Ejercicio 6

Calcular esperanza y varianza:

* $X \sim Geo(p)$

$$p_X(k) = \begin{cases} (1-p)^{k-1} p & \text{si } k \in \{1, 2, \dots\} \\ 0 & \text{si no} \end{cases}$$

$E(X) = ?$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$E(X) = \sum_{k=1}^{\infty} k \, p_X(k)$$

$$= \sum_{k=1}^{\infty} k (1-p)^{k-1} p$$

$$= p \underbrace{\sum_{k=1}^{\infty} k (1-p)^{k-1}}_{} = f(p)$$

$$((1-p)^k)' = -k(1-p)^{k-1}$$

$$f(p) = \sum_{k=1}^{\infty} k (1-p)^{k-1}$$

$$F(p) = \sum_{k=0}^{\infty} -(1-p)^k = -\sum_{k=0}^{\infty} (1-p)^k = -\frac{1}{1-(1-p)} = -\frac{1}{p}$$

$$f(p) = F'(p) = \left(-\frac{1}{p}\right)' = -\frac{-1}{p^2} = \frac{1}{p^2}$$

Entonces: $E(X) = p \underbrace{\sum_{k=1}^{\infty} k (1-p)^{k-1}}_{1/p^2} = p \frac{1}{p^2} = \frac{1}{p}$

$$\boxed{E(X) = \frac{1}{p}}$$

$E(X^2) = ?$

$$E(X^2) = \sum_{k=1}^{\infty} k^2 p_X(k)$$

$$= \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

$$= p \underbrace{\sum_{k=1}^{\infty} k^2 (1-p)^{k-1}}_{g(p)}$$

$$\left((1-p)^k\right)' = -k(1-p)^{k-1}$$

$$k^2 (1-p)^{k-1} = k\, k(1-p)^{k-1}$$

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}$$

$$\left(k(1-p)^k\right)' = -k^2(1-p)^{k-1}$$

tomamos $G(p) = \sum_{k=0}^{\infty} -k(1-p)^k = -\sum_{k=0}^{\infty} k(1-p)^k$

$$= -\sum_{k=0}^{\infty} k(1-p)^{k-1}(1-p)$$

$$= -(1-p)\sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$= -(1-p)\cdot\frac{1}{p^2}$$

$$= -\frac{1}{p^2} + \frac{p}{p^2}$$

$$= \frac{1}{p} - \frac{1}{p^2}$$

$$G(p) = \frac{1}{p} - \frac{1}{p^2} = p^{-1} - p^{-2}$$

$$g(p) = G'(p) = -p^{-2} + 2p^{-3} = -\frac{1}{p^2} + \frac{2}{p^3}$$

Entonces $E(X^2) = p\sum_{k=1}^{\infty} k^2(1-p)^{k-1} = p\left(\frac{2}{p^3} - \frac{1}{p^2}\right) = \frac{2}{p^2} - \frac{1}{p}$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p}$$

$$= \frac{1-p}{p^2}$$

* $X \sim \exp(\lambda)$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{si } x \geq 0 \\ 0 & \text{si no} \end{cases}$$

$E(X) = ?$

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\, dx$$

$$= \int_{0}^{+\infty} x \lambda e^{-\lambda x}\, dx$$

$$f(x) = x \longrightarrow f'(x) = 1$$

$$g'(x) = \lambda e^{-\lambda x} \longrightarrow g(x) = -e^{-\lambda x}$$

$$E(X) = \int_{0}^{+\infty} x \lambda e^{-\lambda x}\, dx$$

$$= -x e^{-\lambda x} \Big|_{0}^{+\infty} - \int_{0}^{+\infty} -e^{-\lambda x}\, dx$$

$$= \int_{0}^{+\infty} e^{-\lambda x}\, dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x} \Big|_{0}^{+\infty}$$

$$\boxed{E(X) = \frac{1}{\lambda}}$$

$$= \frac{1}{\lambda}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx$$

$$= \int_{0}^{+\infty} x^2 \lambda e^{-\lambda x}\, dx$$

$f(x) = x^2 \rightsquigarrow f'(x) = 2x$

$g'(x) = \lambda e^{-\lambda x} \rightsquigarrow g(x) = -e^{-\lambda x}$

$$E(X^2) = \int_0^{+\infty} x^2 \lambda e^{-\lambda x}\, dx$$

$$= -x^2 e^{-\lambda x}\Big|_0^{+\infty} - \int_0^{+\infty} -2x e^{-\lambda x}\, dx$$

$$= 2\int_0^{+\infty} x e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda}\int_0^{+\infty} x \lambda e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \quad \boxed{Var(X) = \frac{1}{\lambda^2}}$$

Ejercicio 9
Se ponen a funcionar en un mismo momento (que tomamos como tiempo 0) dos lamparitas de dos marcas distintas, A y B, que se dejan prendidas hasta que se rompan. Llamemos X al tiempo de duración de la lamparita A e Y al tiempo de duración de la lamparita B. Admitamos que X e Y son independientes, que X sigue una distribución exponencial de parámetro $\lambda_1 > 0$ y que Y sigue una distribución exponencial de parámetro $\lambda_2 > 0$.

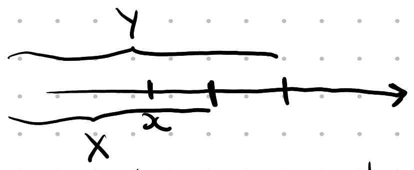
Llamemos S al tiempo en que ocurre la primera rotura de alguna de las dos lamparitas y T al tiempo en que se rompe la restante lamparita.

1. Calcular las funciones de distribución de S y T.
2. Calcular $\mathbf{E}(S)$, $\mathbf{E}(T)$.
3. Calcular $\mathbf{E}(ST)$. ¿Son S y T independientes? Justifique la respuesta.
4. Calcular $\mathbf{P}(S = T)$.

X = tiempo de duración de la lamparita A $\quad X \sim exp(\lambda_1)$

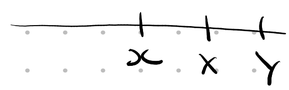
Y = tiempo de duración de la lamparita B $\quad Y \sim exp(\lambda_2)$

X e Y independientes



1) S = tiempo en que ocurre la primera rotura de alguna lampanita

$F_S(x) = ?$ $\qquad S = \min\{X, Y\}$



$$F_S(x) = P(S \leq x) = P(\min\{X,Y\} \leq x)$$

$x \geq 0$

$$= 1 - P(\min\{X,Y\} \geq x\}$$
$$= 1 - P(\{X \geq x\} \cap \{Y \geq x\})$$
$$= 1 - P(X \geq x)\, P(Y \geq x)$$
$$= 1 - (1 - P(X \leq x))(1 - P(Y \leq x))$$
$$= 1 - (1 - (1 - e^{-\lambda_1 x}))(1 - (1 - e^{-\lambda_2 x}))$$
$$= 1 - e^{-\lambda_1 x} e^{-\lambda_2 x}$$
$$= 1 - e^{-(\lambda_1 + \lambda_2) x}$$

$X \sim exp(\lambda_1)$

$$\underset{\substack{\shortparallel \\ P(X \leq x)}}{F_X(x)} = \begin{cases} 1 - e^{-\lambda_1 x} & x \geq 0 \\ 0 & \text{si no} \end{cases}$$

$Y \sim exp(\lambda_2)$

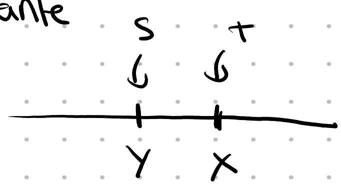
$$F_Y(x) = \begin{cases} 1 - e^{-\lambda_2 x} & \text{si } x \geq 0 \\ 0 & \text{si no} \end{cases}$$

$$F_S(x) = \begin{cases} 1 - e^{-(\lambda_1 + \lambda_2) x} & \text{si } x \geq 0 \\ 0 & \text{si no} \end{cases}$$

$$\Rightarrow S \sim exp(\lambda_1 + \lambda_2)$$

T = tiempo de rotura de la lampanita restante

$F_T(x) = ?$ $\qquad T = \max\{X, Y\}$



$$F_T(x) = P(T \leq x) = P(\max\{X, Y\} \leq x\}$$

$x \geqslant 0$

$$= P(\{X \leq x\} \cap \{Y \leq x\}) \quad X \text{ e } Y \text{ independientes}$$

$$= P(X \leq x) P(Y \leq x)$$

$$= (1 - e^{-\lambda x})(1 - e^{-\lambda_2 x})$$

$$= 1 - e^{-\lambda x} - e^{-\lambda_2 x} + e^{-(\lambda_1 + \lambda_2) x}$$

$$F_T(x) = \begin{cases} 1 - e^{-\lambda_1 x} - e^{-\lambda_2} x + e^{-(\lambda_1 + \lambda_2) x} & \text{si } x \geqslant 0 \\ 0 & \text{si no} \end{cases}$$

2. $E(S) = \dfrac{1}{\lambda_1 + \lambda_2}$ $\qquad (-e^{-\lambda x})' = -e^{-\lambda_1 x}(-\lambda) = \lambda_1 e^{-\lambda x}$

$S \sim \exp(\lambda_1 + \lambda_2)$

$E(T) = ?$

$$f_T(x) = F_T'(x) = \begin{cases} \overbrace{\lambda e^{-\lambda_1 x}}^{f_X} + \overbrace{\lambda_2 e^{-\lambda_2 x}}^{f_Y} \overbrace{-(\lambda + \lambda_2) e^{-(\lambda_1 + \lambda_2) x}}^{f_S} & \text{si } x \geqslant 0 \\ 0 & \text{si no} \end{cases}$$

$$E(T) = \int_{-\infty}^{+\infty} x f_T(x)\,dx = \int_{-\infty}^{+\infty} x f_X(x)\,dx + \int_{-\infty}^{+\infty} x f_Y(x)\,dx - \int_{-\infty}^{+\infty} x f_S(x)\,dx$$

$$S + T = \min\{X, Y\} + \max\{X, Y\} = X + Y$$

$$\Rightarrow T = X + Y - S$$

$$\Rightarrow E(T) = E(X + Y - S) = E(X) + E(Y) - E(S)$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

3. $$E(ST) = E(XY) \overset{X \text{ e } Y \text{ independientes}}{=} E(X) E(Y) = \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2} = \frac{1}{\lambda_1 \lambda_2}$$

$ST = \min\{X, Y\} \max\{X, Y\} = XY$

S y T independientes?

si S y T fueran independientes tendríamos $E(ST) = E(S)E(T)$

$$E(ST) = \frac{1}{\lambda_1 \lambda_2}$$

$$E(S)E(T) = \left(\frac{1}{\lambda_1+\lambda_2}\right)\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2}\right)$$

entonces $E(ST) \neq E(S)E(T)$

$\Rightarrow$ S y T no son independientes