Señales y Sistemas

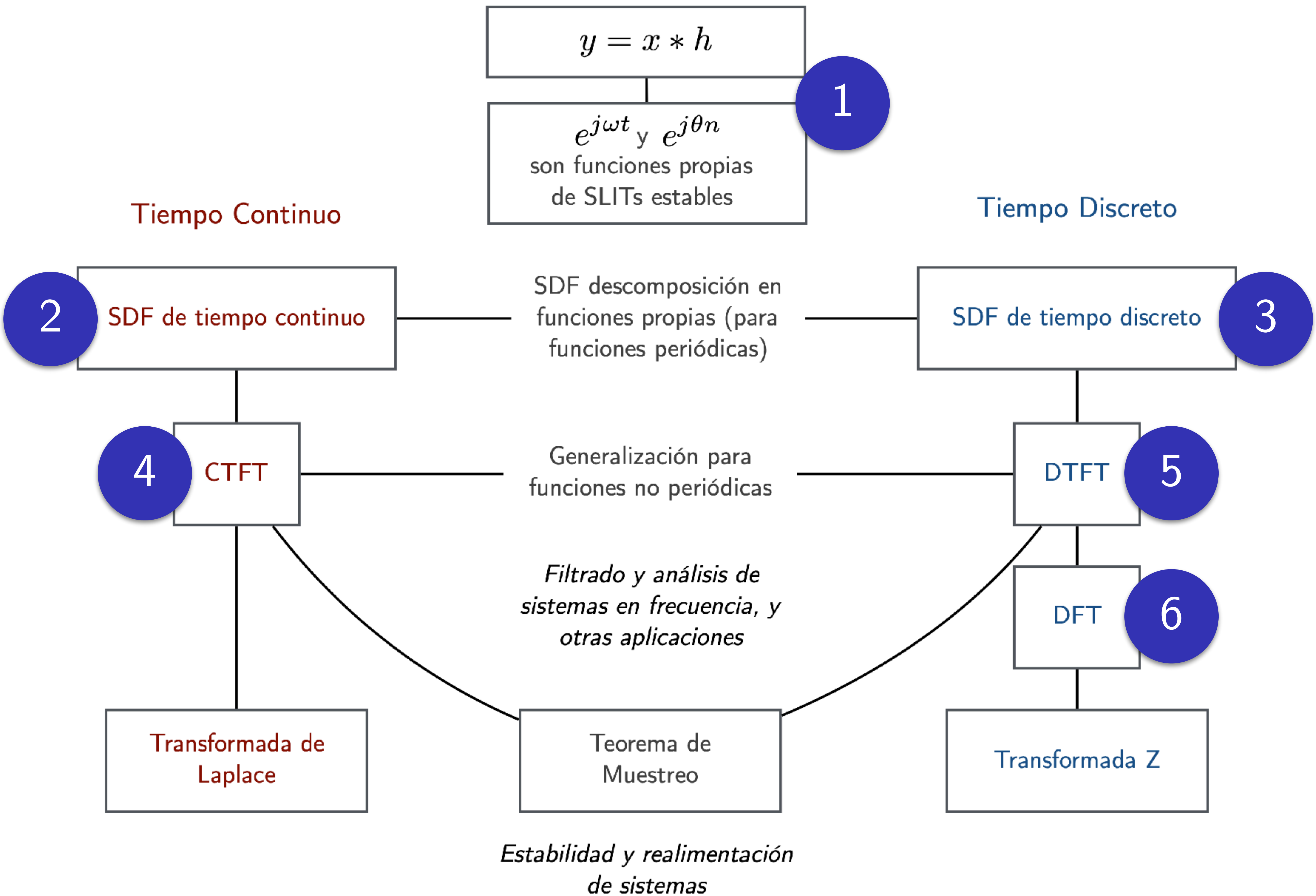
Transformada de Fourier de variable discreta (DTFT)
Transformada Discreta de Fourier (DFT)
Algoritmo Fast Fourier Transform (FFT)

Instituto de Ingeniería Eléctrica

FACULTAD DE INGENIERÍA

UNIVERSIDAD DE LA REPÚBLICA URUGUAY

Señales y sistemas



* tiempo o variable

Series de Fourier de variable discreta

- $x[n]$ periódica, de período N: $x[n+N] = x[n]\ \forall n \qquad (\theta_0 = {}^{2\pi}/_N)$

Serie de Fourier

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\theta_0 n}$$

Síntesis

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\theta_0 n}$$

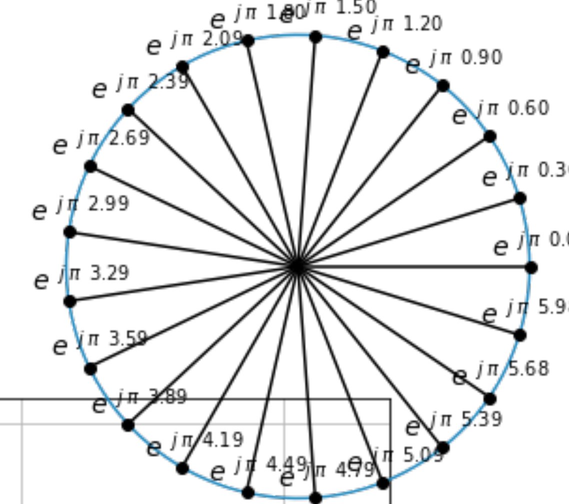
Análisis

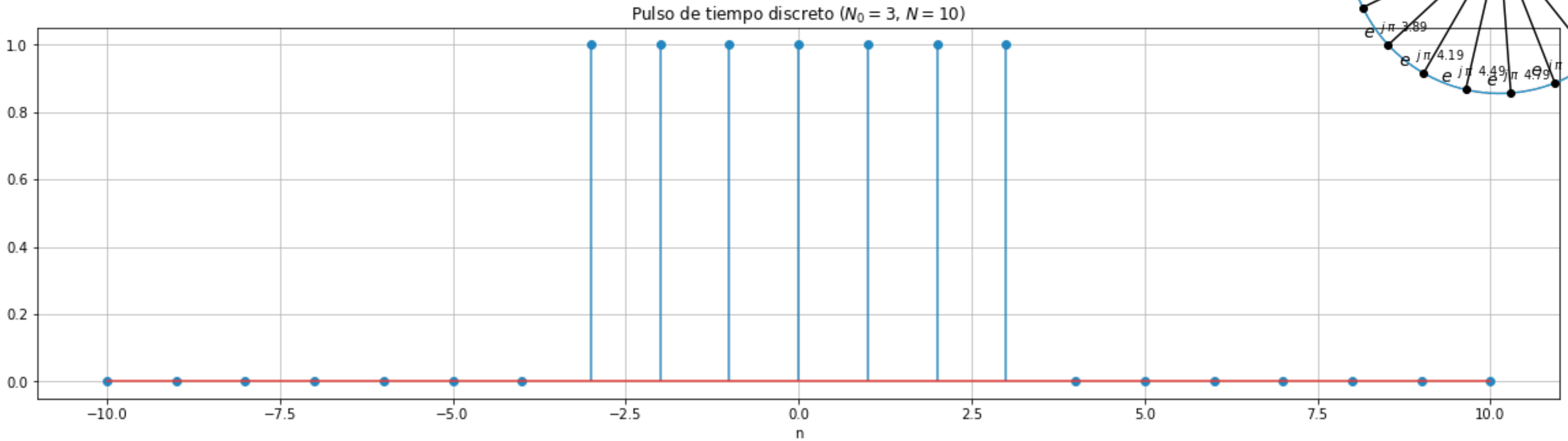
- a_k son periódicos

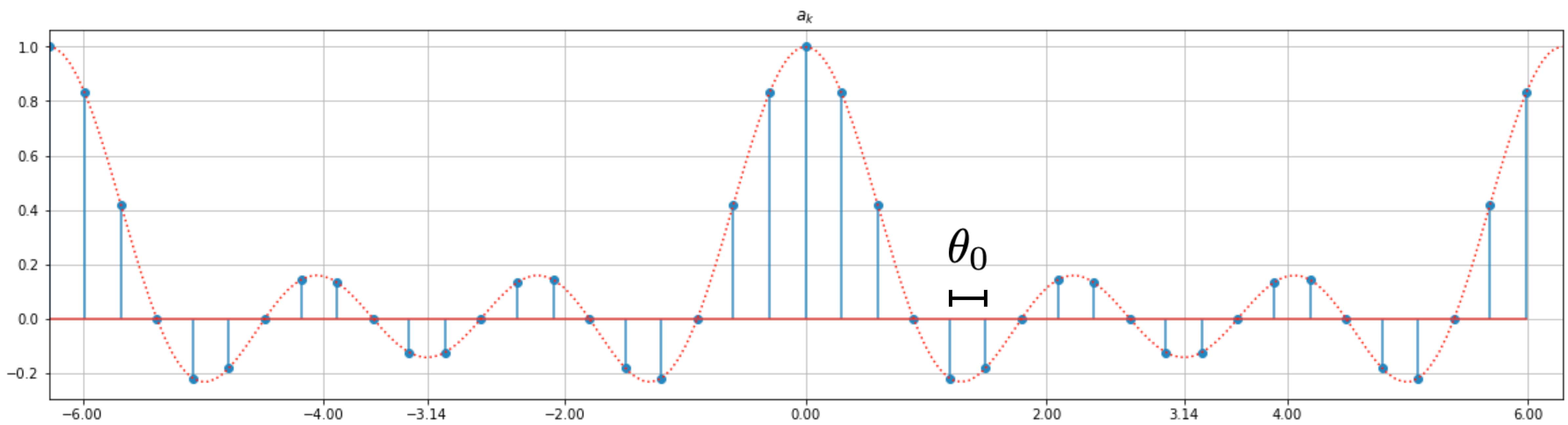
$$\langle e^{jl\theta_0 n}, e^{jk\theta_0 n}\rangle = \frac{1}{N}\sum_{k=0}^{N-1} e^{j(l-k)\theta_0 n} = \begin{cases} 1, & l = k + mN \\ 0, & l \neq k + mN \end{cases}$$

Base ortonormal

Series de Fourier

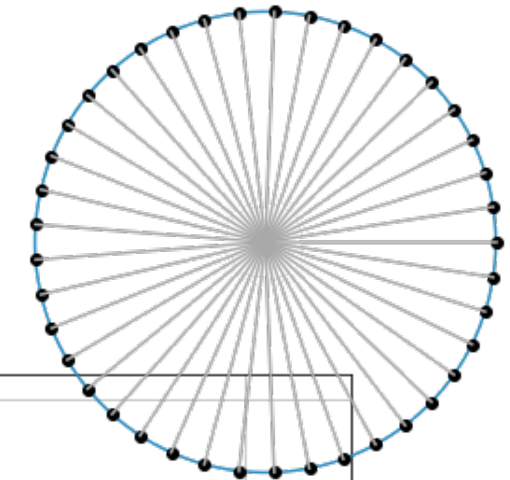


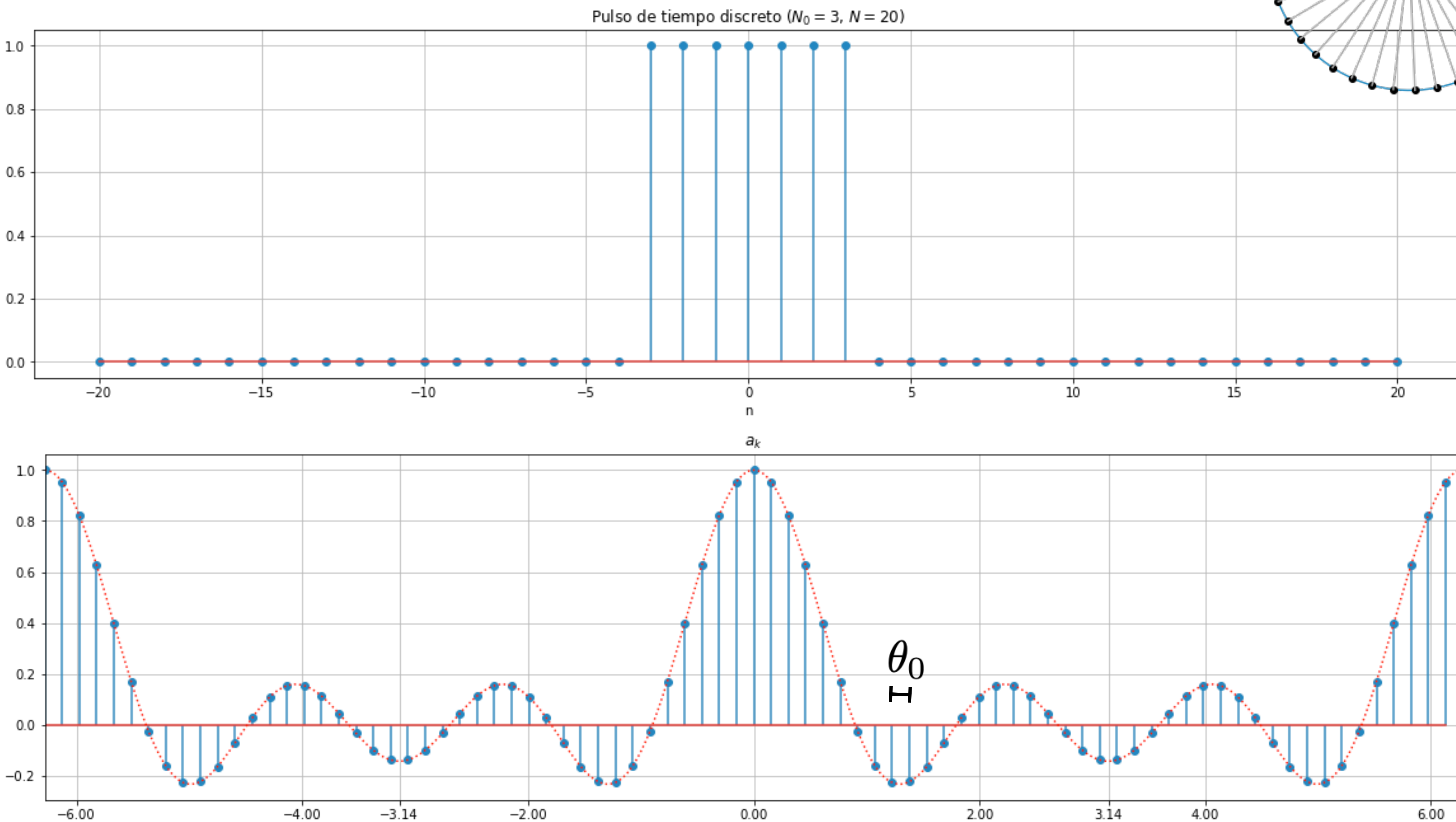




$$x[n] = \Pi\left(\frac{n}{2N_0+1}\right) \overset{\text{SF}}{\longleftrightarrow} a_k = \frac{1}{N}\frac{\sin(1/2\,\theta_0 k(N_0+1))}{\sin(1/2\,\theta_0 k)}$$

Series de Fourier





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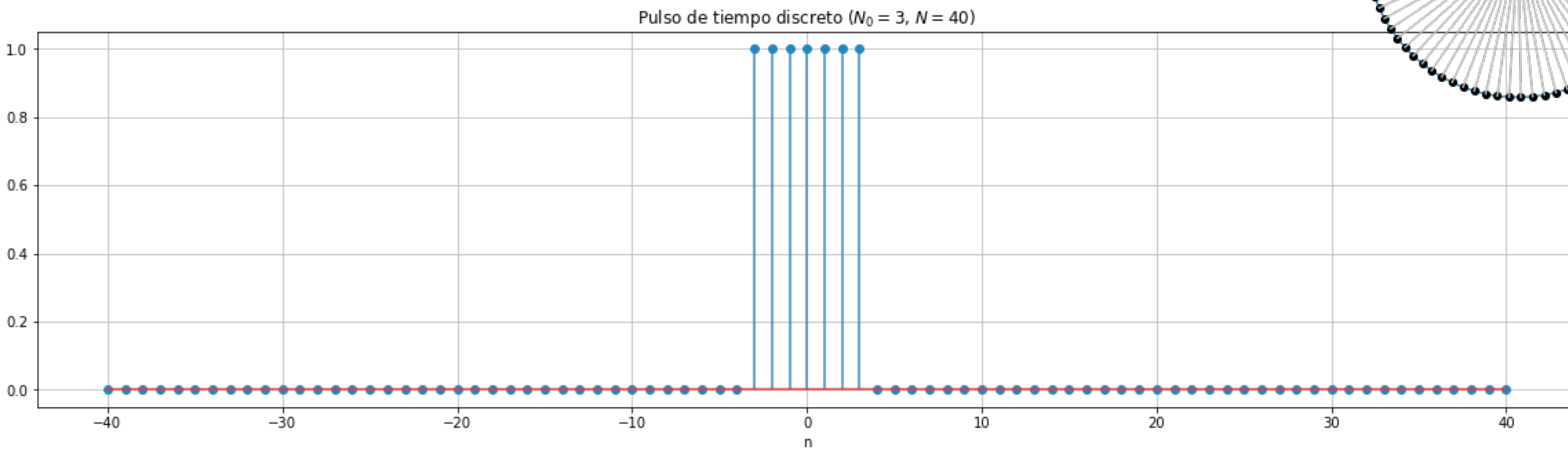
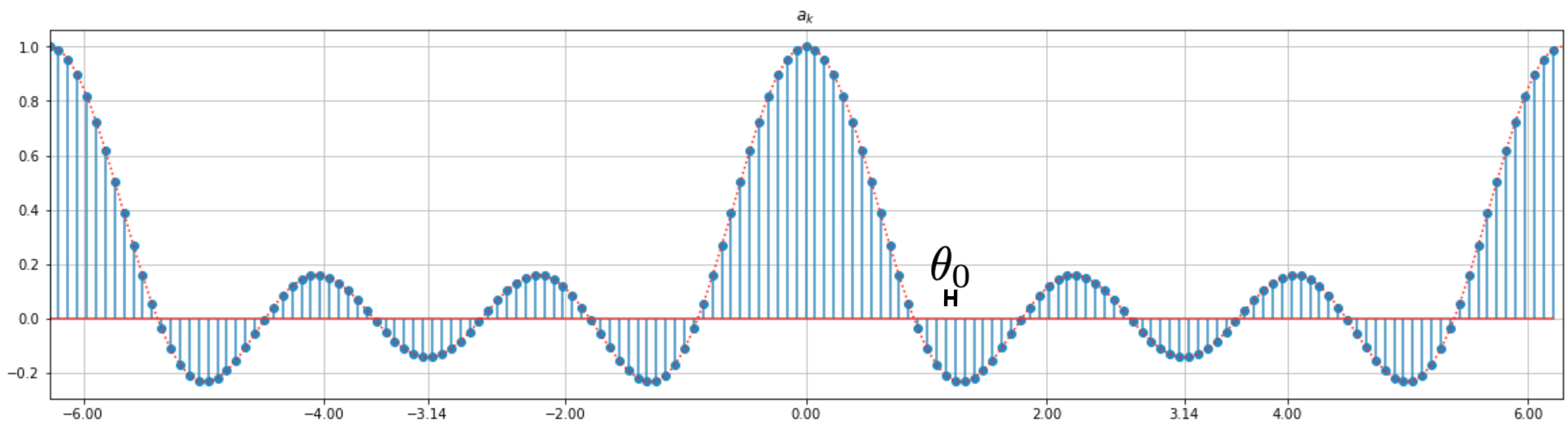
Series de Fourier





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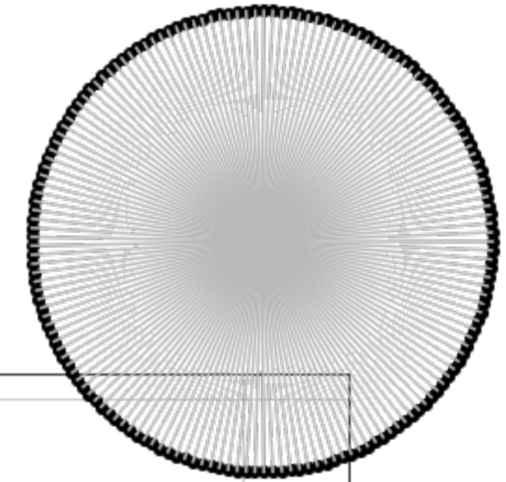
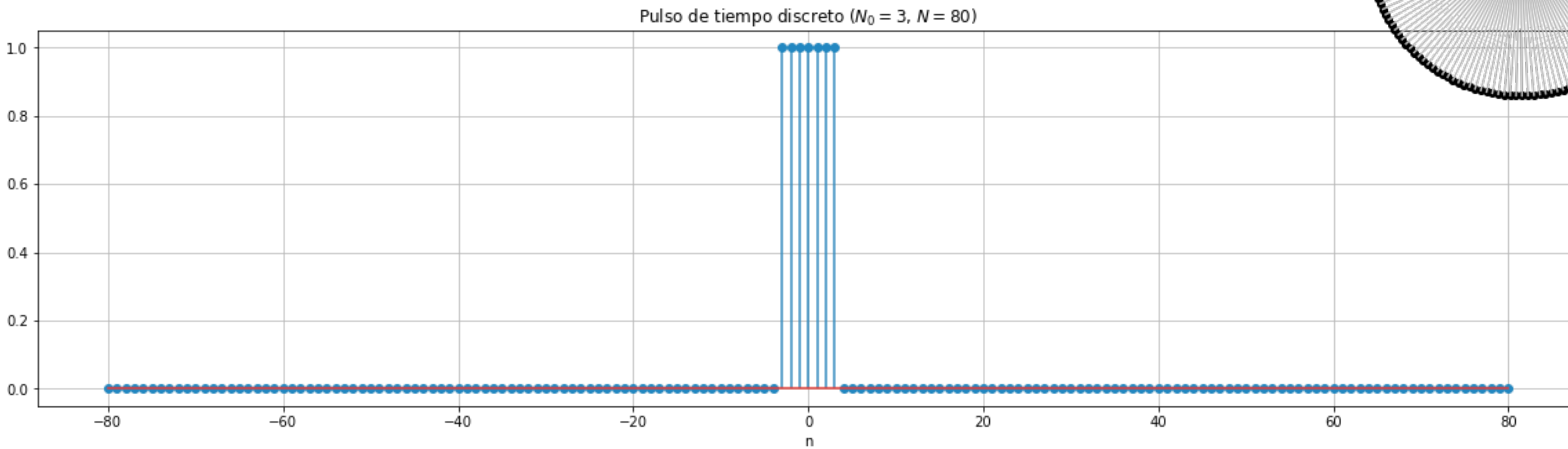
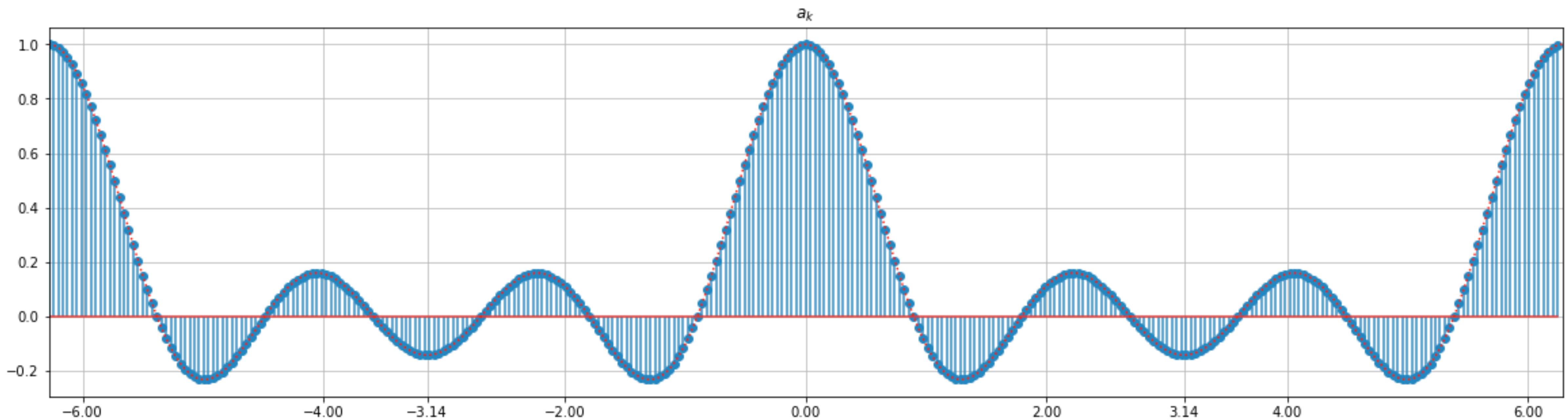
Series de Fourier

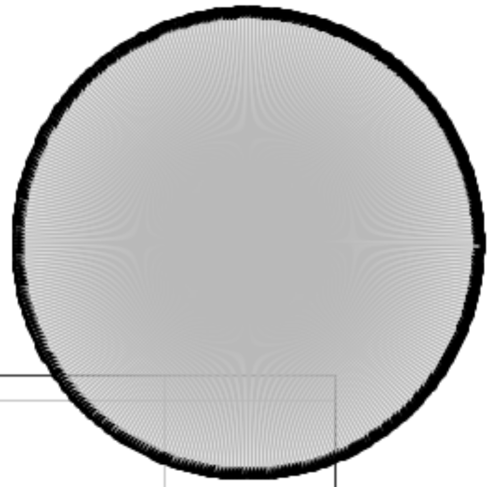


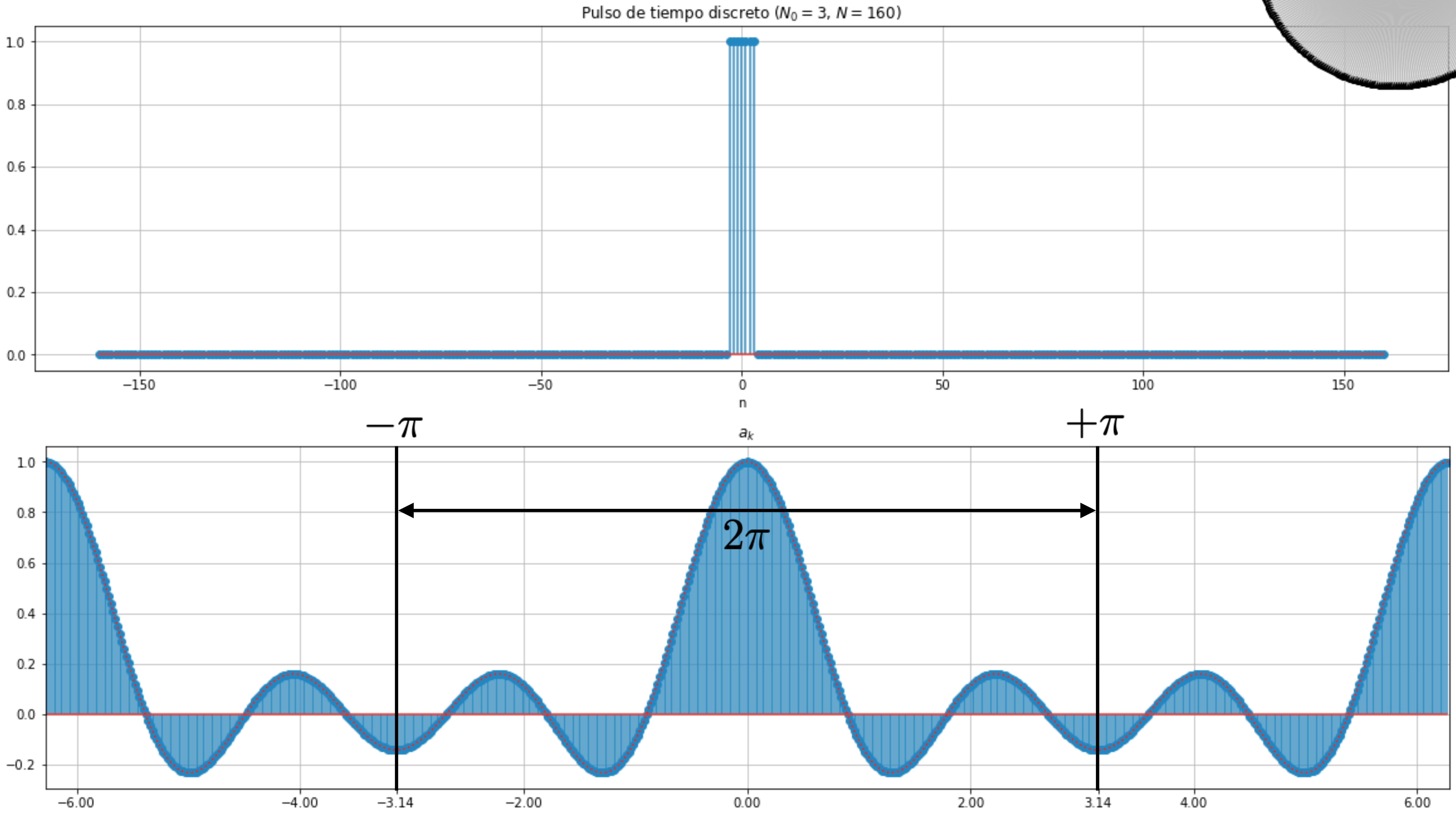




$$x[n] = \Pi\left(\frac{n}{2N_0+1}\right) \stackrel{\text{SF}}{\longleftrightarrow} a_k = \frac{1}{N}\,\frac{\sin({}^1\!/\!_2\,\theta_0 k(N_0+1))}{\sin({}^1\!/\!_2\,\theta_0 k)}$$

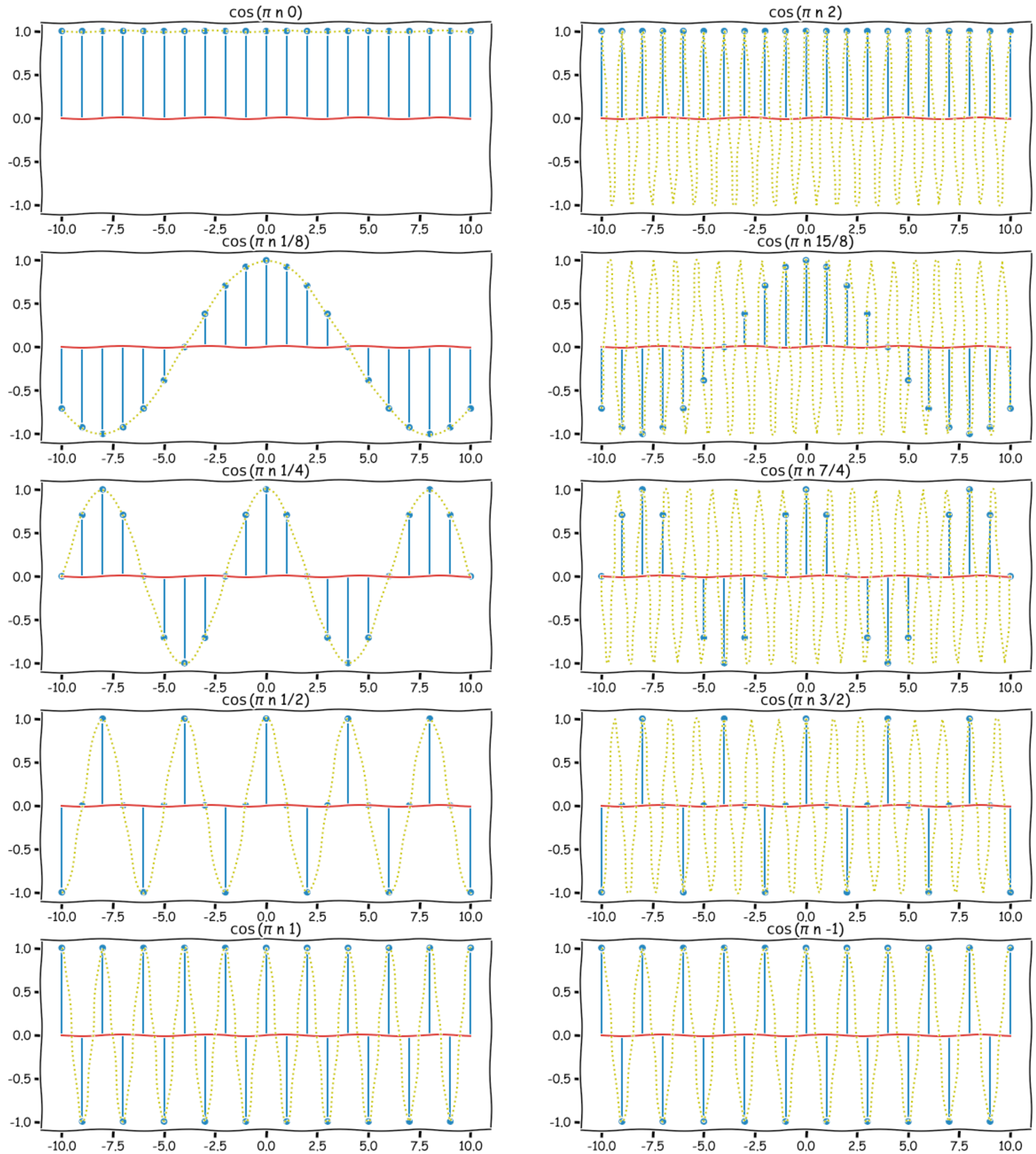
Series de Fourier





$$x[n] = \Pi\left(\frac{n}{2N_0+1}\right) \overset{\text{SF}}{\longleftrightarrow} a_k = \frac{1}{N}\frac{\sin({}^1\!/_2\,\theta_0 k(N_0+1))}{\sin({}^1\!/_2\,\theta_0 k)}$$

Periodicidad de las exponenciales / sinusoidales



De la Serie de Fourier a la DTFT

$$N \to +\infty, \quad \theta_0 = \frac{2\pi}{N} \to 0, \quad \Delta\theta_k = \theta_0(k+1) - \theta_0 k = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N}\sum_{n=N/2}^{N/2} x[n]e^{-j\theta_0 kn} \Rightarrow a_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-j\theta kn}$$

$$X\left(e^{j\theta_k}\right) = Na_k$$

$$x[n] = \sum_{n=N/2}^{N/2} \frac{X\left(e^{j\theta_k}\right)}{N} e^{j\theta_0 kn} = \frac{1}{2\pi}\sum_{n=N/2}^{N/2} X\left(e^{j\theta_k}\right) e^{j\theta_0 kn}\Delta\theta_k =$$

$$\to \frac{1}{2\pi}\int_{-\pi}^{+\pi} X\left(e^{j\theta}\right) e^{j\theta n} d\theta$$

Cuando $N \to \infty$ aumentan en número de fasores $\left(e^{j\theta_k n}\right)$ entre $[0, 2\pi)$ donde se evalúa $X\left(e^{j\theta_k}\right)$. La integral quedan evaluada en un período (cualquiera) de ancho 2π pues $X\left(e^{j\theta_k}\right)$ es periódica 2π.

Transformada de Fourier de variable discreta (DTFT)

- $X\left(e^{j\theta}\right)$ es la TF de $x[n]$ (no es necesario que sea periódica)

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X\left(e^{j\theta}\right) e^{j\theta n} \mathrm{d}\theta \qquad \text{Síntesis}$$

$$X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\theta n} \qquad \text{Análisis}$$

- $X\left(e^{j\theta}\right)$ es periódica 2π.
- ¿Cuál es la DTFT de una señal periódica?
- Convergencia de la DTFT: $x[n]$ es absolutamente sumable

Ejemplos

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, N_2 > N_1, \alpha \neq 1$$

$$x[n] = a^n\, u[n] \quad |a| < 1 \overset{\text{TF}}{\longleftrightarrow} X\left(e^{j\theta}\right) = \sum_{n=0}^{+\infty} a^n e^{-j\theta n} = \frac{1}{1 - a\, e^{-j\theta}}$$

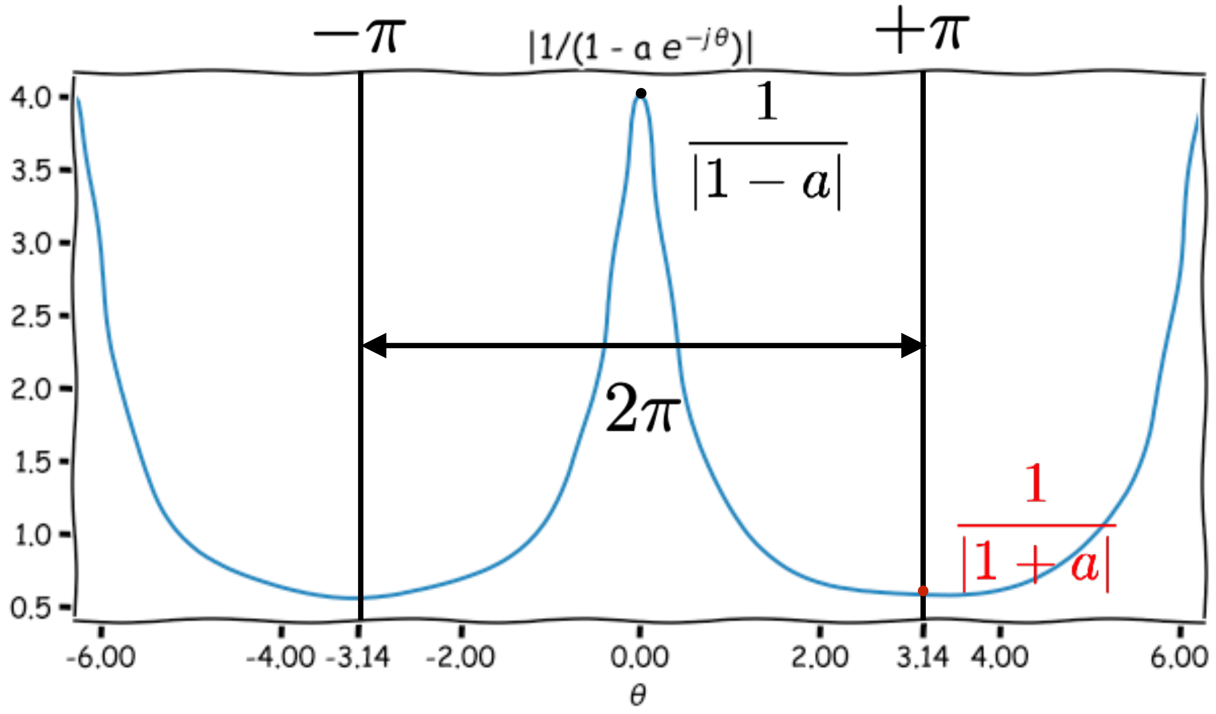
$$x[n] = a^n\, u[n] \overset{\text{TF}}{\longleftrightarrow} X\left(e^{j\theta}\right) = \frac{1}{1 - a\, e^{-j\theta}} \quad |a| < 1$$

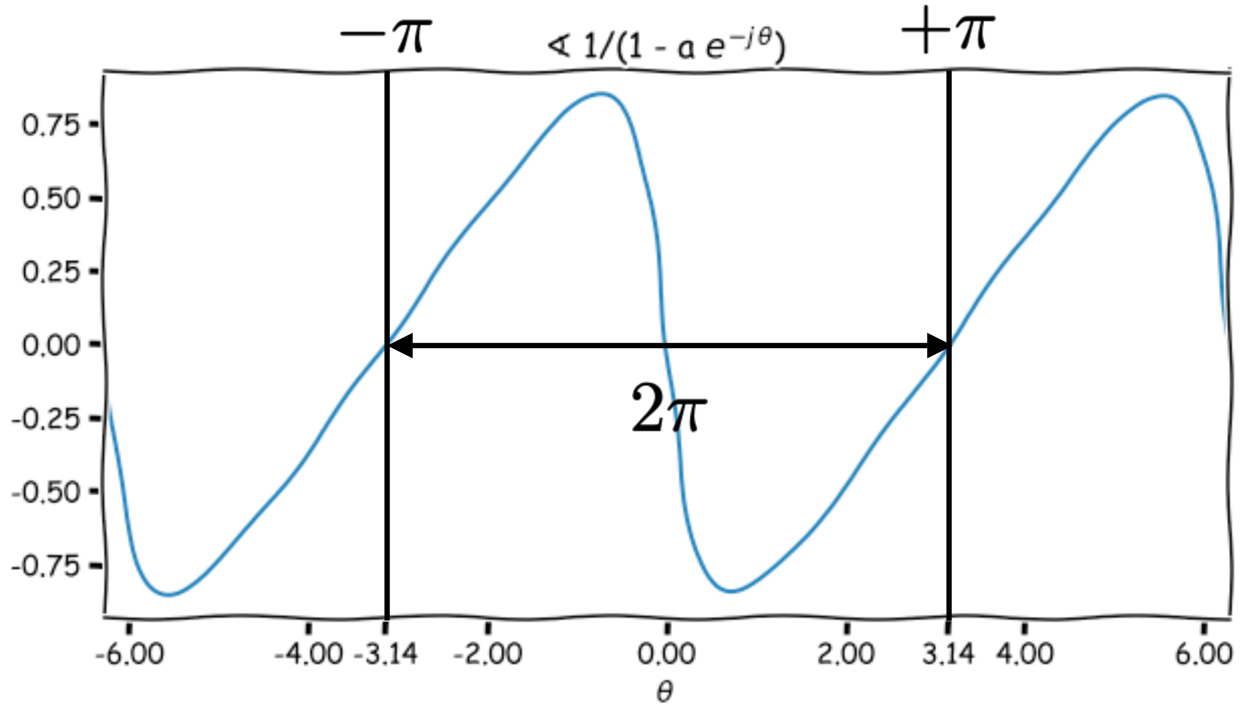
Módulo

$$\frac{1}{|1 - a\, e^{-j\theta}|} \qquad a = 0.75$$

Fase

$$-\sphericalangle(1 - a\, e^{-j\theta})$$





Ejemplos

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, N_2 > N_1,\ \alpha \neq 1$$

$$x[n] = a^n\, u[n] \quad |a| < 1 \overset{\text{TF}}{\longleftrightarrow} X\left(e^{j\theta}\right) = \sum_{n=0}^{+\infty} a^n e^{-j\theta n} = \frac{1}{1 - a\, e^{-j\theta}}$$

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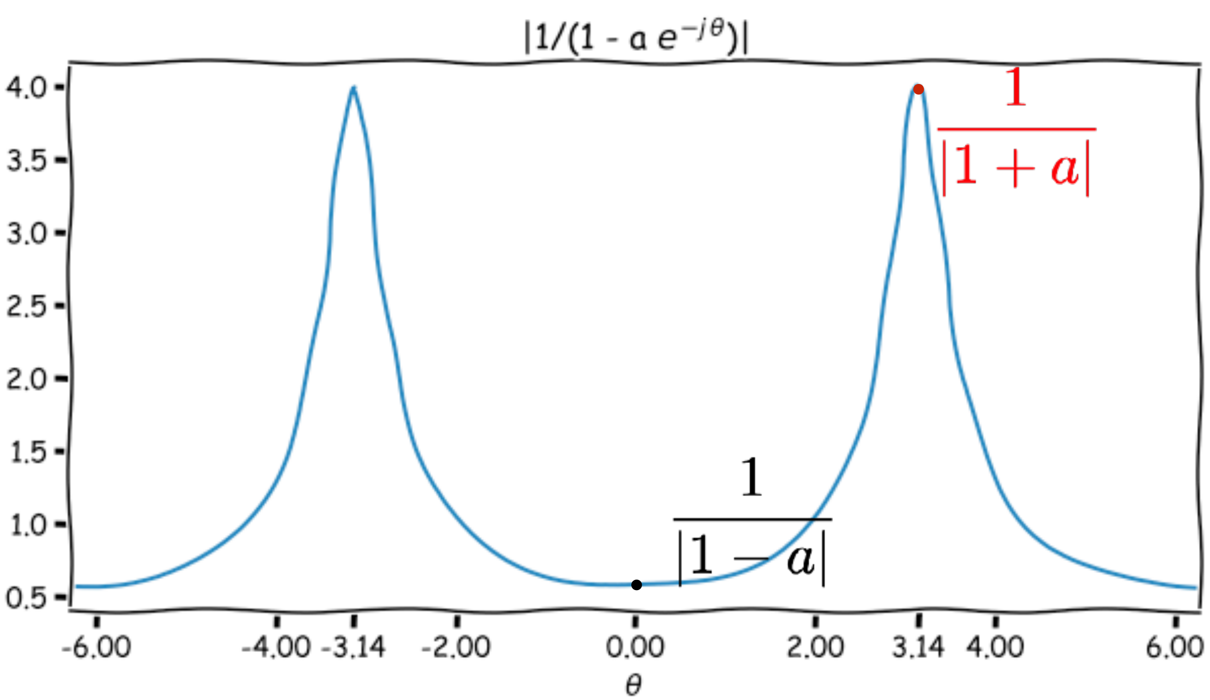
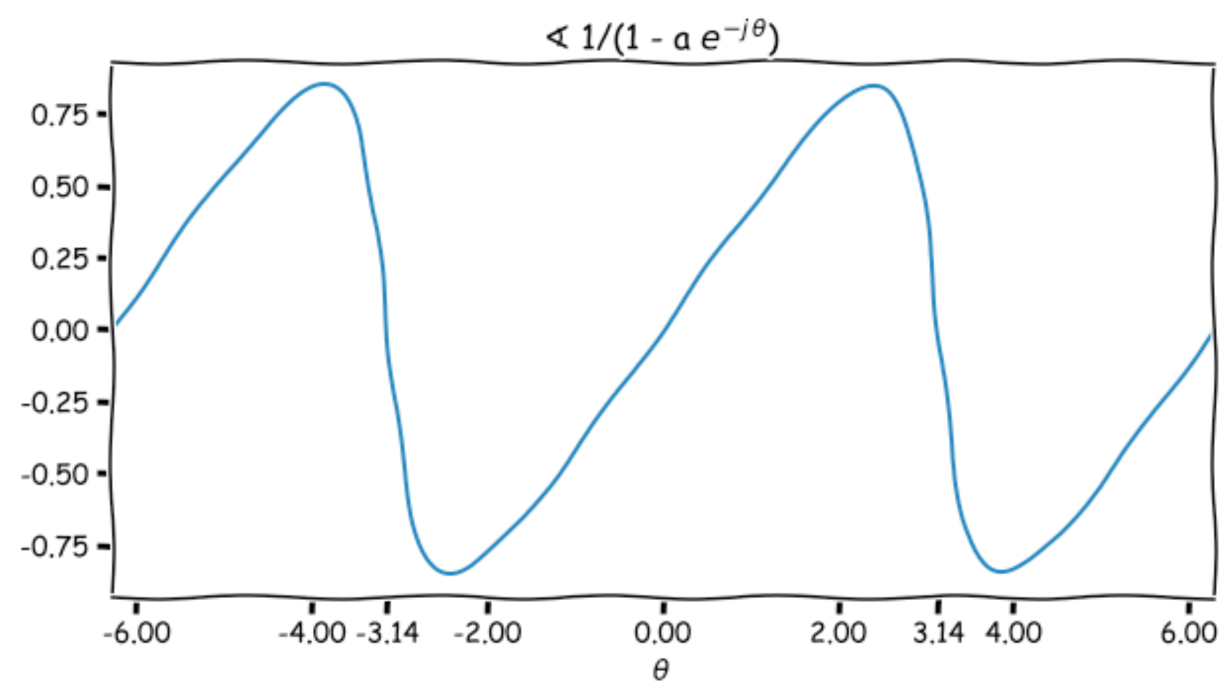
Módulo

$$\frac{1}{|1 - a\, e^{-j\theta}|} \quad a = -0.75$$

Fase

$$-\sphericalangle(1 - a\, e^{-j\theta})$$

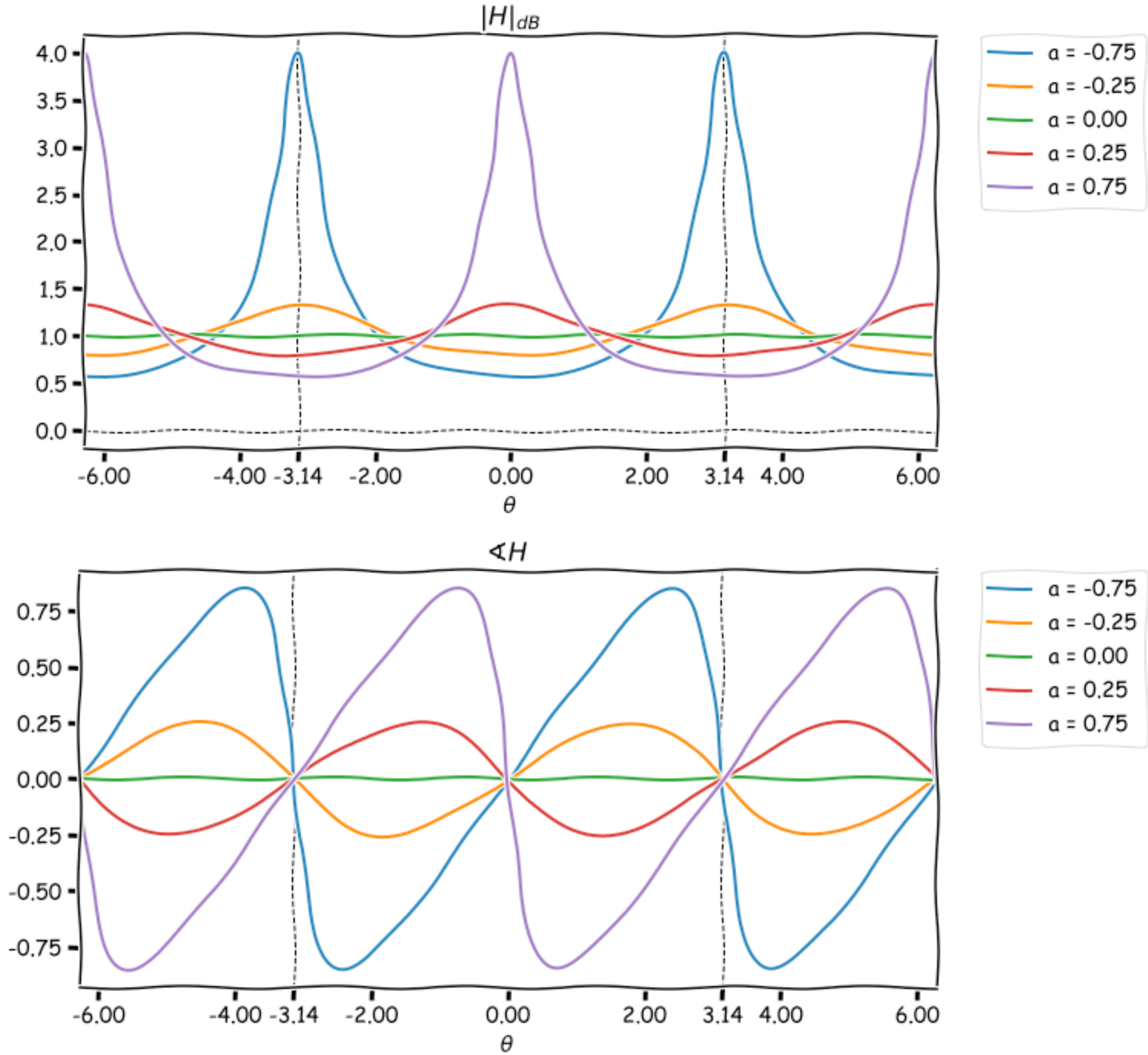




Ejemplos

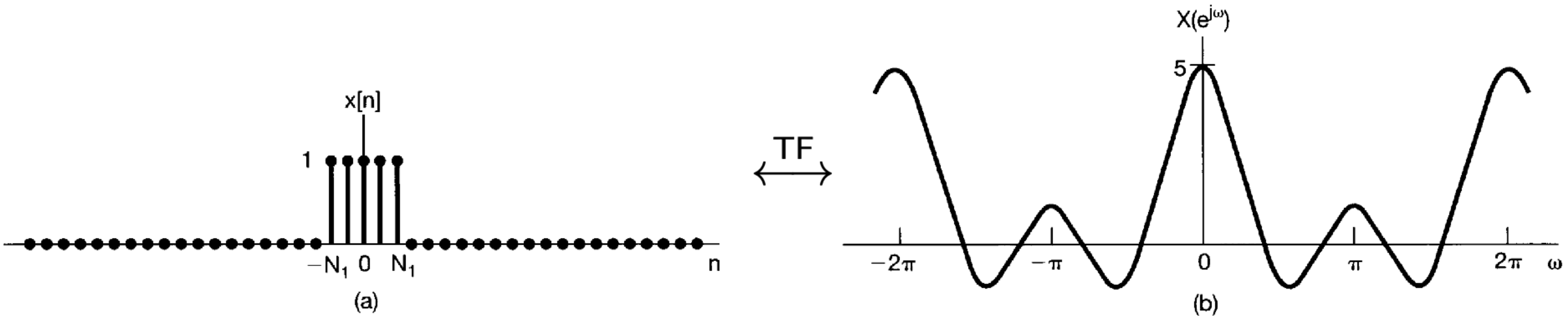
$$X\left(e^{j\theta}\right) = \frac{1}{1 - a\,e^{-j\theta}}$$

$$|a| < 1$$



Ejemplos

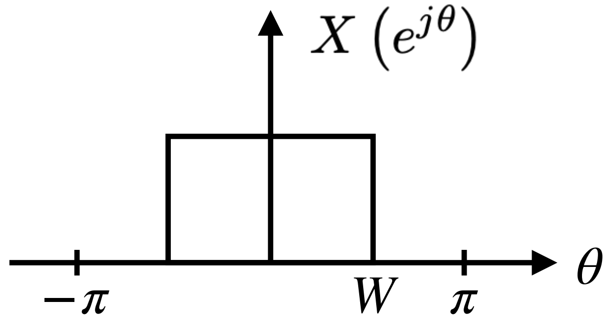
$$x[n] = \Pi\left(\frac{n}{2N_1+1}\right) \overset{\text{TF}}{\longleftrightarrow} X\left(e^{j\theta}\right) = \frac{\sin {}^1\!/_2\,\theta(N_1+1)}{\sin {}^1\!/_2\,\theta}$$

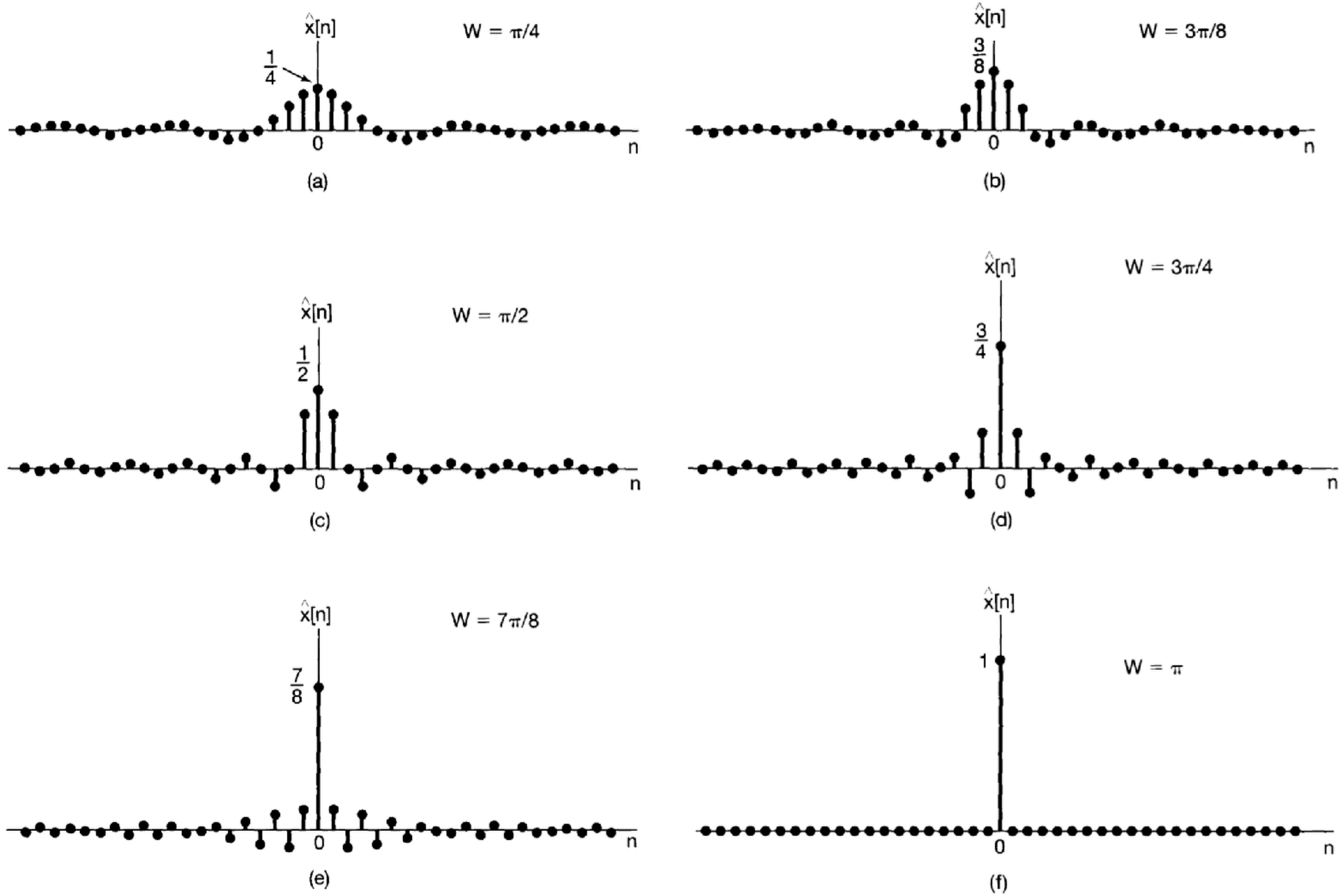


Atención: El libro también usa ω para la frecuencia angular en la DTFT, nosotros usaremos θ para diferenciarla de la frecuencia de la CTFT.

Ejemplos

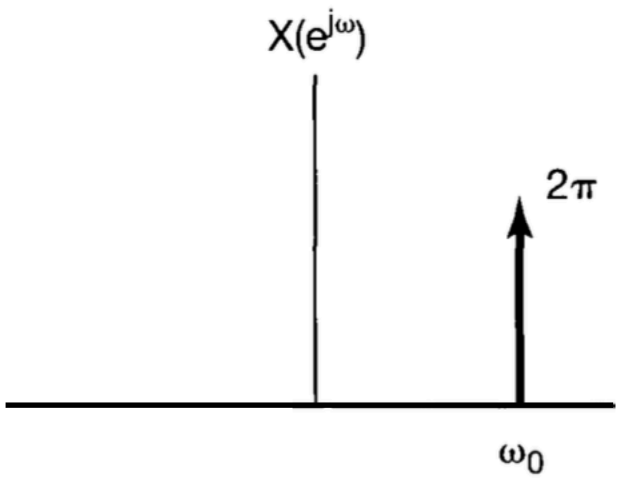
$$X\left(e^{j\theta}\right) = \Pi\left(\theta/2W\right)$$





DTFT de una señal periódica

$x[n] = e^{j\theta_0 n} \quad \Rightarrow x[n]$ solo tiene componentes en $\theta = \theta_0$, $X\left(e^{j\theta}\right)$ es una $\delta(\theta - \theta_0)$



La DTFT es periódica 2π.

$$X\left(e^{j\theta}\right) = \sum_{l=-\infty}^{+\infty} 2\pi\delta\left(\theta - \theta_0 - 2\pi l\right)$$

$$\frac{1}{2\pi}\int_{2\pi} X\left(e^{j\theta}\right) e^{j\theta n} d\theta = e^{j(\theta_0 + 2\pi r)n} = e^{j\theta_0 n}$$

DTFT de una señal periódica

- Consideremos una $x[n]$ periódica, su Serie de Fourier es

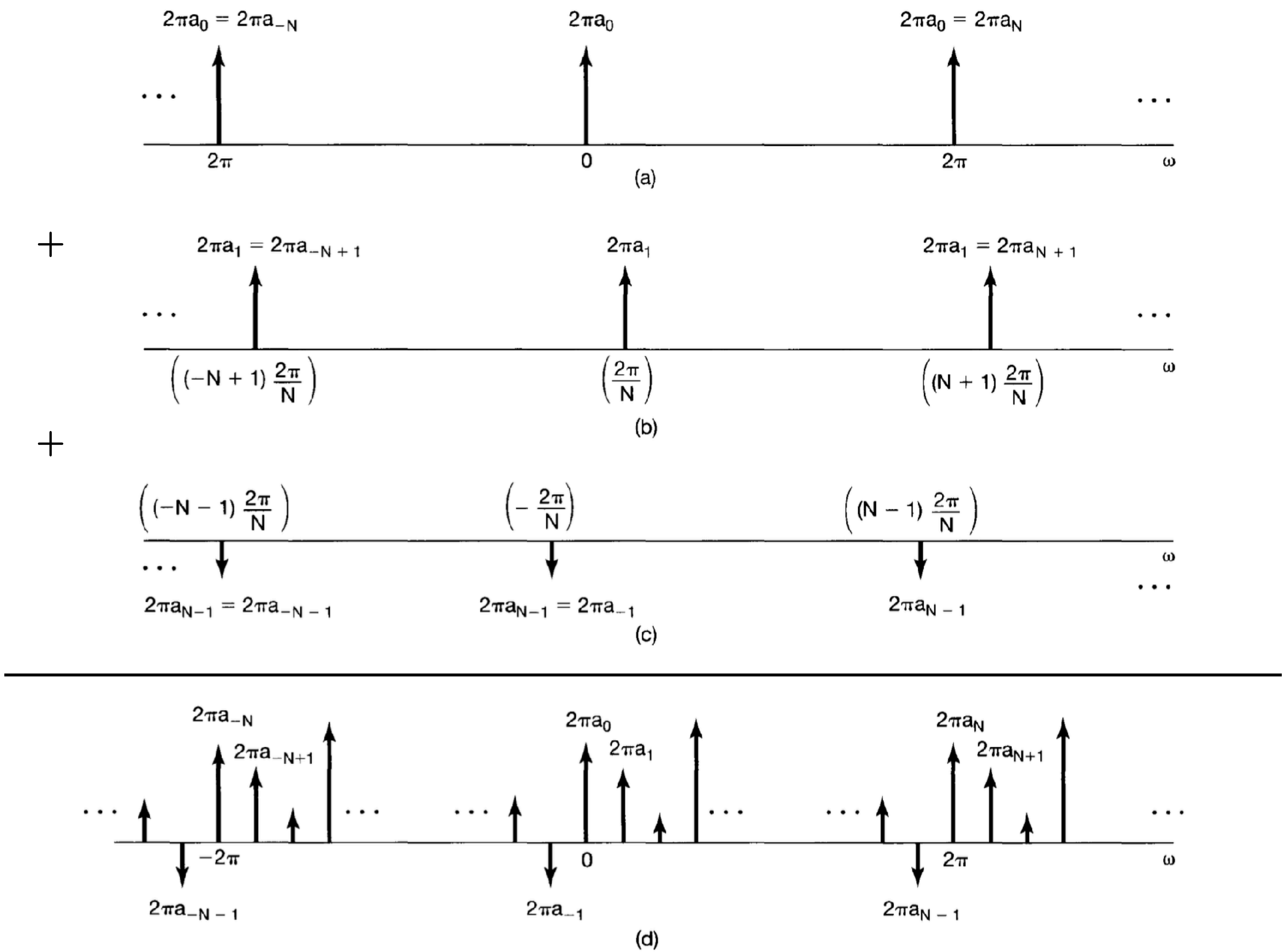
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

a_k: coeficiente de SdeF

$$X\left(e^{j\theta}\right) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\theta - \frac{2\pi k}{N}\right)$$

La DTFT de una señal periódica es un tren de impulsos en el dominio de la frecuencia, con las áreas de los impulsos proporcionales (2π) a los coeficientes de la Serie de Fourier.

DTFT de una señal periódica

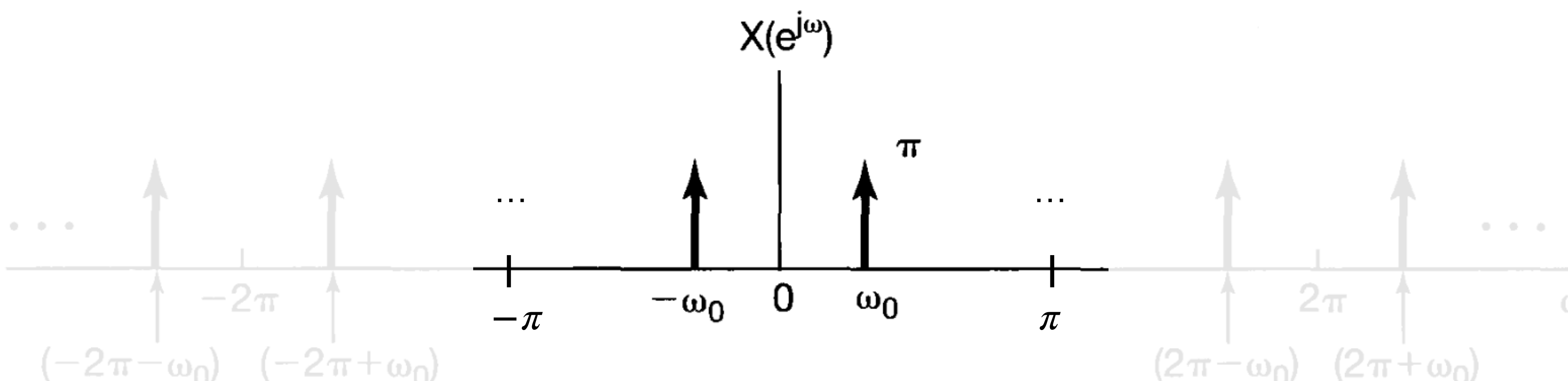


Ejemplos

$$x[n] = \cos\theta_0 n = \frac{1}{2}e^{j\theta_0 n} + \frac{1}{2}e^{-j\theta_0 n}$$

$$X\left(e^{j\theta}\right) = \pi\delta\left(\theta - \theta_0\right) + \pi\delta\left(\theta + \theta_0\right), \quad -\pi \le \theta < \pi$$

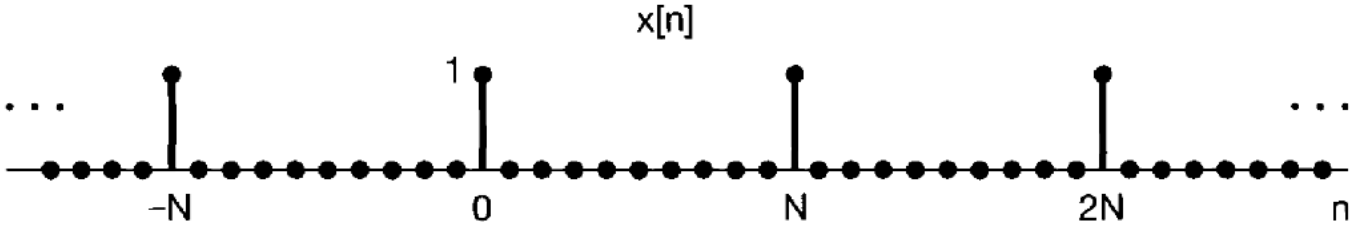
$$X\left(e^{j\theta}\right) = \sum_{l=-\infty}^{+\infty} \pi\delta\left(\theta - \theta_0 - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\delta\left(\theta + \theta_0 - 2\pi l\right)$$



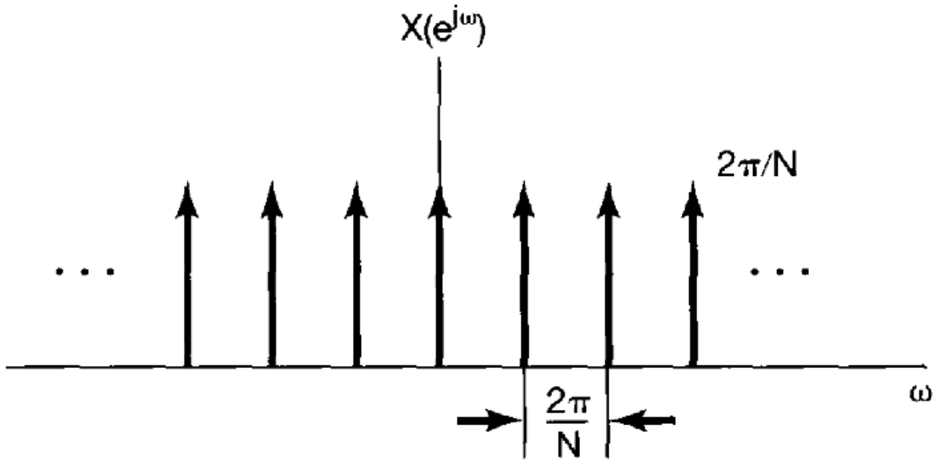
Se debe dejar claro en los bosquejos que se está graficando una señal periódica.

- Peine de Dirac o Tren de Impulsos

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN] \Rightarrow a_k = \frac{1}{N}\,\forall k$$



$$X\left(e^{j\theta}\right) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\theta - \frac{2\pi k}{N}\right)$$



DTFT: propiedades

$$x_1[n] \overset{\text{TF}}{\longleftrightarrow} X_1\left(e^{j\theta}\right)$$

$$x_2[n] \overset{\text{TF}}{\longleftrightarrow} X_2\left(e^{j\theta}\right)$$

- Linealidad

$$ax_1[n] + bx_2[n] \overset{\text{TF}}{\longleftrightarrow} aX_1\left(e^{j\theta}\right) + bX_2\left(e^{j\theta}\right)$$

- Desplazamiento temporal y frecuencial

$$x[n-n_0] \overset{\text{TF}}{\longleftrightarrow} e^{-j\theta n_0} X\left(e^{j\theta}\right) \qquad e^{j\theta_0 n} x[n] \overset{\text{TF}}{\longleftrightarrow} X\left(e^{j(\theta-\theta_0)}\right)$$

- Conjugación

$$x^*[n] \overset{\text{TF}}{\longleftrightarrow} X^*\left(e^{-j\theta}\right)$$

- Diferenciación

$$x[n] - x[n-1] \overset{\text{TF}}{\longleftrightarrow} \left(1 - e^{-j\theta}\right) X\left(e^{j\theta}\right)$$

- Acumulación

$$\sum_{m=-\infty}^{m=n} x[m] \overset{\text{TF}}{\longleftrightarrow} \frac{1}{1-e^{-j\theta}} X\left(e^{j\theta}\right) + \pi X\left(e^{j0}\right) \sum_k \delta(\theta - 2\pi k)$$

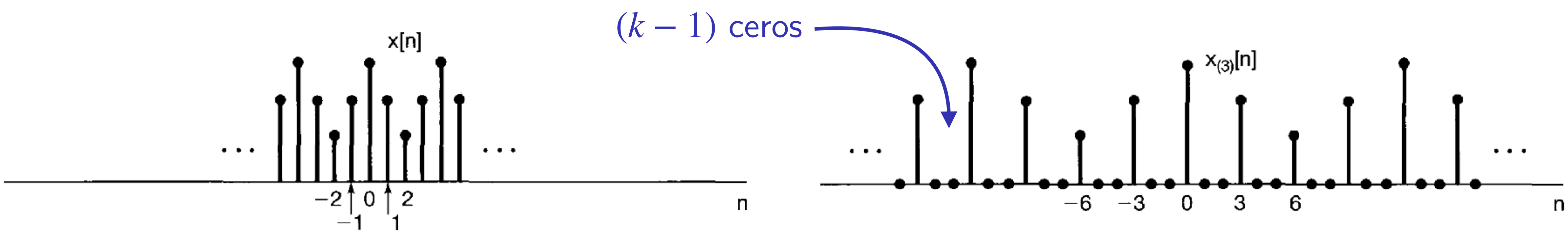
- Reversión temporal

$$x[-n] \overset{\text{TF}}{\longleftrightarrow} X\left(e^{-j\theta}\right)$$

DTFT: propiedades

- Escalado temporal

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{si } n \text{ es múltiplo de } k \ (n/k \in \mathbb{Z}) \\ 0 & \text{si } n \text{ no es múltiplo de } k \ (n/k \notin \mathbb{Z}) \end{cases}$$

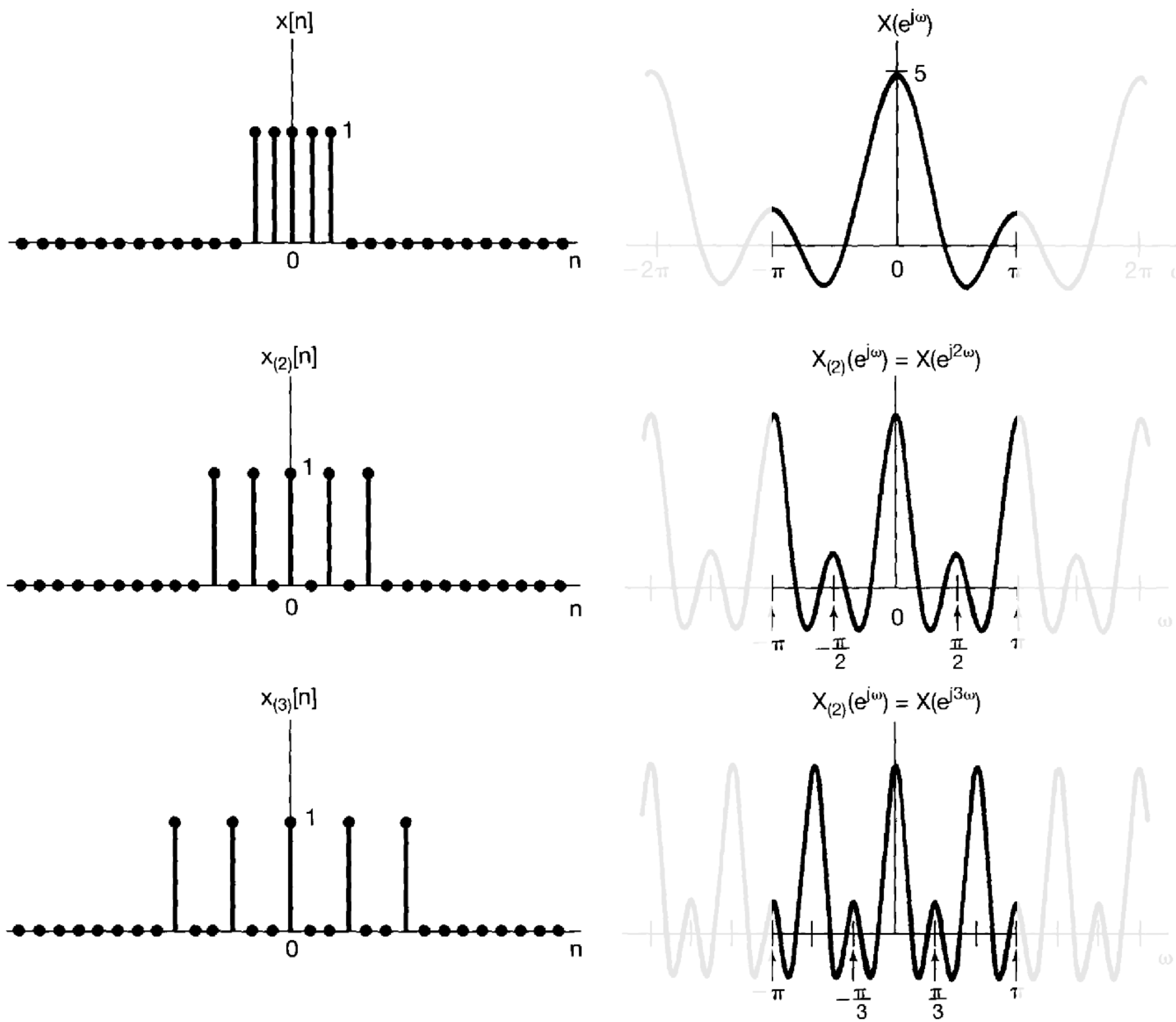


$$X_{(k)}\left(e^{j\theta}\right) = \sum_k x_{(k)}[n] e^{-j\theta n} = \sum_r \overbrace{x_{(k)}[rk]}^{x[r]} e^{-j\theta r k} = X\left(e^{j\theta k}\right)$$

Se comprime el espectro; en $[-\pi, \pi)$ hay k copias de $X\left(e^{j\theta}\right)$.

DTFT: propiedades

- Escalado temporal (ejemplo)



DTFT: propiedades

- Identidad de Parseval

$$\sum_n |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X\left(e^{j\theta}\right)\right|^2 d\theta$$

- Convolución (y filtrado)

$$y[n] = x[n] * h[n] \stackrel{\text{TF}}{\longleftrightarrow} Y\left(e^{j\theta}\right) = X\left(e^{j\theta}\right) H\left(e^{j\theta}\right)$$

- Multiplicación

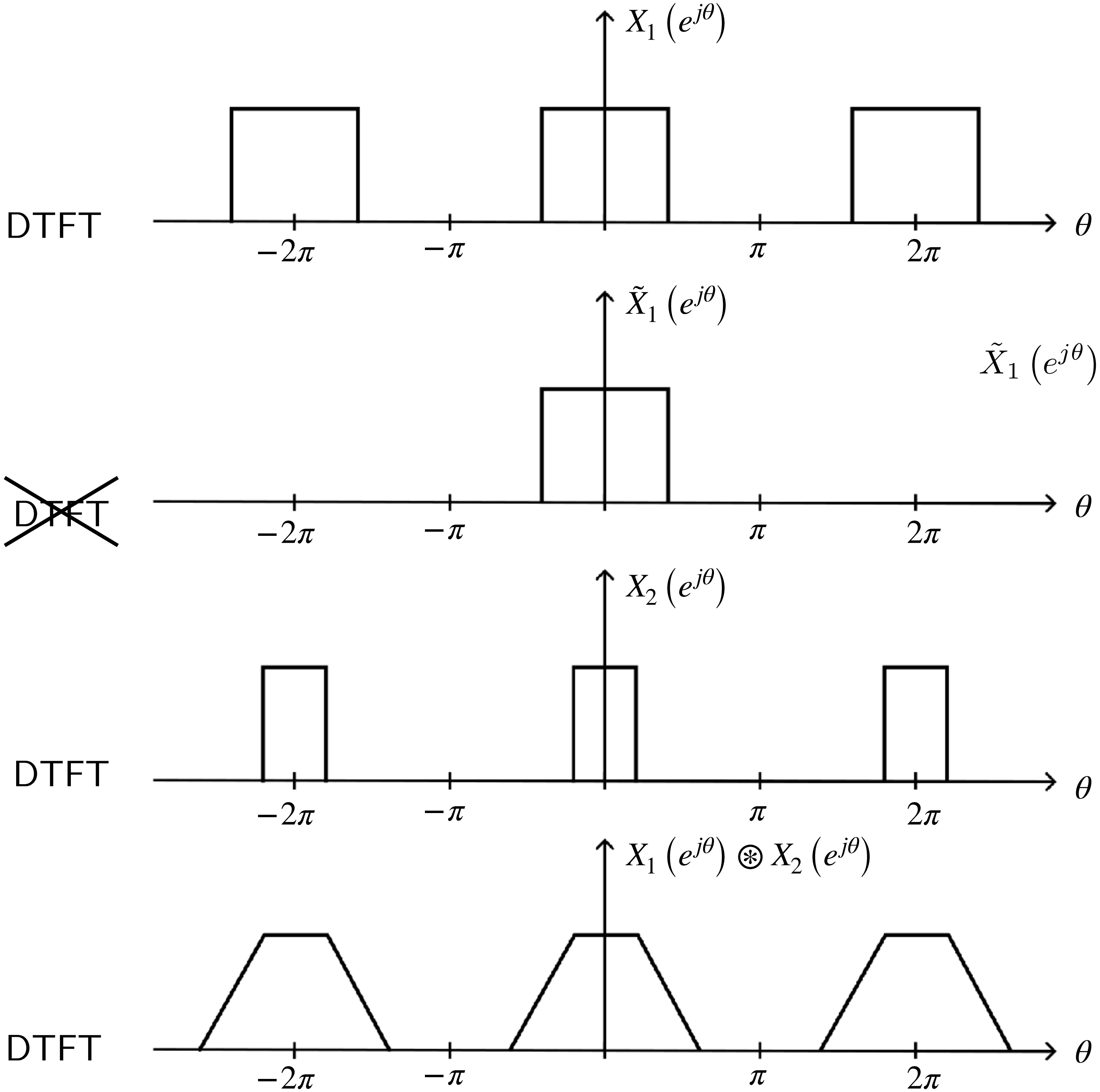
$$y[n] = x_1[n]\, x_2[n] \stackrel{\text{TF}}{\longleftrightarrow} Y\left(e^{j\theta}\right) = \frac{1}{2\pi}\int_{2\pi} X_1\left(e^{j\lambda}\right) X_2\left(e^{j(\theta-\lambda)}\right) d\lambda$$

$$= X_1\left(e^{j\theta}\right) \circledast X_2\left(e^{j\theta}\right)$$

Convolución circular

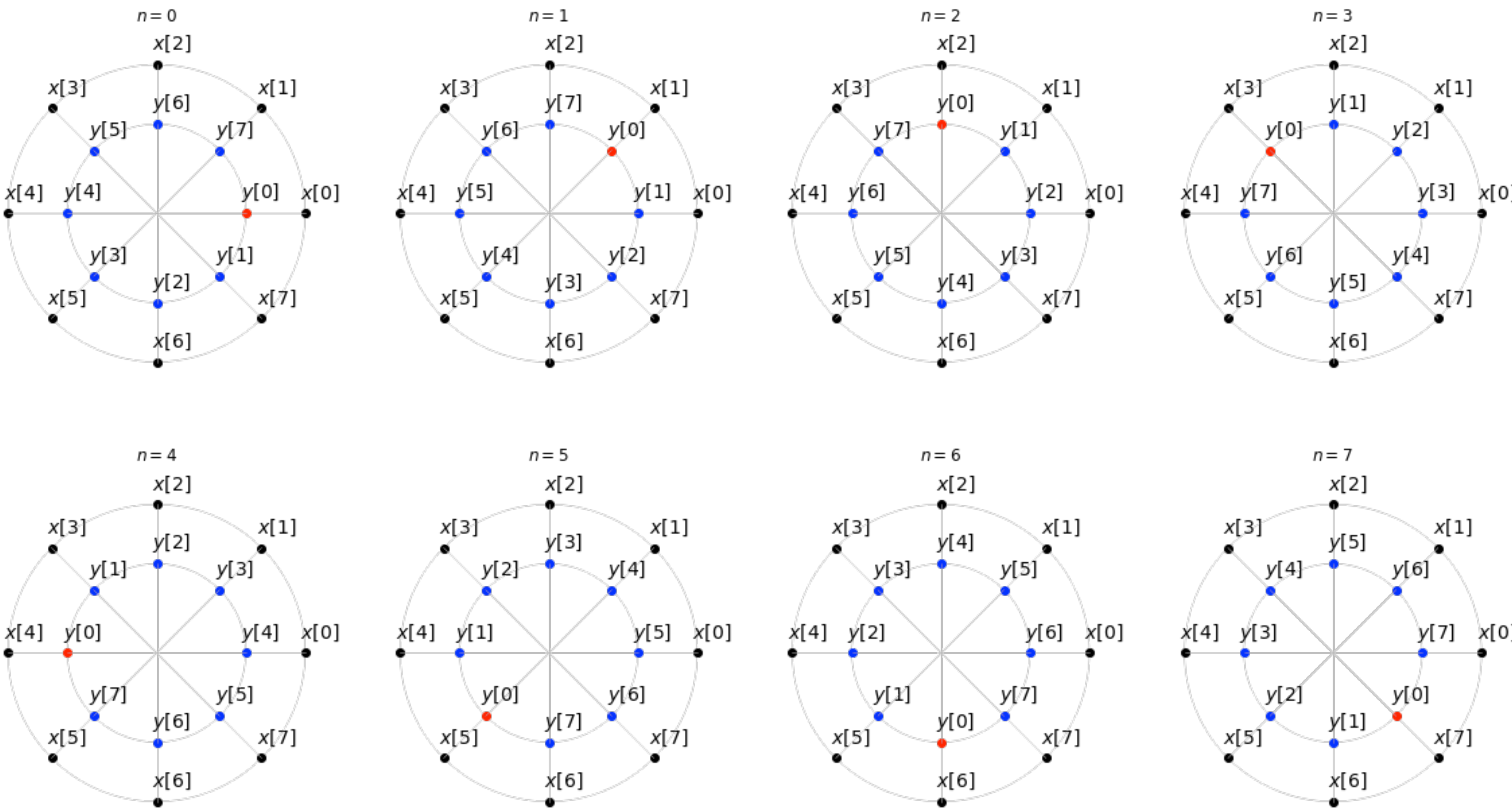
Convolución circular (variable continua)

$$X_1\left(e^{j\theta}\right) \circledast X_2\left(e^{j\theta}\right) = \tilde{X}_1\left(e^{j\theta}\right) * X_2\left(e^{j\theta}\right)$$



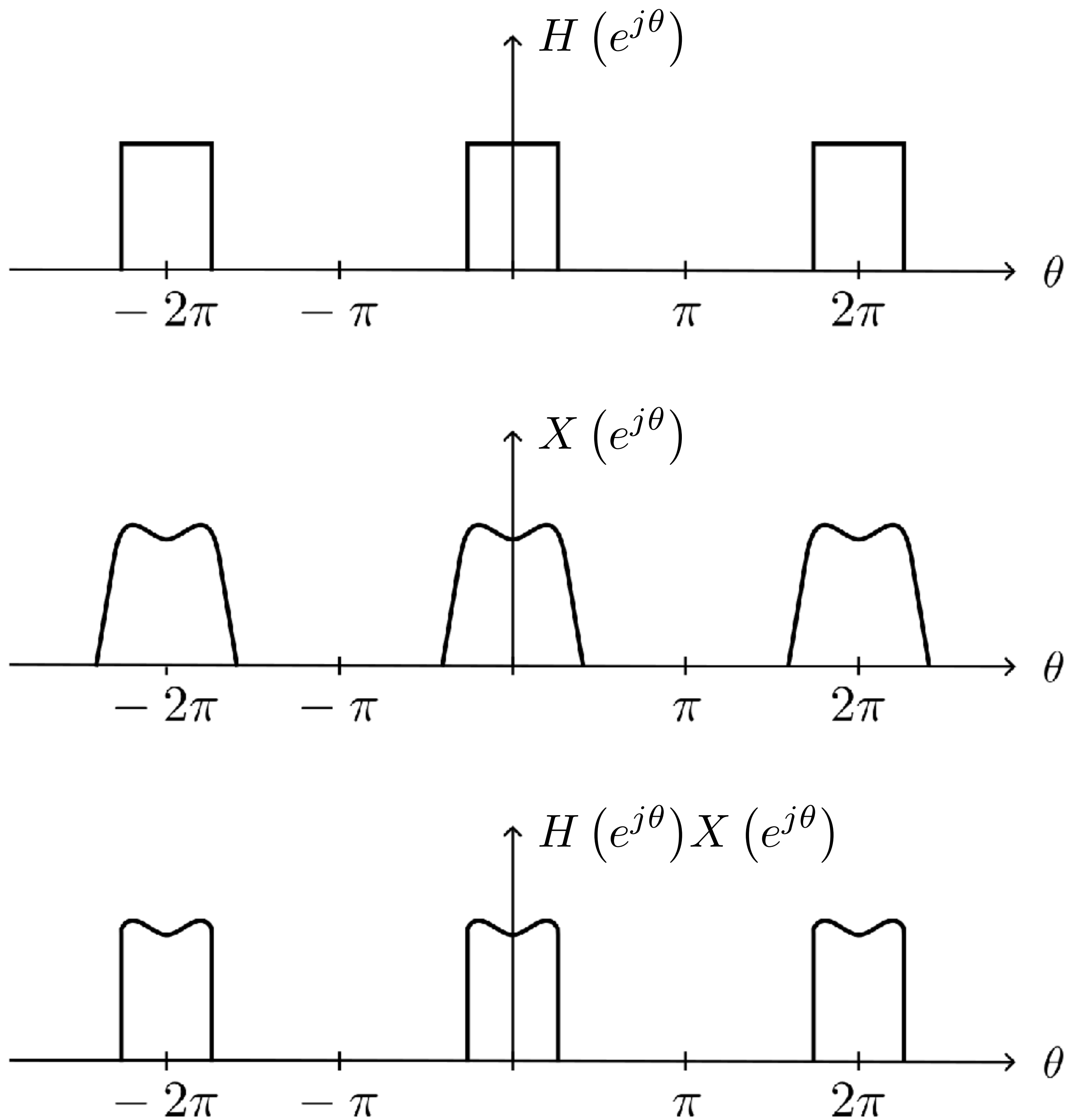
$$\tilde{X}_1\left(e^{j\theta}\right) = \begin{cases} X_1\left(e^{j\theta}\right) & -\pi < \theta < \pi \\ 0 & \text{en otro caso} \end{cases}$$

Convolución circular (variable discreta)



Filtrado

- Respuesta frecuencia de un filtro $H\left(e^{j\theta}\right)$ periódica 2π (módulo y fase).



LPF

Filtrado

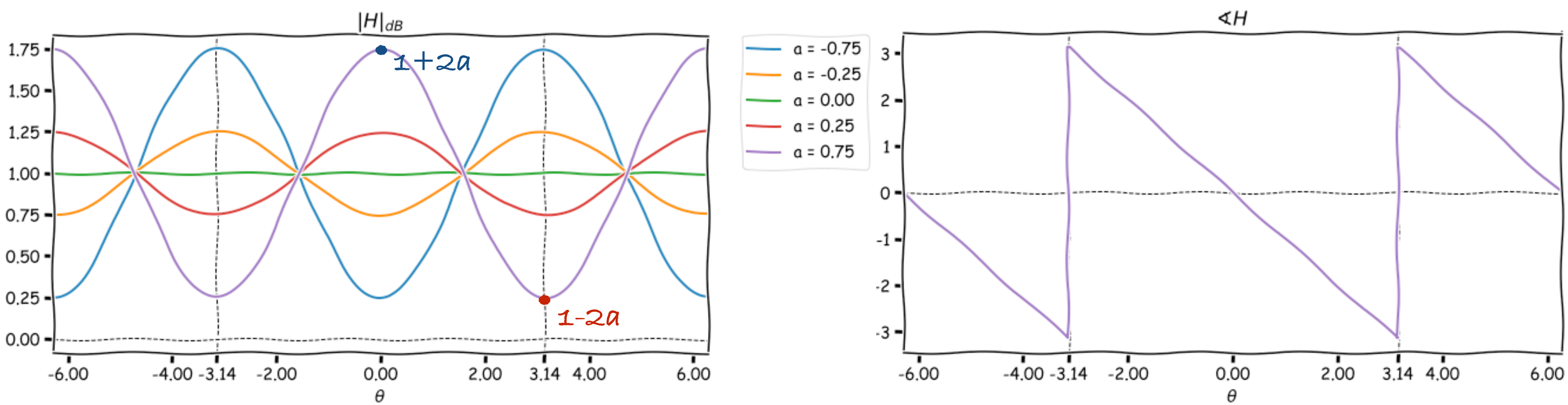
$$y[n] = a\,x[n-2] + x[n-1] + a\,x[n] \overset{TF}{\longleftrightarrow} Y(e^{j\theta}) = a e^{-j2\theta} X(e^{j\theta}) + e^{-j\theta} X(e^{j\theta}) + a X(e^{j\theta})$$

$$= X(e^{j\theta}) e^{-j\theta} \left(1 + a\left(e^{-j\theta} + e^{j\theta}\right)\right)$$

$$= X(e^{j\theta}) e^{-j\theta} \left(1 + 2a \cos\theta\right)$$

$$\Rightarrow H(e^{j\theta}) = e^{-j\theta}\left(1 + 2a\cos\theta\right) \overset{TF}{\longleftrightarrow} h[n] = a\,\delta[n-2] + \delta[n-1] + a\,\delta[n]$$

FIR: Finite Impulse Response



LPF para $a > 0$, HPF para $a < 0$. Fase lineal.

Filtrado

$$y[n] = a\,y[n-1] + x[n] \overset{TF}{\longleftrightarrow} Y(e^{j\theta}) = a\,e^{-j\theta}\,Y(e^{j\theta}) + X(e^{j\theta})$$

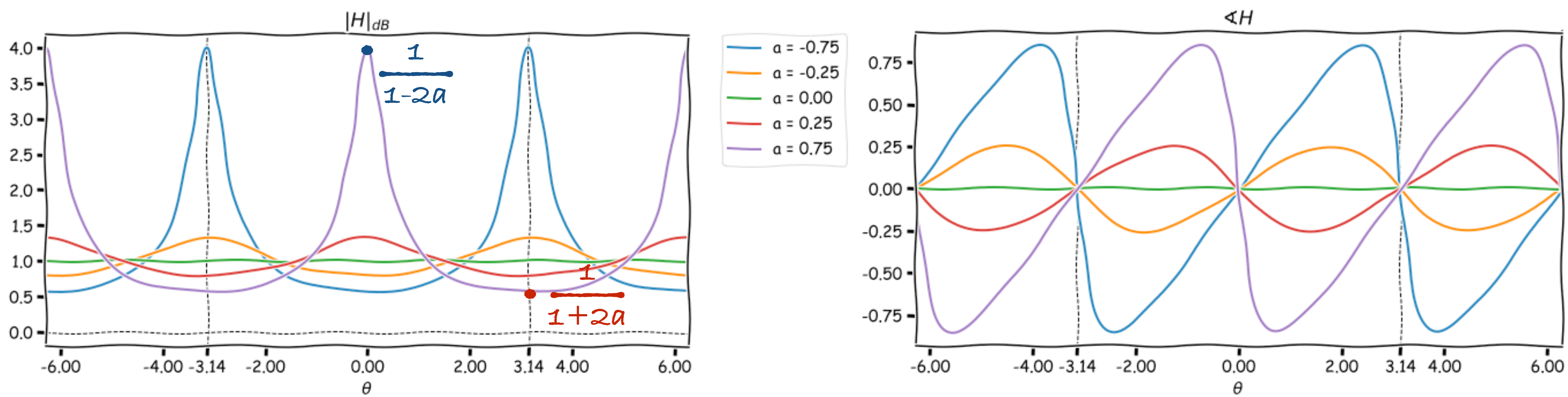
$$Y(e^{j\theta})\left(1 - a e^{-j\theta}\right) = X(e^{j\theta})$$

$$Y(e^{j\theta}) = \frac{X(e^{j\theta})}{1+ae^{-j\theta}}$$

$$H(e^{j\theta}) = \frac{1}{1-ae^{-j\theta}} \quad |a|<1 \overset{TF}{\longleftrightarrow} h[n] = a^n u[n]$$

IIR: Infinite Impulse Response

VEREMOS MÁS SOBRE FILTROS DIGITALES LUEGO DE TZ

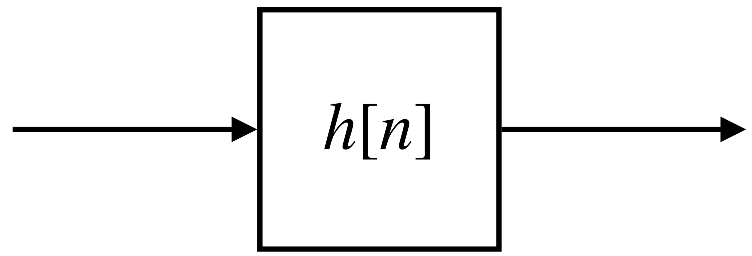


LPF para $a > 0$, HPF para $a < 0$. Fase no lineal.

Filtrado de señales periódicas con DTFT

- La salida del filtrado de una señal periódica es una señal periódica.
  - $a_x[k]$ periódico N

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_x[k] e^{j\theta_0 k n}$$



$$\tilde{y}[n] = \sum_{k=0}^{N-1} a_y[k] e^{j\theta_0 k n}$$

$$\boxed{a_y[k] = a_x[k] H(e^{j\theta_0 k})}$$

Dualidad DTFT $\leftrightarrow$ CT Fourier Series

DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\theta}\right) e^{j\theta n} d\theta$$

$$X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\theta n}$$

CTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt.$$

- Representación en SF de la señal $X\left(e^{j\theta}\right)$ periódica 2π.
  - Es la suma de armónicos de exponenciales $e^{j\theta n}$
  - Los coeficientes son $x[-n]$

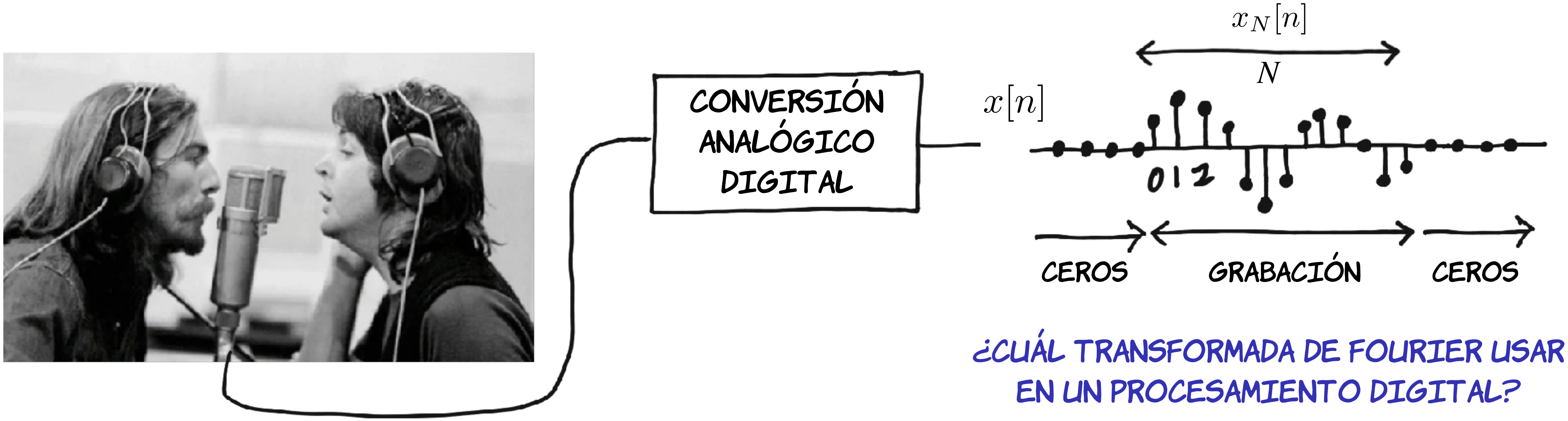
La familia de la Transformada de Fourier

	Frecuencia discreta	Frecuencia continua
Tiempo continuo	**FS** $x(t) = \sum_{k=-\infty}^{+\infty} a_x[k] e^{j\omega_0 kt}$ periódico T_0 $\omega_0 = \frac{2\pi}{T_0}$ $a_x[k] = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-j\omega_0 t} dt$	**CTFT** $x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$
Tiempo discreto		**DTFT** $x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\theta}\right) e^{j\theta n} d\theta$ $X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\theta n}$ periódico 2π

⟷ Dualidad

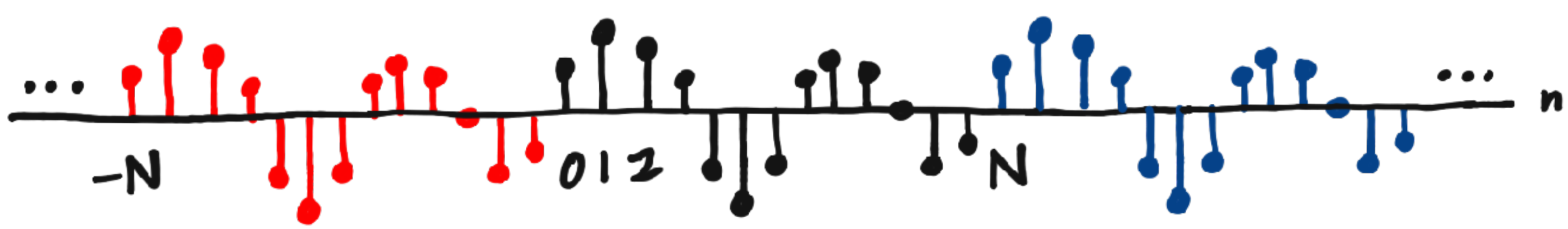
Transforma Discreta de Fourier (DFT)

Transforma Discreta de Fourier (DFT)



- ¿TF con tiempo y frecuencia discretas ($x[n] \overset{\text{TF}}{\longleftrightarrow} X[k]$)?
- Consideremos $x_N[n]$ la señal de N muestras y $\tilde{x}[n]$ periódica tal que

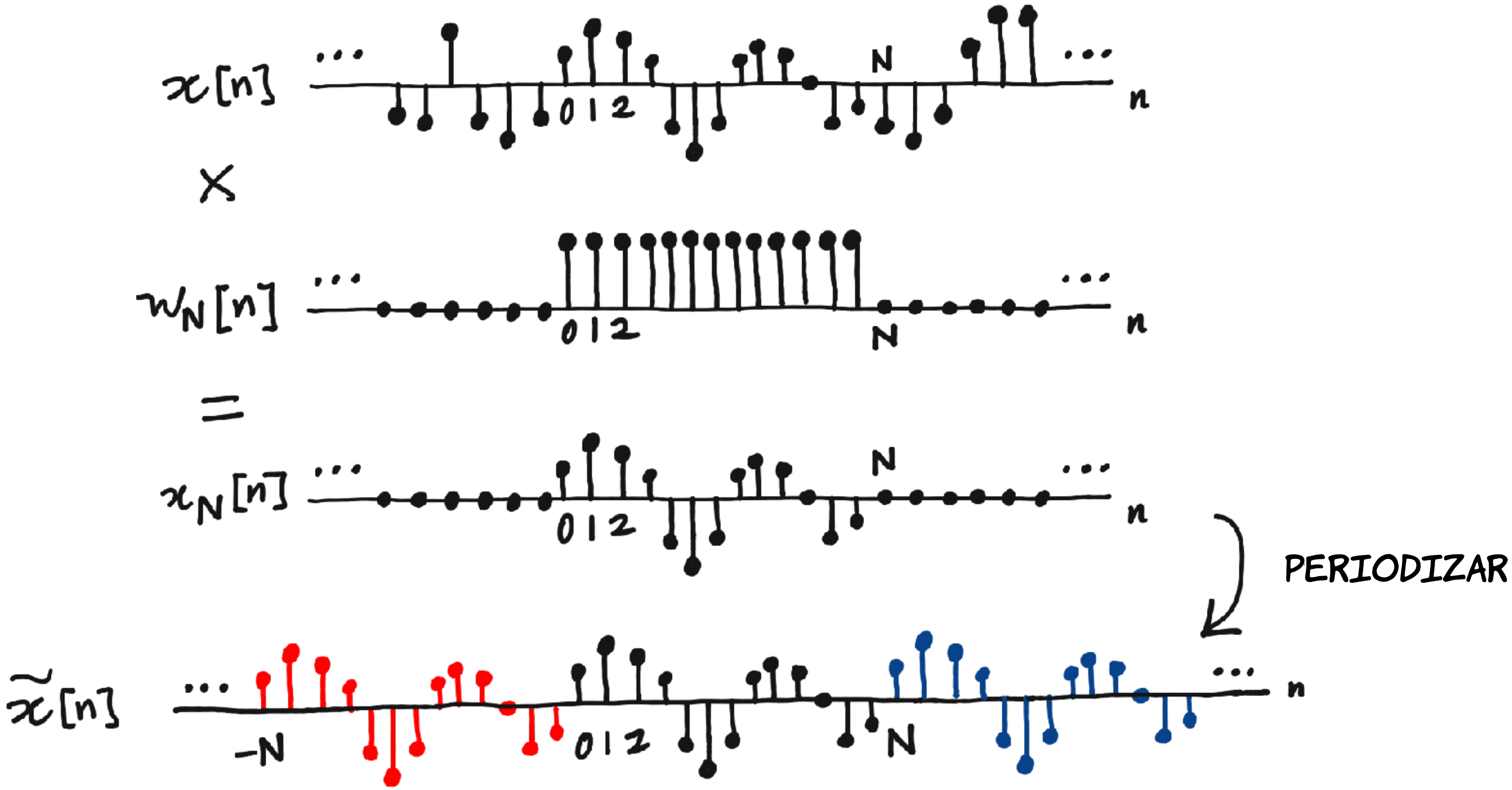
$$\tilde{x}[n] = x_N[n] \quad n = 0, \cdots, N-1$$



- La SdeF de $\tilde{x}[n]$ es

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a[k] e^{j\frac{2\pi}{N}kn} \quad \text{con} \quad a[k] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} \quad (\text{periódica } N)$$

Enventanado de una señal



- La señal a la que se le calcula la DFT $\tilde{x}[n]$ es la periodización N de $x_N[n]$ obtenida como el producto entre la señal original $x[n]$ y una función de ventana $w_N[n]$.
  - Efectos en el espectro de multiplicar y periodizar la señal.

VEREMOS MÁS DEL ENVENTANADO LUEGO DE LA TZ

Transforma Discreta de Fourier (DFT)

DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\theta_0 kn} \qquad \text{Síntesis} \qquad \theta_0 = \frac{2\pi}{N}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\theta_0 kn} \qquad \text{Análisis} \qquad n, k \in [0, 1, \cdots, N-1]$$

- $x[n]$ periódica N.
- $X[k]$ periódica N.
- Definiendo los vectores

$$\mathbf{x} = (x[0], x[1], \cdots, x[N-1])^T$$

$$\mathbf{X} = (X[0], X[1], \cdots, X[N-1])^T$$

podemos escribir una relación matricial entre ambos. Definamos

$$W_N = e^{j\theta_0} = e^{j\frac{2\pi}{N}}$$

REPRESENTA UNA ROTACIÓN DE ÁNGULO θ_0

donde W_N^{kn} son los coeficientes de la matriz $\mathbf{W}_N$.

Transforma Discreta de Fourier (DFT)

- $x[n]$ y $X[k]$ son vectores en $\mathbb{R}^N$, y $\mathbf{W}_N \in \mathbb{R}^{N\times N}$

$$\mathbf{W}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^n & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^{2\,2} & \cdots & W_N^{2n} & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^k & W_N^{2k} & \cdots & W_N^{kn} & \cdots & W_N^{k(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2N-1} & \cdots & W_N^{n(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

ES UNA MATRIZ DE VANDERMONDE

- $\mathbf{W}_N$ es ortonormal

$$\mathbf{W}_N^{\star}\mathbf{W}_N = I \;\Rightarrow \mathbf{W}_N^{-1} = \mathbf{W}_N^{\star}$$

- La DFT queda

$$\mathbf{x} = \frac{1}{\sqrt{N}}\mathbf{W}_N^{*}\,\mathbf{X} \qquad \mathbf{X} = \sqrt{N}\,\mathbf{W}_N\,\mathbf{x}$$

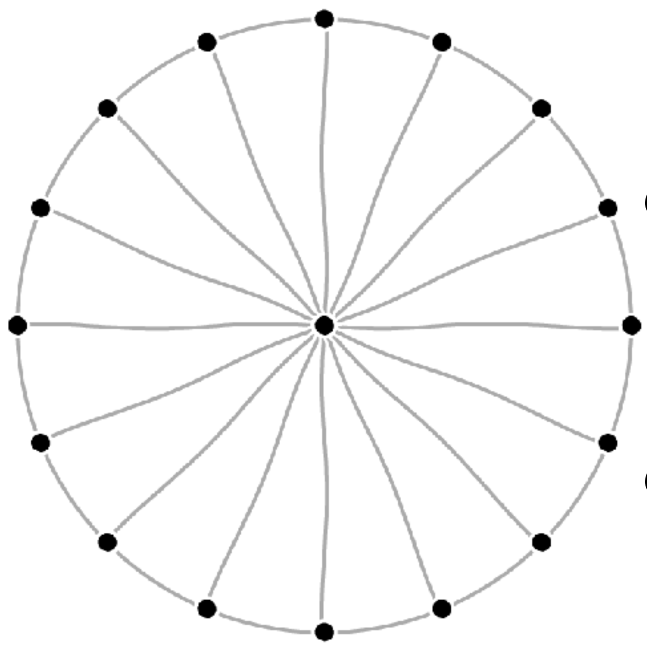
La DFT es el muestreo de la DTFT en N puntos equiespaciados

$$x_N[n] \overset{\mathsf{TF}}{\longleftrightarrow} X[k] = \sum_{n=0}^{N-1} x_N[n] e^{-j\theta_0 kn}$$

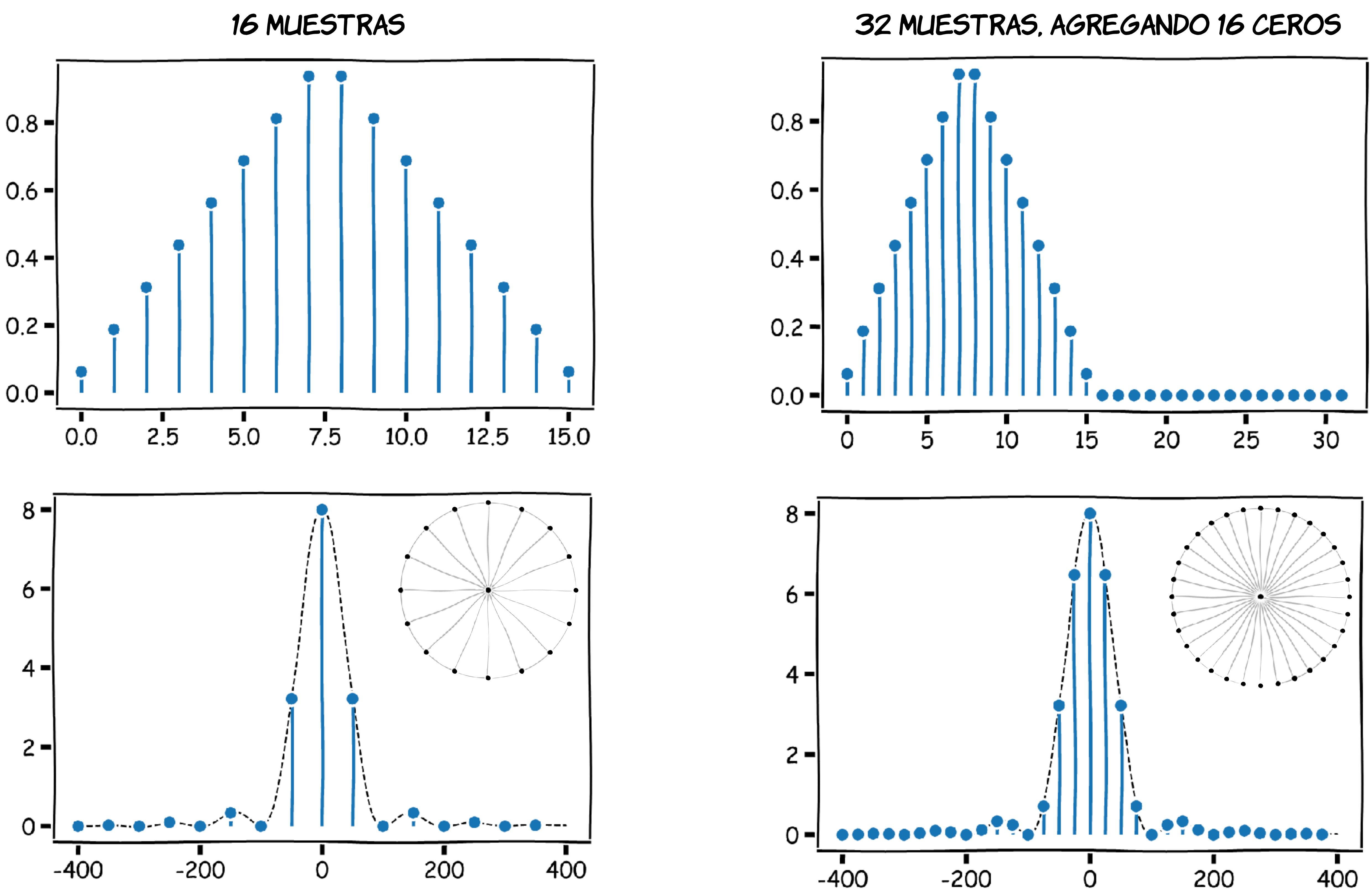
$$x[n] \overset{\mathsf{TF}}{\longleftrightarrow} X\left(e^{j\theta}\right) = \sum_{n=0}^{N-1} x[n] e^{-j\theta n}$$

$$X[k] = X\left(e^{j\theta}\right)\big|_{\theta=\theta_0 k}, \quad k = 0, \cdots, N-1$$

$N = 16$



Zero-padding aumenta la resolución en θ



DTFT EVALUADA EN 16 PUNTOS

DTFT EVALUADA EN 32 PUNTOS

$$\Delta\theta = \theta_0 = \frac{2\pi}{N}$$

La familia de la Transformada de Fourier

	Frecuencia discreta	Frecuencia continua
Tiempo continuo	**FS** $x(t) = \sum_{k=-\infty}^{+\infty} a_x[k] e^{j\omega_0 kt}$ $a_x[k] = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) e^{-j\omega_0 t} dt$ periódico T_0 $\omega_0 = \frac{2\pi}{T_0}$	**CTFT** $x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$
Tiempo discreto	**DFT** $x[n] = \sum_{k=\langle N \rangle} a_x[k] e^{j\theta_0 kn}$ $a_x[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\theta_0 kn}$ periódico N $\theta_0 = \frac{2\pi}{N}$ periódico N	**DTFT** $x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\theta}\right) e^{j\theta n} d\theta$ $X\left(e^{j\theta}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\theta n}$ periódico 2π

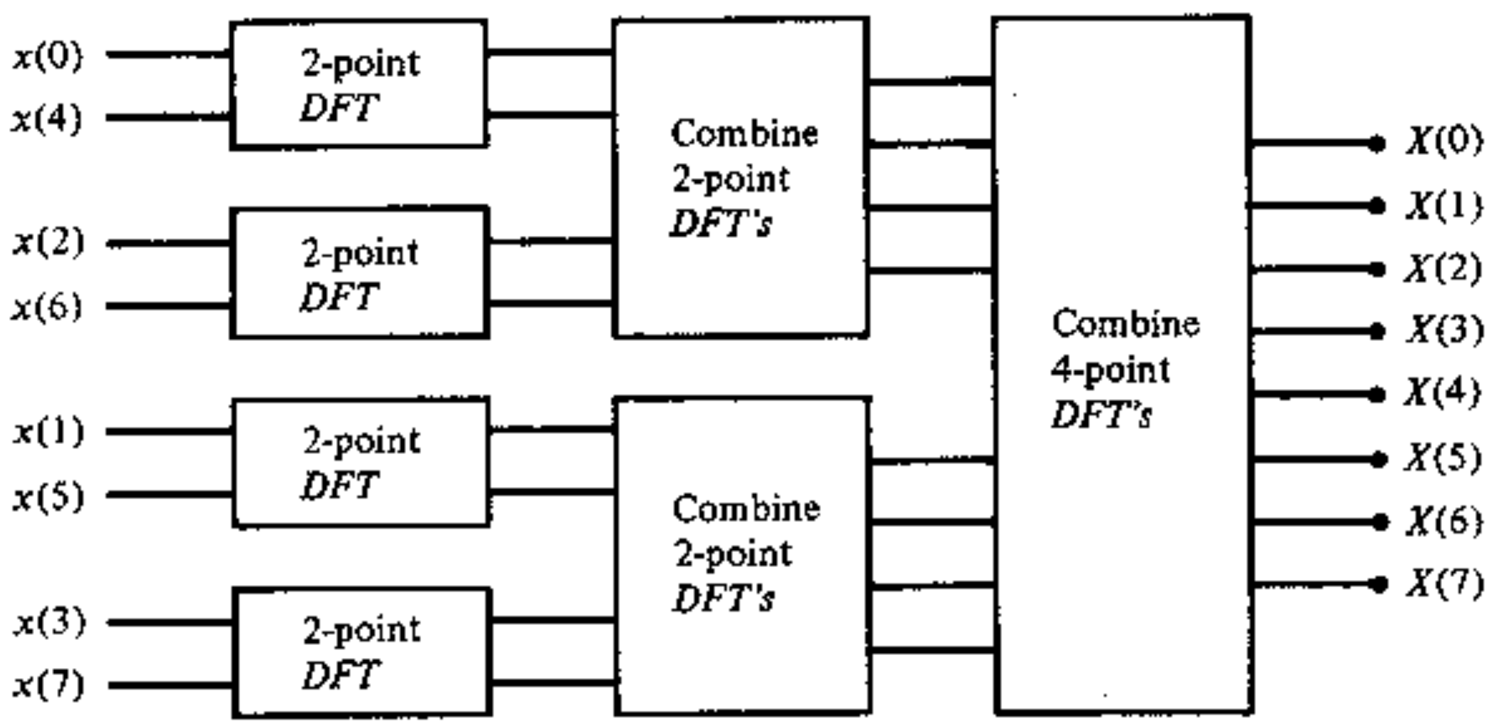
⟷ Dualidad

Fast Fourier Transform (FFT)

An Algorithm for the Machine Calculation of Complex Fourier Series

By James W. Cooley and John W. Tukey

An efficient method for the calculation of the interactions of a 2^m factorial experiment was introduced by Yates and is widely known by his name. The generalization to 3^m was given by Box et al. [1]. Good [2] generalized these methods and gave elegant algorithms for which one class of applications is the calculation of Fourier series. In their full generality, Good's methods are applicable to certain problems in which one must multiply an N-vector by an $N \times N$ matrix which can be factored into m sparse matrices, where m is proportional to $\log N$. This results in a procedure requiring a number of operations proportional to $N \log N$ rather than N^2. These methods are applied here to the calculation of complex Fourier series. They are



Cooley, J. W., & Tukey, J. W. (1965). An algorithm for the machine calculation of complex Fourier series. Mathematics of computation, 19(90), 297-301.

Fast Fourier Transform (FFT)

- Algoritmo recursivo: Divide & Conquer (& Combine)
- Descomposición en pares e impares

$$X[k] = X_p[k] + X_i[k]e^{-j\frac{\pi k}{N}}$$
$$X[k + {}^{N}/_{2}] = X_p[k] - X_i[k]e^{-j\frac{\pi k}{N/2}}$$

- Evaluación de polinomios en N las raíces complejas de 1

$$W_N^k = e^{j\frac{2\pi}{N}k}$$

- La inversa IFFT es el mismo algoritmo con

$$W_N^{-k} = e^{-j\frac{2\pi}{N}k}$$

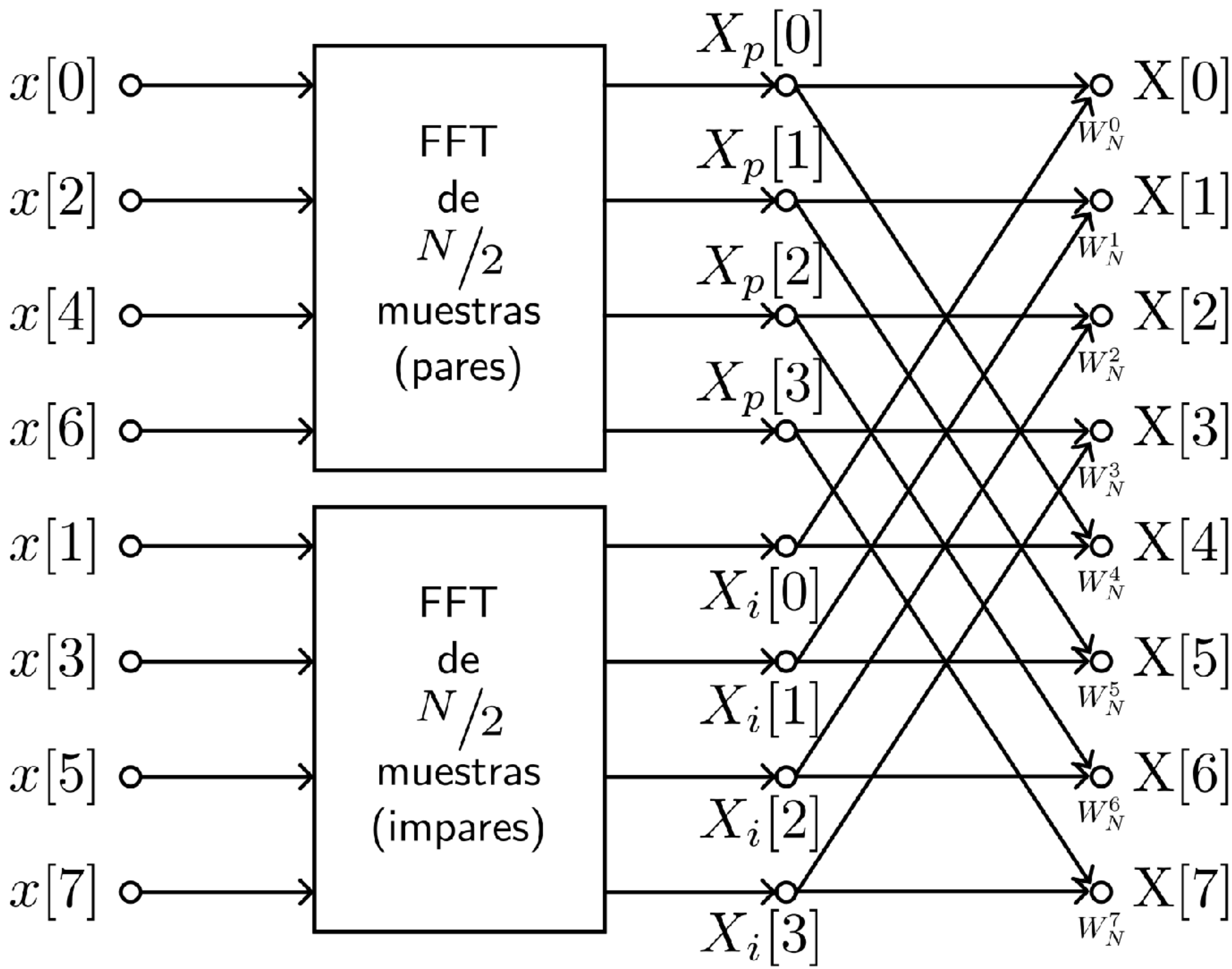
- Más eficiente si $N = 2^m$, sino se puede hacer zero-padding
- Complejidad: la FFT es $O(N \log N)$ vs. $O(N^2)$ de la DFT

	N		
	10^3	10^6	10^9
N^2	10^6	10^{12}	10^{18}
$N \log N$	10^3	20×10^6	30×10^9

- N = 1.00e+09 muestras (a 44.10 kHz) son 0 años, 0 meses, 0 días, 6 horas, 17 minutos, 55 segundos de audio.
- La FFT se computa en 29.90 segundos.
- La DFT se computa en 1.00e+09 segundos o 32 años, 1 meses, 24 días, 1 horas, 46 minutos, 40 segundos.

Cooley, J. W., & Tukey, J. W. (1965). An algorithm for the machine calculation of complex Fourier series. Mathematics of computation, 19(90), 297-301.

Fast Fourier Transform (FFT)



$$W_N = e^{j\frac{2\pi}{N}}$$