Variable aleatoria discreta

$$X : \Omega \longrightarrow \mathbb{R}$$

X toma valores en un conjunto discreto

* función de probabilidad puntual

$$P_X : \mathbb{R} \longrightarrow [0,1]$$

$$P_X(k) = P(X = k)$$

* función de distribución acumulada

$$F_X : \mathbb{R} \to [0,1]$$

$$F_X(k) = P(X \leq k)$$

$$= \sum_{i \leq k} P_X(i)$$

Variable aleatoria (absolutamente) continua

$$X : \Omega \longrightarrow \mathbb{R}$$

X toma valores en un conjunto que no es discreto

por ejemplo: un intervalo

$$P(X = 1) = 0$$

$$P(X = x) = 0$$

1,0000000 . . .

$1/10 \quad 1/10 \quad 1/10$

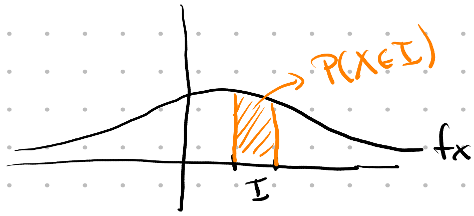
* función de densidad de probabilidad

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

tal que:
- integrable
- positiva : $f(x) \geq 0 \ \forall x$
- integra 1 : $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

* X es absolutamente continua si existe una función densidad f_X tal que

$$P(X \in I) = \int_I f_X(x)\, dx$$



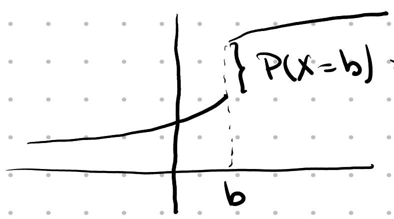
* función de distribución acumulada

$$F_X : \mathbb{R} \longrightarrow [0,1]$$

$$F_X(y) = P(X \leq y) = \int_{-\infty}^{y} f_X(x)\, dx$$

$\rightarrow F_X' = f_X$

$\rightarrow F_X$ es continua



$$P(X = b) = \int_b^b f_X(x)\, dx = 0$$

$$\rightarrow P(a \leq X \leq b) = P(a < X \leq b)$$
$$= P(a < X < b)$$
$$= P(a \leq X < b)$$
$$= F_X(b) - F_X(a)$$

Ejercicio 2

Se considera la variable aleatoria X absolutamente continua con densidad:

$$f_X(x) = \begin{cases} 0 & \text{si } x < 0 \\ bx & \text{si } x \in (0,1] \\ ae^{-x} & \text{si } x > 1 \end{cases}$$

Hallar a y b sabiendo que $\mathbf{P}(X \in [0,2]) = 2\mathbf{P}(X \in [2,4])$.

$$* \int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{+\infty} f_X(x)dx = \int_{-\infty}^{0} \overbrace{f_X(x)}^{=0} dx + \int_{0}^{1} \overbrace{f_X(x)}^{=bx} dx + \int_{1}^{+\infty} ae^{-x} dx$$

$$= \int_0^1 bx\,dx + \int_1^{+\infty} ae^{-x} dx$$

$$= b\frac{x^2}{2}\Big|_0^1 + (-ae^{-x})\Big|_1^{+\infty}$$

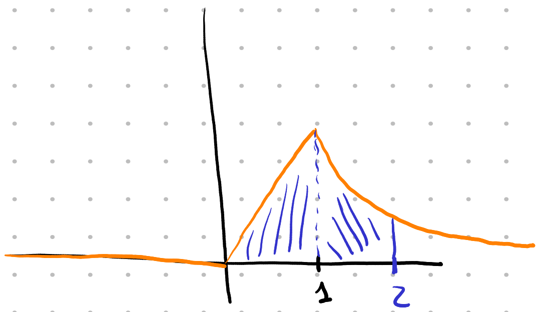
$$(-ae^{-x})\Big|_1^{+\infty} = \underbrace{-ae^{-\infty}}_{=0} - (-ae^{-1})$$

$$= \frac{b}{2} + ae^{-1}$$

$$\boxed{1 = \frac{b}{2} + \frac{a}{e}}$$

$$* \ P(X \in [0,2]) = 2P(X \in [2,4])$$

$$P(X \in [0,2]) = \int_0^2 f_X(x)dx$$



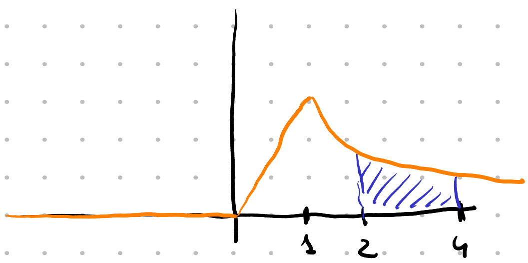
$$= \int_0^1 bx\,dx + \int_1^2 ae^{-x} dx$$

$$= b\frac{x^2}{2}\Big|_0^1 + (-ae^{-x})\Big|_1^2$$

$$= \frac{b}{2} - ae^{-2} - (-ae^{-1})$$

$$= \frac{b}{2} - ae^{-2} + ae^{-1}$$

$$P(X \in [2,4]) = \int_2^4 ae^{-x} dx$$



$$= -ae^{-x}\Big|_2^4$$

$$= -ae^{-4} - (-ae^{-2})$$

$$= -ae^{-4} + ae^{-2}$$

$$P(X\in[0,2]) = 2\,P(X\in[2,4])$$

$$\Rightarrow \frac{b}{2} + ae^{-1} - ae^{-2} = 2(-ae^{-4} + ae^{-2})$$

$$\Rightarrow \frac{b}{2} + ae^{-1} - ae^{-2} = -2ae^{-4} + 2ae^{-2}$$

$$\Rightarrow \frac{b}{2} + ae^{-1} - ae^{-2} + 2ae^{-4} - 2ae^{-2} = 0$$

$$\Rightarrow \boxed{\frac{b}{2} + a(e^{-1} - 3e^{-2} + 2e^{-4}) = 0} \quad (2)$$

$$\boxed{\frac{b}{2} + ae^{-1} = 1} \quad (1)$$

$$(1) - (2):\ a\cancel{e^{-1}} - a(\cancel{e^{-1}} - 3e^{-2} + 2e^{-4}) = 1$$

$$\Rightarrow -a(-3e^{-2} + 2e^{-4}) = 1$$

$$\Rightarrow a(3e^{-2} - 2e^{-4}) = 1$$

$$\Rightarrow a = \frac{1}{3e^{-2} - 2e^{-4}}$$

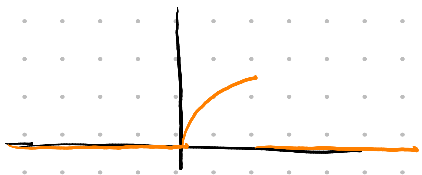
$$(1)\quad \frac{b}{2} + ae^{-1} = 1 \Rightarrow b = 2(1 - ae^{-1})$$

$$b = 2\left(1 - \frac{e^{-1}}{3e^{-2} - 2e^{-4}}\right)$$

Ejercicio 3

Se consideran las siguientes funciones reales:

$$f_1(x) = \begin{cases} c_1\sqrt{x} & \text{si } x \in (0,1) \\ 0 & \text{si } x \notin (0,1) \end{cases}$$



* c_1 para que f_1 sea una función densidad

$$1 = \int_{-\infty}^{+\infty} f_1(x)\,dx = \int_{-\infty}^{0} \overbrace{f_1(x)}^{=0}\,dx + \int_0^1 f_1(x)\,dx + \int_1^{+\infty} \overbrace{f_1(x)}^{=0}\,dx$$

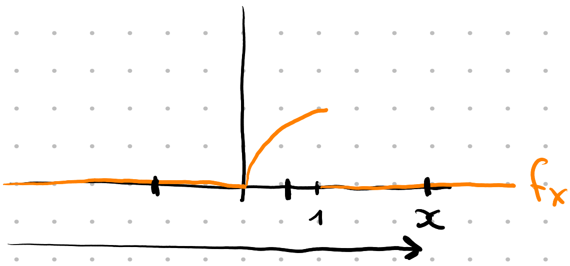
$$\underbrace{P(X\in(-\infty,+\infty))}_{=1} = \int_{-\infty}^{+\infty} f_1(x)\,dx$$

$$= \int_0^1 c_1 x^{1/2}\,dx$$

$$= c_1 \frac{x^{3/2}}{3/2}\Big|_0^1 = c_1 \frac{1}{3/2} \Rightarrow c_1 = 3/2$$

* $F_X(x) = ?$

$$F_X(x) = P(\underbrace{X \leq x}_{X \in (-\infty, x)}) = \int_{-\infty}^{x} f_1(t)\,dt$$



- si $x \leq 0$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} \overbrace{f_1(t)}^{=0}\,dt = 0$$

- si $x \in (0,1)$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_1(t)\,dt$$

$$= \int_{-\infty}^{0} \overbrace{f_1(t)}^{=0}\,dt + \int_{0}^{x} \overbrace{f_1(t)}^{=\frac{3}{2}\sqrt{t}}\,dt$$

$$= \int_{0}^{x} \frac{3}{2}\sqrt{t}\,dt$$

$$= \int_{0}^{x} \frac{3}{2} t^{1/2}\,dt$$

$$= \frac{\cancel{3}}{\cancel{2}} \frac{t^{3/2}}{\cancel{3/2}} \Big|_{0}^{x}$$

$$= x^{3/2}$$

- si $x \geq 1$

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_1(t)\,dt$$

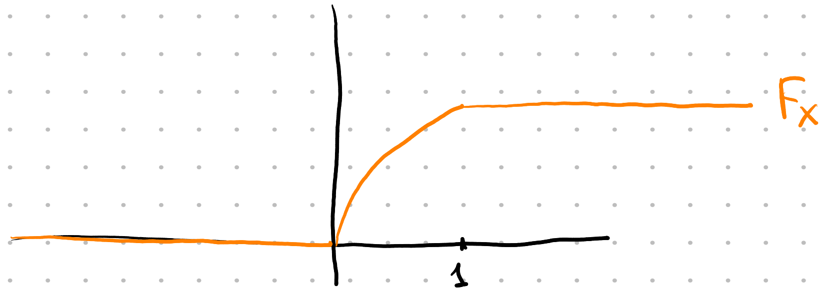
$$= \int_{-\infty}^{0} \overbrace{f_1(t)}^{=0}\,dt + \int_{0}^{1} \overbrace{f_1(t)}^{=\frac{3}{2}\sqrt{t}}\,dt + \int_{1}^{x} \overbrace{f_1(t)}^{=0}\,dt$$

$$= \int_{0}^{1} \frac{3}{2}\sqrt{t}\,dt$$

$$= \frac{3}{2} \left. \frac{t^{3/2}}{3/2} \right|_0^1$$

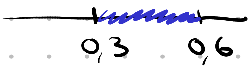
$$= 1$$

$$F_X(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ x^{3/2} & \text{si } 0 < x < 1 \\ 1 & \text{si } 1 \leq x \end{cases}$$



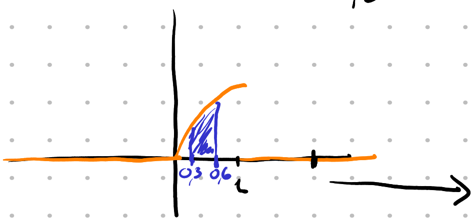
Calcular $\mathbf{P}(0{,}3 < X < 0{,}6)$, $\mathbf{P}(X > 2)$ y $\mathbf{P}\left(\frac{1}{2} < X < \frac{3}{2}\right)$.

- $P(0{,}3 < X < 0{,}6) = P(X < 0{,}6) - P(X \leq 0{,}3)$



$$= P(X \leq 0{,}6) - P(X \leq 0{,}3)$$
$$= F_X(0{,}6) - F_X(0{,}3)$$
$$= 0{,}6^{3/2} - 0{,}3^{3/2}$$

$$P(0{,}3 < X < 0{,}6) = \int_{0{,}3}^{0{,}6} f_1(x)\,dx$$



- $P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - 1 = 0$

$$P(X > 2) = \int_2^{+\infty} \overbrace{f_1(x)}^{=0}\,dx = 0$$