 4 | 0.271 | 0.741 | 1.533 | 2.132 | 2.776 | 3.747 |
| 5 | 0.267 | 0.727 | 1.476 | 2.015 | 2.571 | 3.365 |
| 6 | 0.265 | 0.718 | 1.440 | 1.943 | $\boxed{2.447}$ | 3.143 |
| 7 | 0.263 | 0.711 | 1.415 | 1.895 | 2.365 | 2.998 |
| 8 | 0.262 | 0.706 | 1.397 | 1.860 | 2.306 | 2.896 |
| 9 | 0.261 | 0.703 | 1.383 | 1.833 | 2.262 | 2.821 |

$$\left[\overline{X_n} - \frac{t_{\alpha/2}(n-1)}{\sqrt{n}} S_n \, , \, \overline{X_n} + \frac{t_{\alpha/2}(n-1)}{\sqrt{n}} S_n\right]$$

$n = 7$

$t_{\alpha/2}(n-1) = 2{,}447$

$\overline{X_n} = 10$

$S_n = \sqrt{0{,}08} = 0{,}28$

$$\frac{t_{\alpha/2}(n-1)}{\sqrt{n}} S_n = \frac{2{,}447}{\sqrt{7}} 0{,}28 = 0{,}26$$

$\Rightarrow$ el intervalo de confianza es

$$[9{,}74 \, ; \, 10{,}26]$$

---

$X \sim \mathcal{N}(\mu, \sigma^2)$

$X_1, \dots, X_n$ MAS de $X$

intervalo de confianza $1-\alpha$ para $\sigma^2$

$$\left[ \frac{(n-1)\, S_n^2}{\chi^2_{\alpha/2}(n-1)} \, , \, \frac{(n-1)\, S_n^2}{\chi^2_{1-\alpha/2}(n-1)} \right]$$

grados de libertad

**Ejercicio 10**
Un fabricante de baterías para automóvil asegura que sus baterías duran, en promedio, 3 años con una varianza de un año. Si 5 de estas baterías tienen duraciones de 1,9, 2,4, 3,0, 3,5 y 4,2 años, determine un intervalo de confianza 0,95 para $\sigma^2$ e indique si es válida la afirmación del fabricante de que $\sigma^2 = 1$. Se supone que la población de las duraciones de las baterías se distribuye aproximadamente en forma normal.

$X$ = duración de una batería

$X \sim \mathcal{N}(\mu, \sigma^2)$

buscamos un intervalo de confianza $\overbrace{0{,}95}^{1-\alpha}$ por $\sigma^2$

el intervalo de confianza va a tener la forma

$$\left[ \frac{(n-1)\, S_n^2}{\chi^2_{\alpha/2}(n-1)} \, , \, \frac{(n-1)\, S_n^2}{\chi^2_{1-\alpha/2}(n-1)} \right]$$

| | $X_i$ | $X_i^2$ |
|---|---|---|
| | 1,9 | 3,61 |
| | 2,4 | 5,76 |
| | 3 | 9 |
| | 3,5 | 12,25 |
| | 4,2 | 17,64 |
| suma | 15 | 48,26 |

$$\overline{X}_n = \frac{15}{5} = 3$$

$$S_n^2 = \frac{1}{n-1}\left(\left(\sum_{i=1}^{n} X_i^2\right) - n\overline{X}_n^2\right)$$

$$= \frac{1}{4}\left(48{,}26 - 5 \cdot 3^2\right)$$

$$= 0{,}815$$

$$\boxed{S_n^2 = 0{,}815}$$

como buscamos un intervalo de confianza 0,95 tenemos $\alpha = 0{,}05$

$\alpha = 0{,}05$

$n = 5$

$\chi^2_{\alpha/2}(n-1) = \chi^2_{0{,}025}(4) = 11{,}14$

$\chi^2_{1-\alpha/2}(n-1) = \chi^2_{0{,}975}(4) = 0{,}484$

| | 0.010 | 0.025 | 0.050 | 0.100 | 0.900 | 0.950 | 0.975 |
|---|---|---|---|---|---|---|---|
| $r$ | $\chi^2_{0.99}(r)$ | $\chi^2_{0.975}(r)$ | $\chi^2_{0.95}(r)$ | $\chi^2_{0.90}(r)$ | $\chi^2_{0.10}(r)$ | $\chi^2_{0.05}(r)$ | $\chi^2_{0.025}(r)$ |
| 1 | 0.000 | 0.001 | 0.004 | 0.016 | 2.706 | 3.841 | 5.024 |
| 2 | 0.020 | 0.051 | 0.103 | 0.211 | 4.605 | 5.991 | 7.378 |
| 3 | 0.115 | 0.216 | 0.352 | 0.584 | 6.251 | 7.815 | 9.348 |
| 4 | 0.297 | $\boxed{0.484}$ | 0.711 | 1.064 | 7.779 | 9.488 | $\boxed{11.14}$ |
| 5 | 0.554 | 0.831 | 1.145 | 1.610 | 9.236 | 11.07 | 12.83 |

$$\left[ \frac{(n-1)\,S_n^2}{\chi^2_{\alpha/2}(n-1)} , \frac{(n-1)\,S_n^2}{\chi^2_{1-\alpha/2}(n-1)} \right]$$

$$\frac{(n-1)\,S_n^2}{\chi^2_{\alpha/2}(n-1)} = \frac{4 \cdot 0{,}815}{11{,}14} = 0{,}2926$$

$$\frac{(n-1)\,S_n^2}{\chi^2_{1-\alpha/2}(n-1)} = \frac{4 \cdot 0{,}815}{0{,}484} = 6{,}7355$$

el intervalo de confianza es $[0{,}2926 \; ; \; 6{,}7355]$

1 pertenece al intervalo de confianza entonces la afirmación $\sigma^2 = 1$ es válida.

---

intervalo de confianza para una proporción p

$X \sim Ber(p)$

buscamos intervalo de confianza para p

Recorrido de X es $\{1, 0\}$

$P_X(1) = p \qquad P_X(0) = 1-p \qquad E(X) = p$

LFGN: $\overline{X_n} \xrightarrow{cs} \underbrace{E(X)}_{p}$

$\overline{X_n}$ es un estimador consistente de $p$

buscamos un intervalo de confianza $1-\alpha$ de la forma

$$[\overline{X_n}-k, \overline{X_n}+k]$$

es decir queremos que $P(p \in [\overline{X_n}-k, \overline{X_n}+k]) = 1-\alpha$

por el teorema central del límite $\overline{X_n} \sim \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)$

$$E(\overline{X_n}) = E(X_1) = p$$

$$Var(\overline{X_n}) = Var\left(\frac{X_1+\dots+X_n}{n}\right) = \frac{1}{n^2} Var(X_1+\dots+X_n) = \frac{1}{n^2}(Var(X_1)+Var(X_2)+\dots Var(X_n))$$

$$= \frac{1}{n} Var(X_1) = \frac{p(1-p)}{n}$$

$$P(p \in [\overline{X_n}-k, \overline{X_n}+k]) = P(\overline{X_n}-k \leq p \leq \overline{X_n}+k)$$

$$= P(-k \leq p-\overline{X_n} \leq k)$$

$$= P(-k \leq \overline{X_n}-p \leq k)$$

$$= P\left(\frac{-k}{\sqrt{p(1-p)}}\sqrt{n} \leq \overbrace{\frac{\overline{X_n}-p}{\sqrt{p(1-p)}}\sqrt{n}}^{\mathcal{N}(0,1)} \leq \frac{k}{\sqrt{p(1-p)}}\sqrt{n}\right)$$

$$= \phi\left(\frac{k}{\sqrt{p(1-p)}}\sqrt{n}\right) - \phi\left(\frac{-k}{\sqrt{p(1-p)}}\sqrt{n}\right)$$

$$= \phi\left(\frac{k}{\sqrt{p(1-p)}}\sqrt{n}\right) - \left(1-\phi\left(\frac{k}{\sqrt{p(1-p)}}\sqrt{n}\right)\right)$$

$$= 2\phi\left(\frac{k}{\sqrt{p(1-p)}}\sqrt{n}\right) - 1$$

buscamos $k$ tal que

$$2\phi\left(\frac{k}{\sqrt{p(1-p)}}\sqrt{n}\right) - 1 = 1-\alpha$$

$$\phi\left(\frac{k}{\sqrt{p(1-p)}}\sqrt{n}\right) = 1-\frac{\alpha}{2}$$

$$\frac{k}{\sqrt{p(1-p)}}\sqrt{n} = z_{\alpha/2} \quad \Longrightarrow \quad k = \frac{z_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}}$$

el intervalo de confianza $1-\alpha$ de $p$ es

$$\left[\overline{X_n} - \frac{z_{\alpha/2}\sqrt{\overline{X_n}(1-\overline{X_n})}}{\sqrt{n}},\ \overline{X_n} + \frac{z_{\alpha/2}\sqrt{\overline{X_n}(1-\overline{X_n})}}{\sqrt{n}}\right]$$

**Ejercicio 11**

1. Al probar 100 barras de acero que fabricó la compañía A se encuentra que 12 no cumplieron con las especificaciones.

   *a*) Determinar un intervalo de confianza 95 % para $\overbrace{\text{la proporción verdadera}}^{P}$ de las barras de acero que no cumplen las especificaciones.

   *b*) Si se desea estimar la proporción verdadera que no cumple con las especificaciones con una exactitud de 0,05 y una confianza de 0,95. ¿Cuántas barras se deben examinar?

$X \sim Ber(p)$

$$X = \begin{cases} 1 & \text{si la barra no cumple las especificaciones} \\ 0 & \text{si cumple las especificaciones} \end{cases}$$

$n = 100$

$$\overline{X_n} = \frac{X_1 + X_2 + \dots + X_{100}}{100} = \frac{12}{100} = 0{,}12$$

$$\alpha = 0{,}05 \longrightarrow z_{\alpha/2} = \phi^{-1}\left(1-\frac{\alpha}{2}\right) = \phi^{-1}(0{,}975) = 1{,}96$$

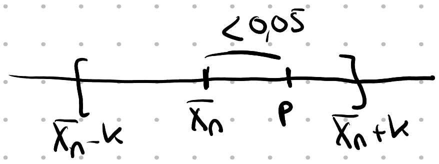
| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | ( |
|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0. |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0. |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0. |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0. |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0. |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0. |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0. |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7703 | 0.7734 | 0.7764 | 0.7794 | 0. |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0. |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0. |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0. |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0. |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0. |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0. |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0. |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0. |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0. |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0. |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0. |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | $\boxed{0.9750}$ | 0.9756 | 0. |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0. |

$$\frac{z_{\alpha/2}\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} = \frac{1{,}96\sqrt{0{,}12(1-0{,}12)}}{\sqrt{100}} = 0{,}064$$

el intervalo de confianza es $[0{,}12 - 0{,}064 \ ; \ 0{,}12 + 0{,}064]$

$$[0{,}056 \ ; \ 0{,}184]$$

b) buscamos estimar $p$ con exactitud de $0{,}05$ y cofianza $95\%$.




buscamos un intervalo de confianza de la forma

$$\left[\overline{X}_n - \frac{z_{\alpha/2}\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} \ , \ \overline{X}_n + \frac{z_{\alpha/2}\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}}\right]$$

tal que $\dfrac{z_{\alpha/2}\sqrt{\overline{X}_n(1-\overline{X}_n)}}{\sqrt{n}} \leq 0{,}05$

$$\Rightarrow \quad \frac{\sqrt{n}}{z_{\alpha/2}\sqrt{\overline{X}_n(1-\overline{X}_n)}} \geq \frac{1}{0{,}05}$$

$$\Rightarrow \quad n \geq \frac{z_{\alpha/2}^2 \ \overline{X}_n(1-\overline{X}_n)}{0{,}05^2} = \frac{1{,}96^2 \ \overline{X}_n(1-\overline{X}_n)}{0{,}05^2}$$

si $x \in [0,1] \Rightarrow x(1-x) \leq \frac{1}{4}$ $\qquad f(x) = x(1-x)$

$$n \geq \frac{1{,}96^2 \cdot 1/4}{0{,}05^2} = 384{,}16$$

entonces tomamos $n = 385$