Señales y Sistemas

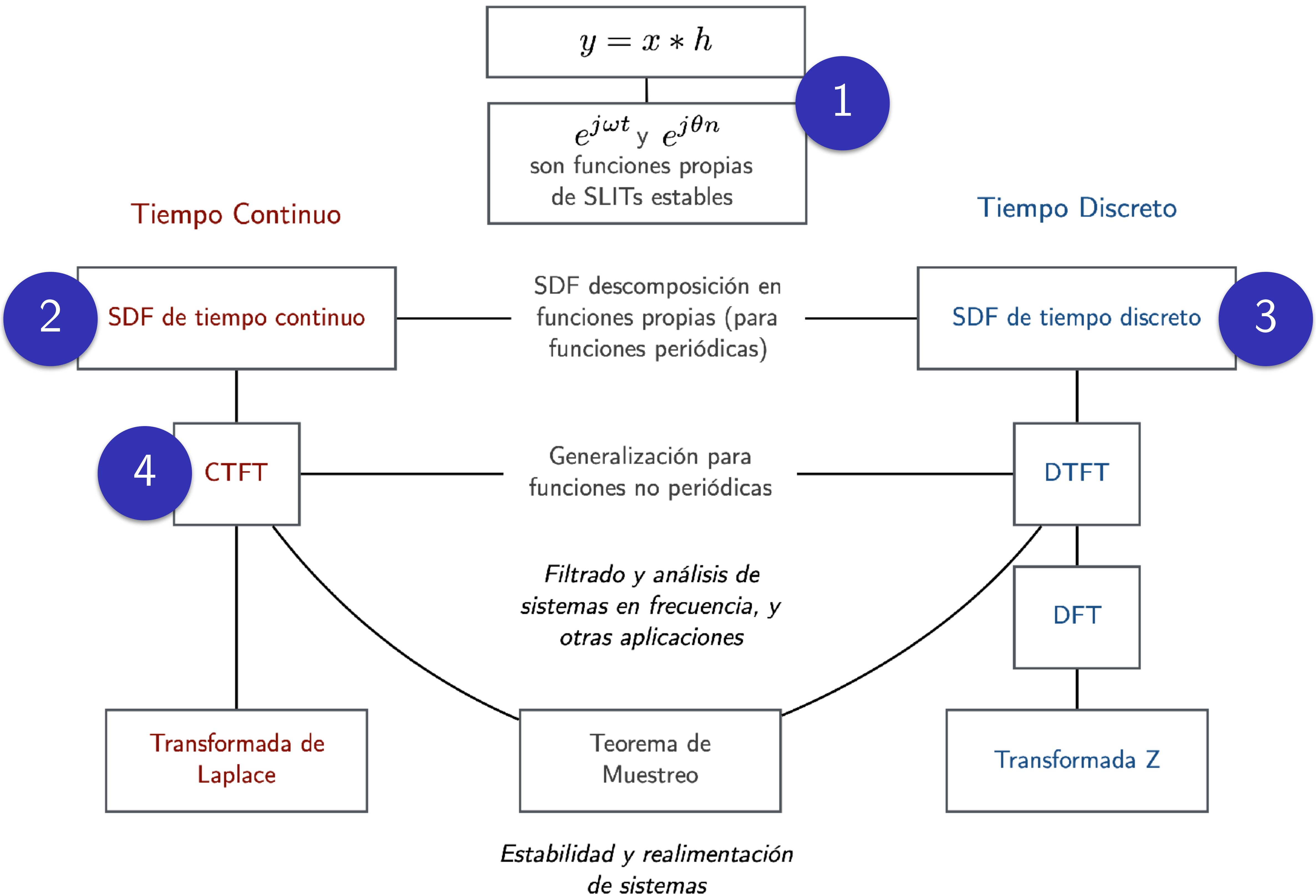
Transformada de Fourier de variable continua (CTFT)

Instituto de Ingeniería Eléctrica

FACULTAD DE INGENIERÍA

UNIVERSIDAD DE LA REPÚBLICA URUGUAY

Señales y sistemas



* tiempo o variable

Series de Fourier

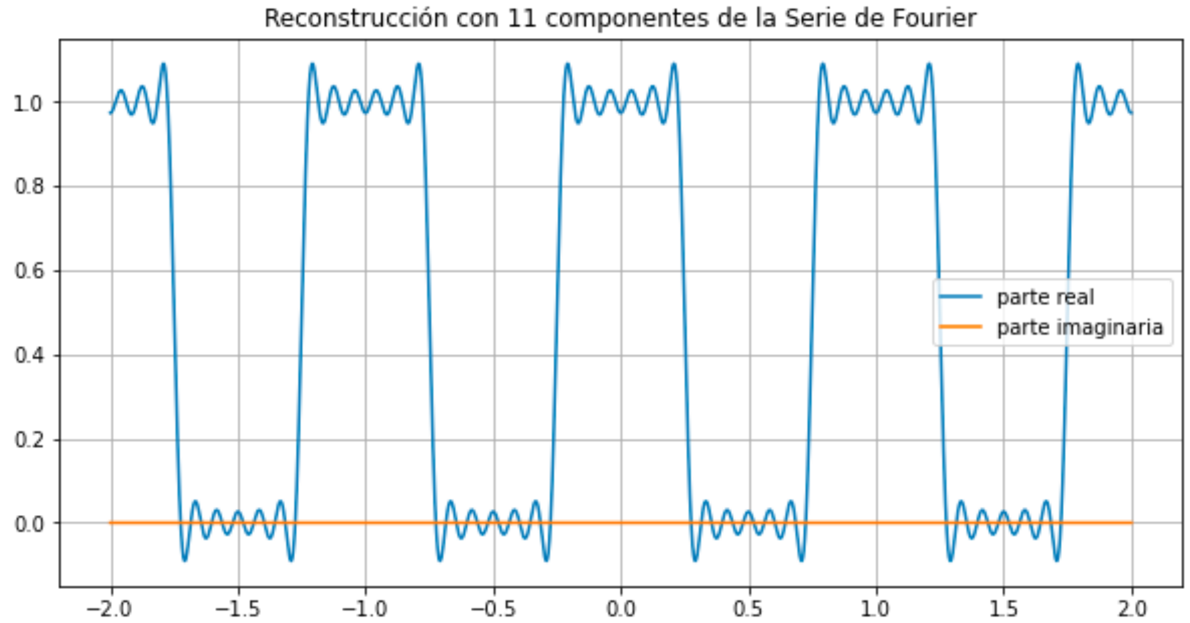
- $x(t)$ **periódica**, de período T: $x(t+T) = x(t)\ \forall t \qquad (\omega_0 = {}^{2\pi}/_{T})$

Serie de Fourier

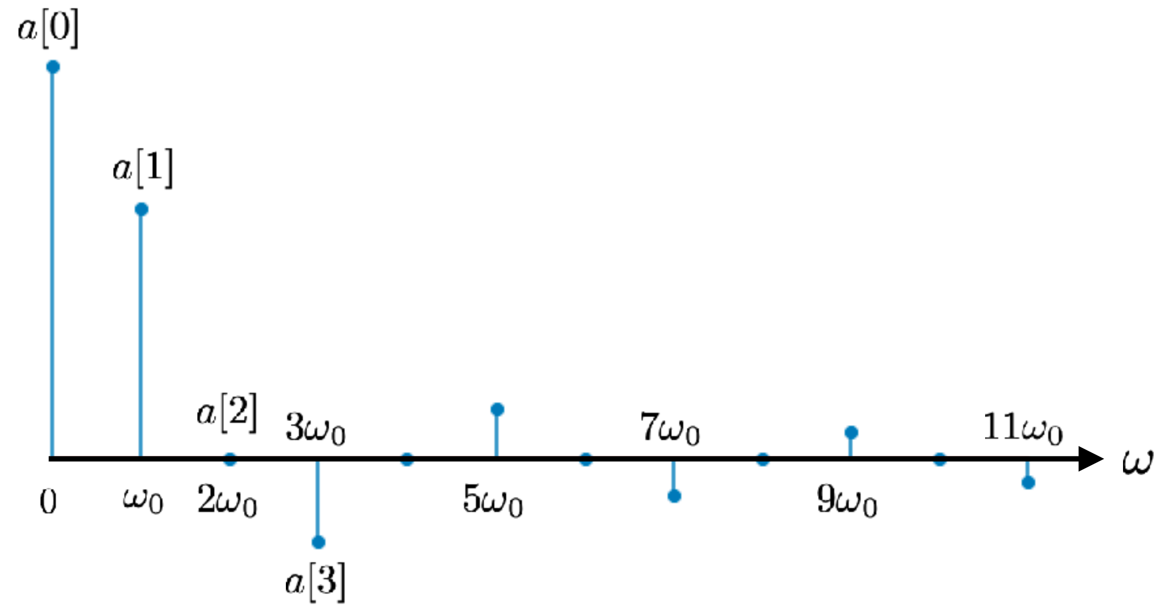
$$x(t) = \sum_{k=-\infty}^{+\infty} a[k] e^{jk\omega_0 t} \qquad \text{Síntesis}$$

$$a[k] = \frac{1}{T} \int_{\langle T \rangle} x(\tau) e^{-jk\omega_0 \tau} d\tau \qquad \text{Análisis}$$

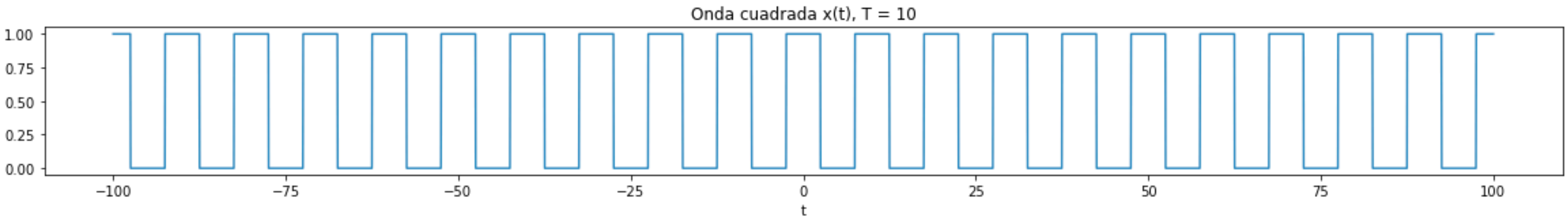
$x(t)$

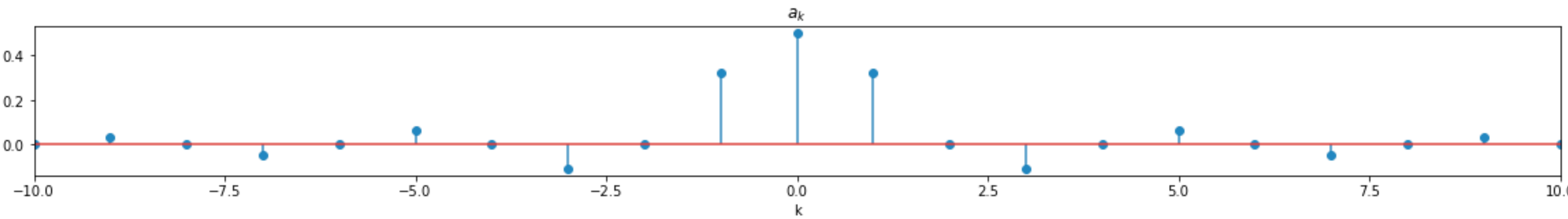


$X(j\omega)$



Series de Fourier

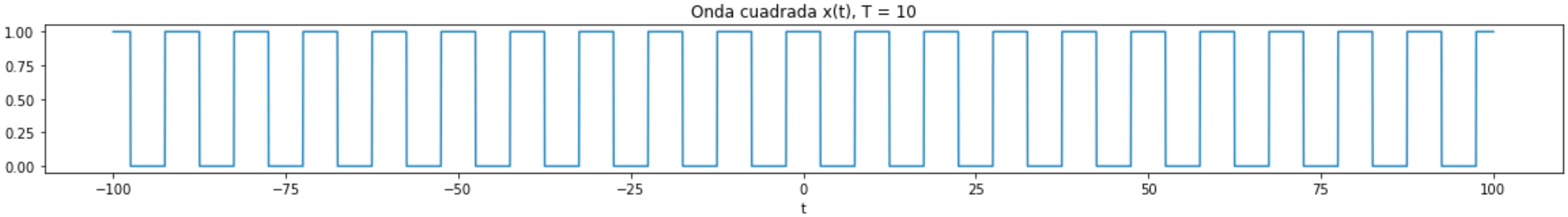


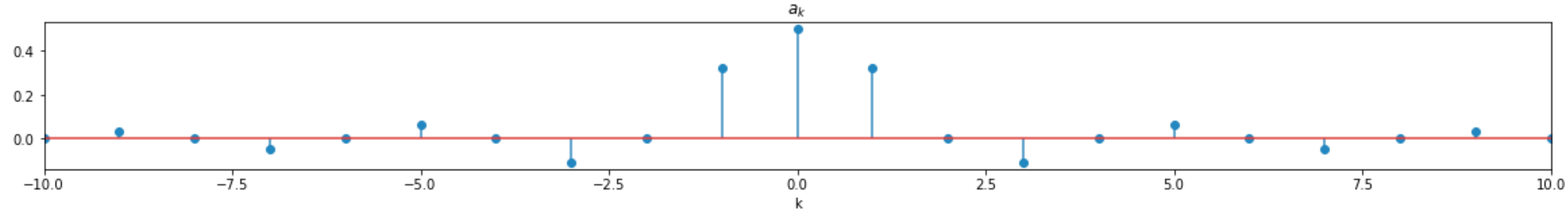


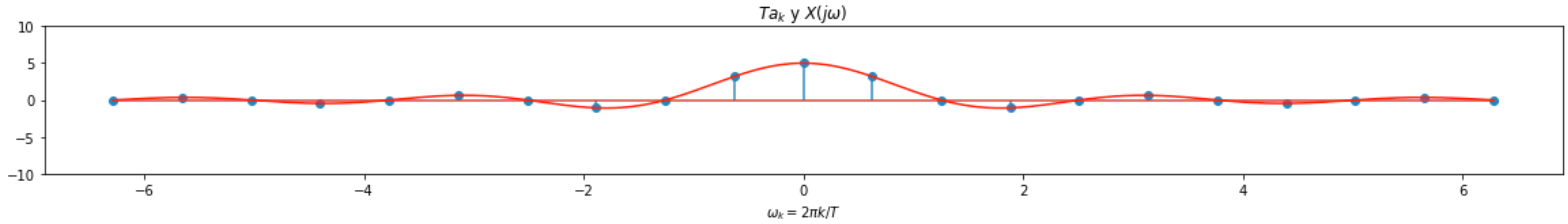
$$x(t) = \sum_k \Pi\left(\frac{t - kT}{2t_0}\right) \qquad a_k = \frac{1}{k\pi}\sin(\omega_k t_0), \quad \omega_k = k\omega_0 \qquad T\uparrow \Rightarrow \omega_0 \downarrow$$

$$T\,a_k = \frac{2}{\omega_k}\sin(\omega_k t_0)$$

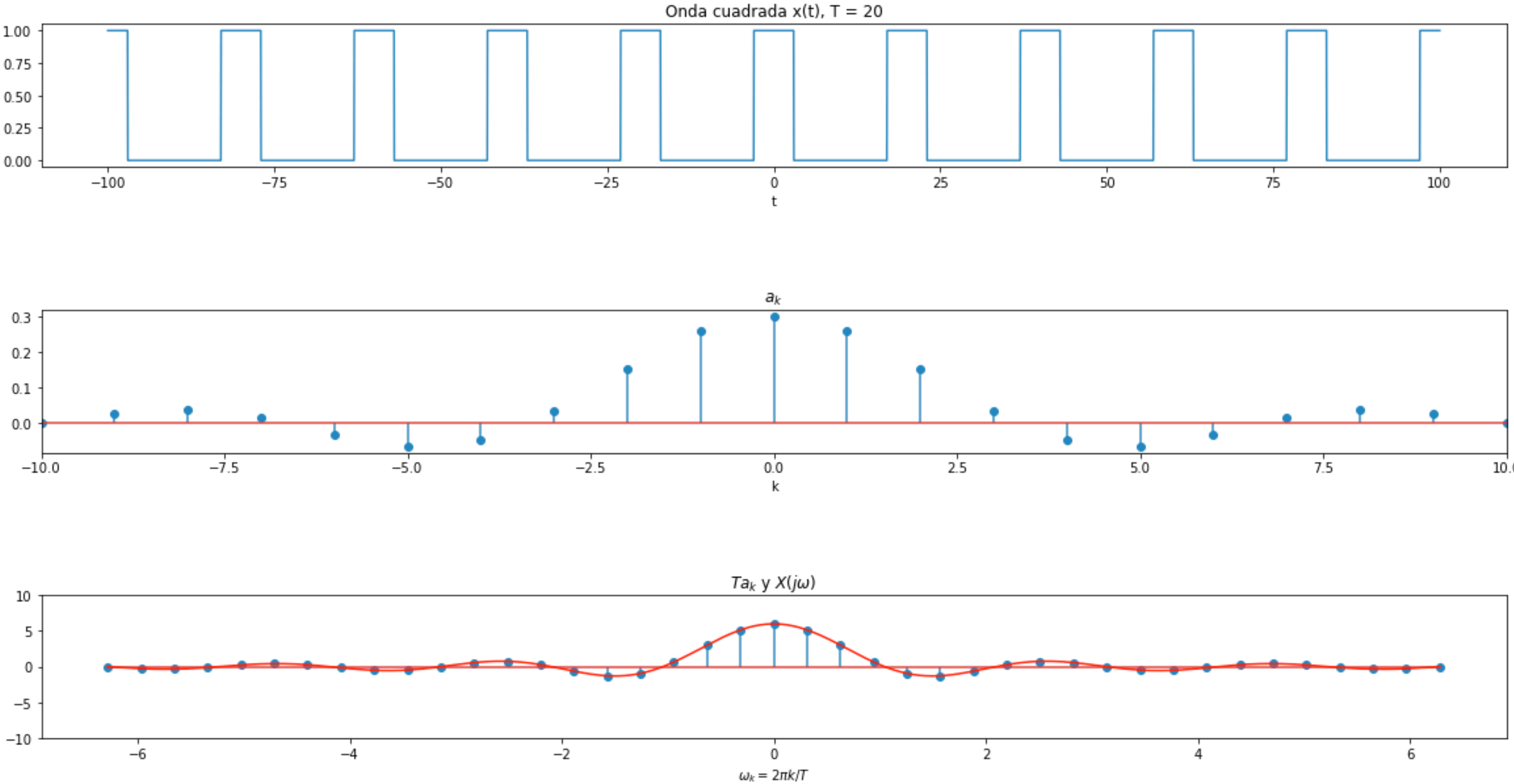
Series de Fourier



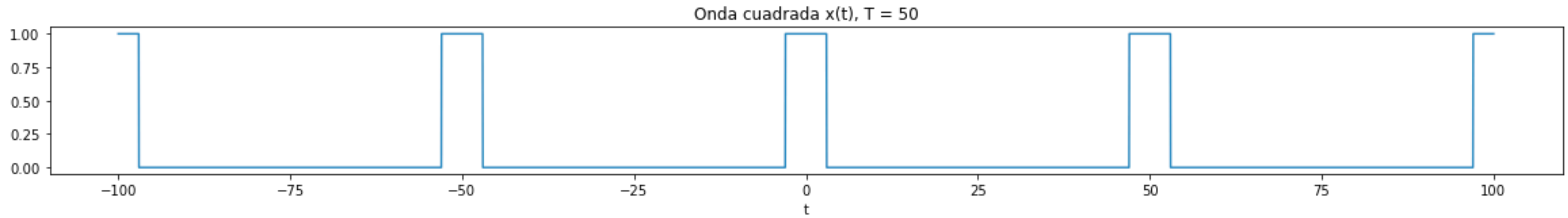


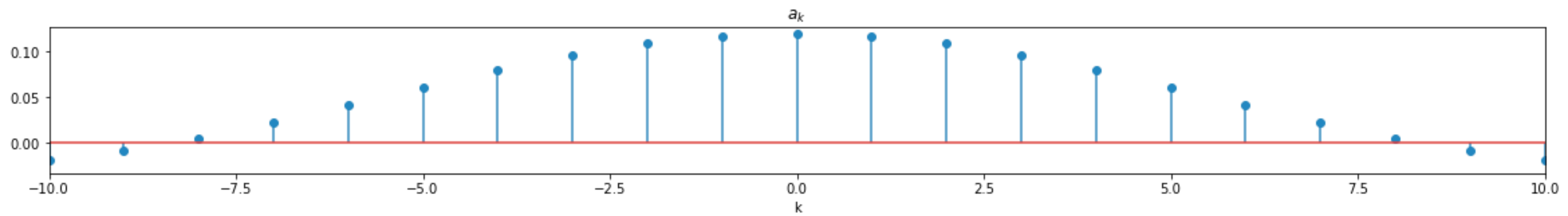


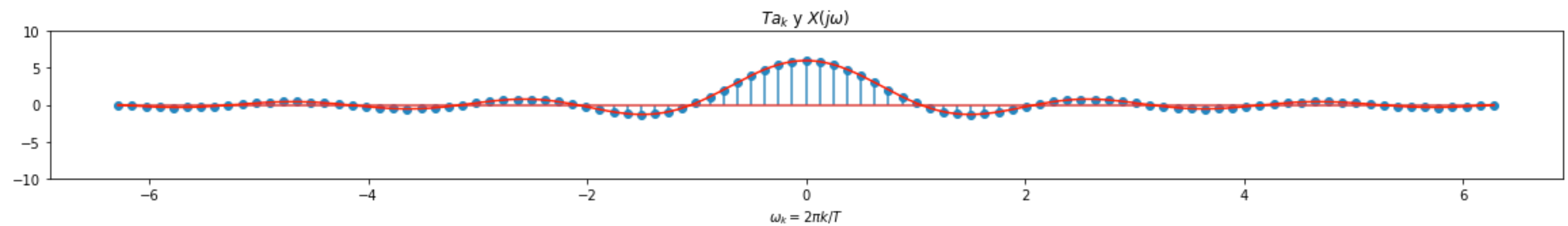
Series de Fourier



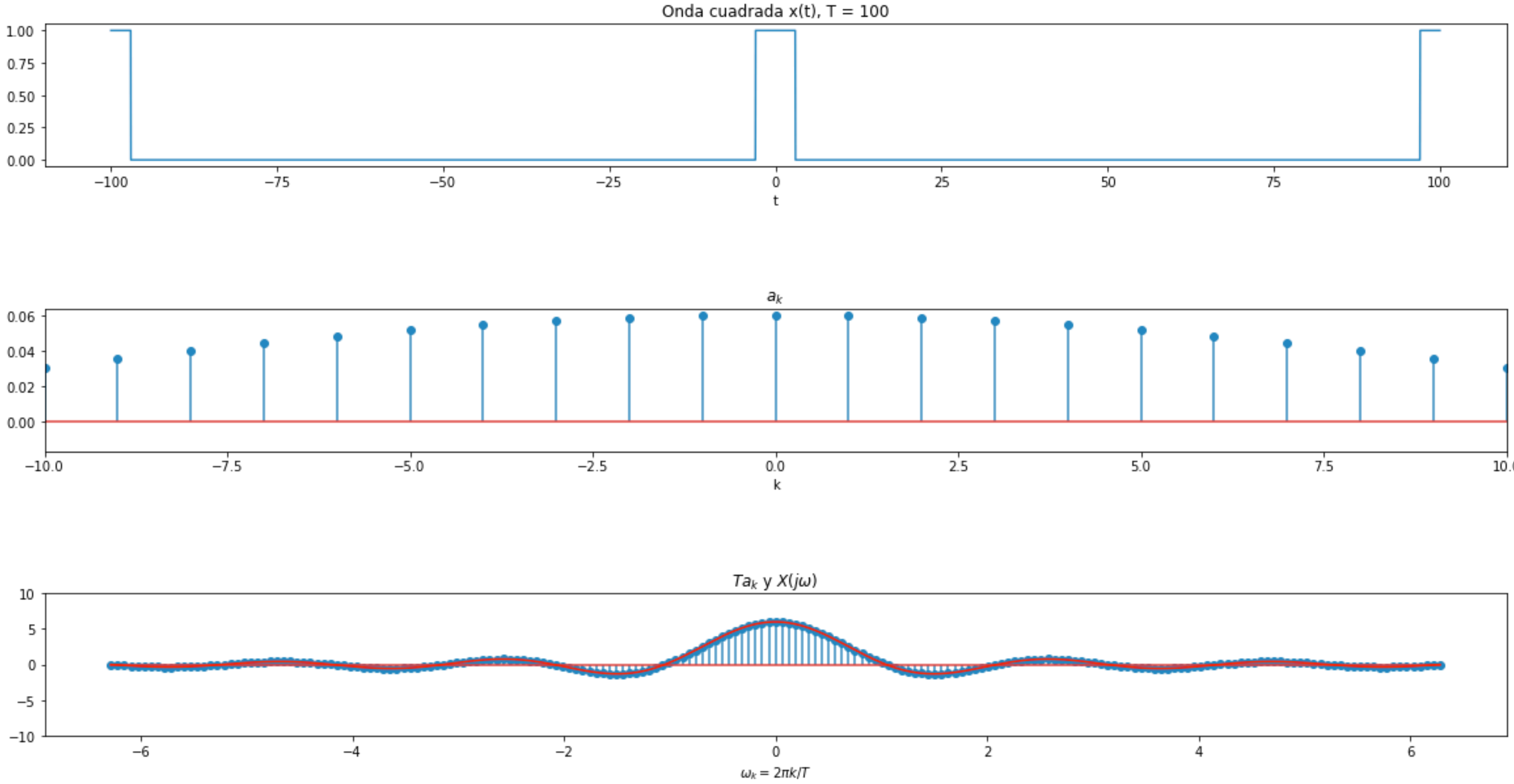
Series de Fourier



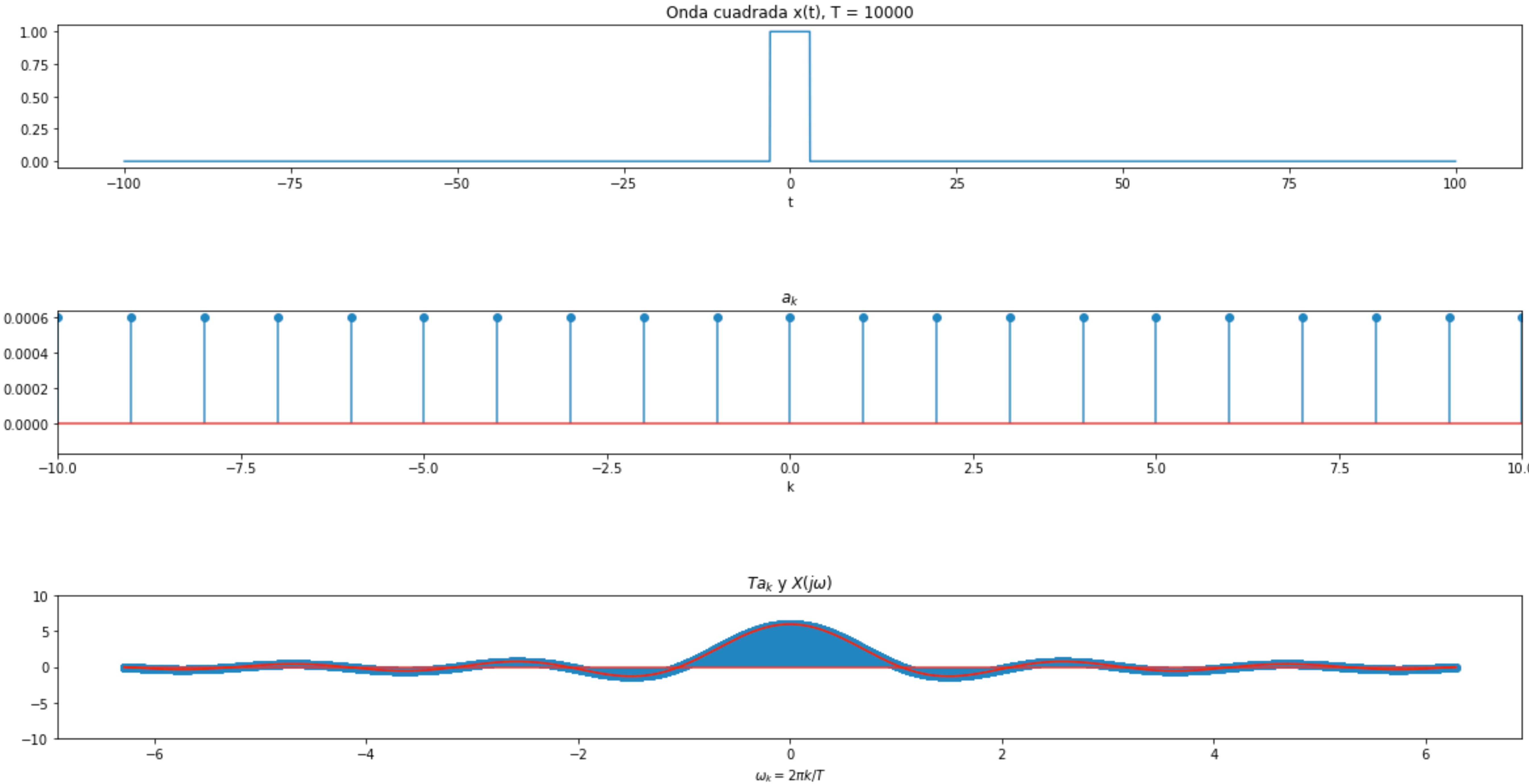




Series de Fourier



Series de Fourier



De la Serie de Fourier a la CTFT

$$x(t) = \frac{1}{T}\sum_k T a_k e^{j\omega_0 k t} = \frac{1}{T}\sum_k X(j\omega_k)e^{j\omega_k t} \quad \omega_k = \omega_0 k$$

$$= \frac{1}{2\pi}\sum_k X(j\omega_k)e^{j\omega_k t}\Delta\omega_k \quad \Delta\omega_k = \frac{2\pi}{T}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega\tau}d\omega \quad T\uparrow\infty, \Delta\omega_k \to 0$$

Transformada de Fourier de variable continua (CTFT)

- $X(j\omega)$ es la TF de $x(t)$ (no es necesario que sea periódica)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau \qquad \text{Análisis}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{Síntesis}$$

- La relación entre la CTFT y la SF es estrecha.
- ¿Cuál es la CTFT de una señal periódica?
- Convergencia de la CTFT: son las condiciones de Dirichlet
  1. $x(t)$ es absolutamente integrable,
  2. en cada intervalo finito hay finitos máximos o mínimos,
  3. en cada intervalo finito hay finitas discontinuidades finitas.
  - 2 y 3 son necesarias para que se cumpla la ecuación de análisis en todo t excepto en las discontinuidades (recordar fenómeno de Gibbs).

Ejemplos

$$x(t) = e^{-at}u(t) \quad a > 0 \qquad X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

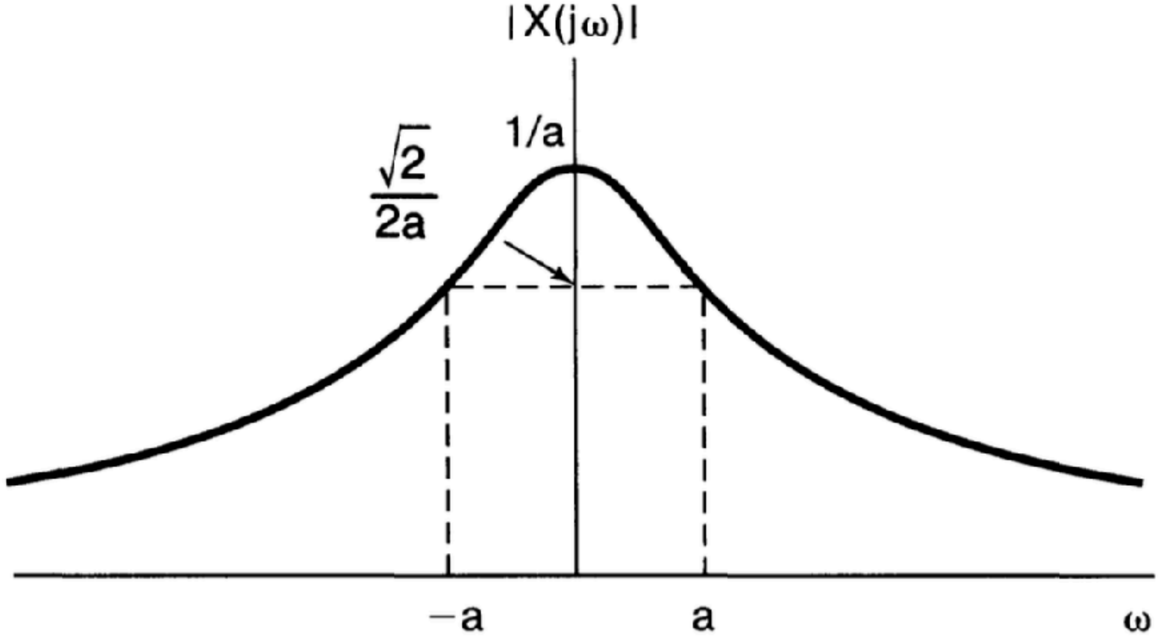
$$x(t) = e^{-at}u(t), \quad a > 0 \quad \overset{\text{TF}}{\longleftrightarrow} \quad X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0$$

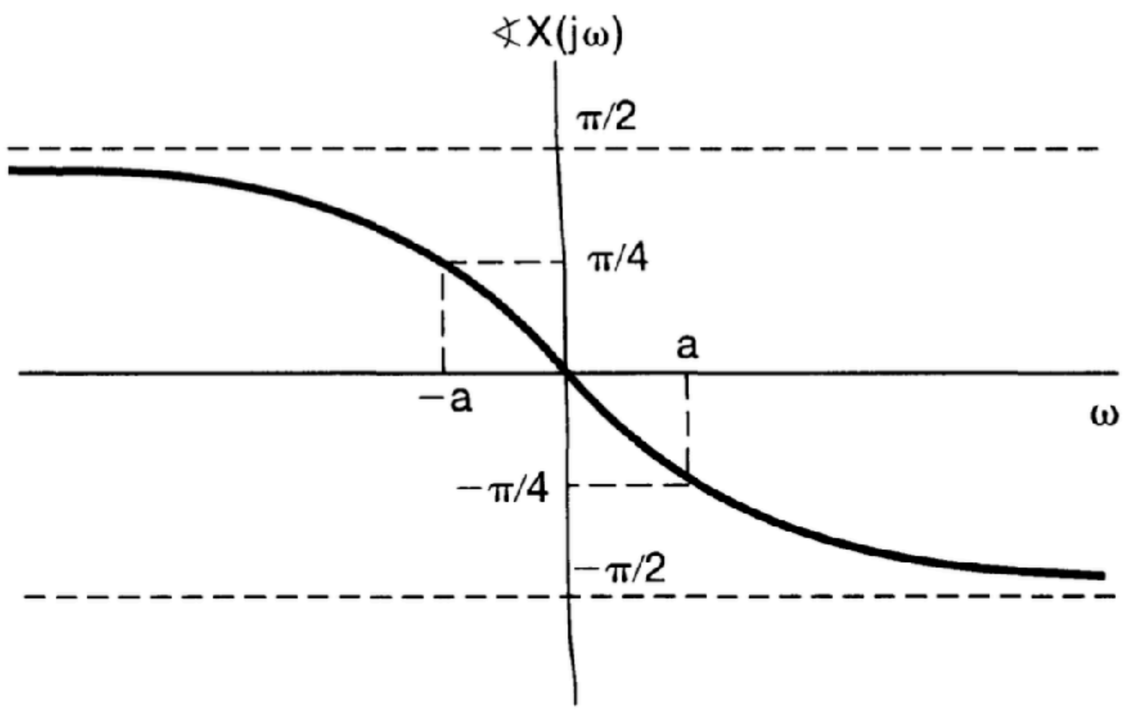
Módulo

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

Fase

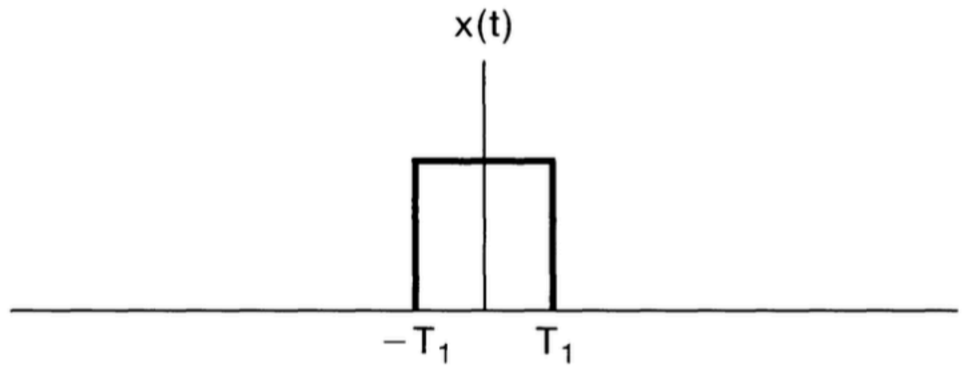
$$\sphericalangle X(j\omega) = -\arctan\left(\frac{\omega}{a}\right)$$



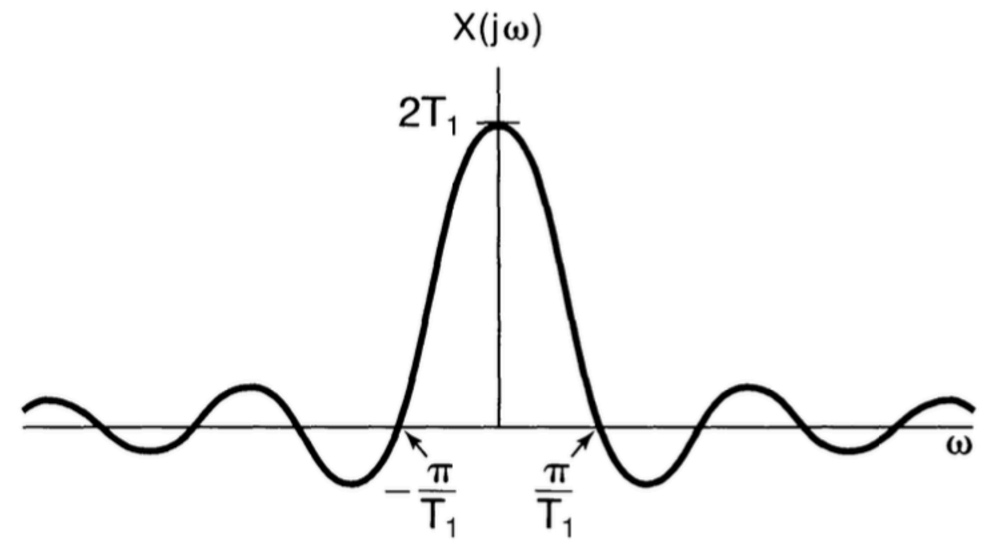


Ejemplos

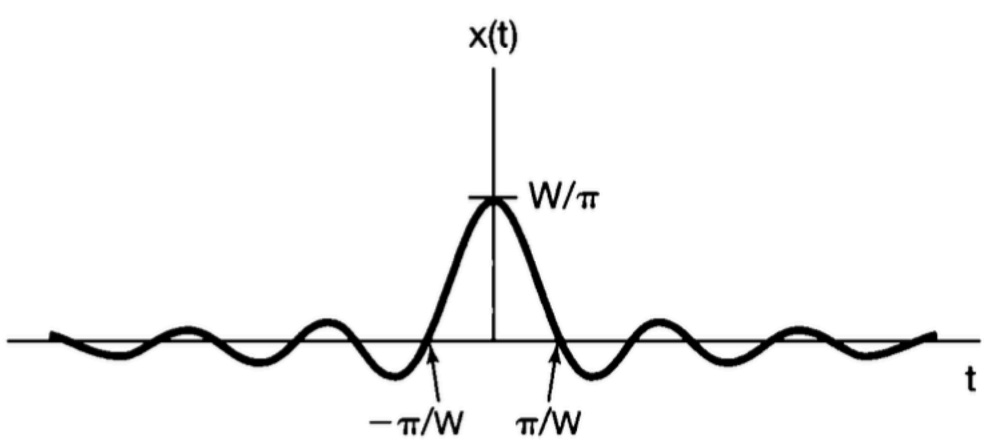
Ejemplo 4.4
$$x(t) = \Pi\left(\frac{t}{2T_1}\right) \overset{\text{TF}}{\longleftrightarrow} X(j\omega) = 2\frac{\sin(\omega T_1)}{\omega}$$



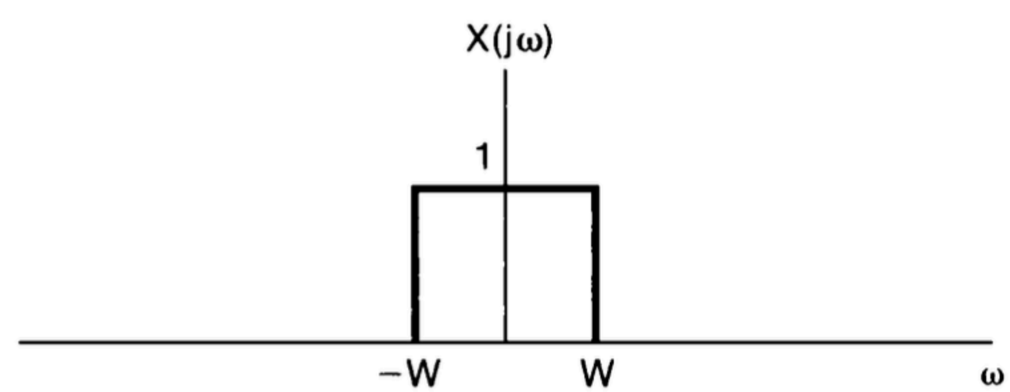
$\overset{\text{TF}}{\longleftrightarrow}$



Ejemplo 4.5
$$X(j\omega) = \Pi\left(\frac{\omega}{2W}\right) \overset{\text{TF}}{\longleftrightarrow} x(t) = \frac{\sin(Wt)}{\pi t}$$

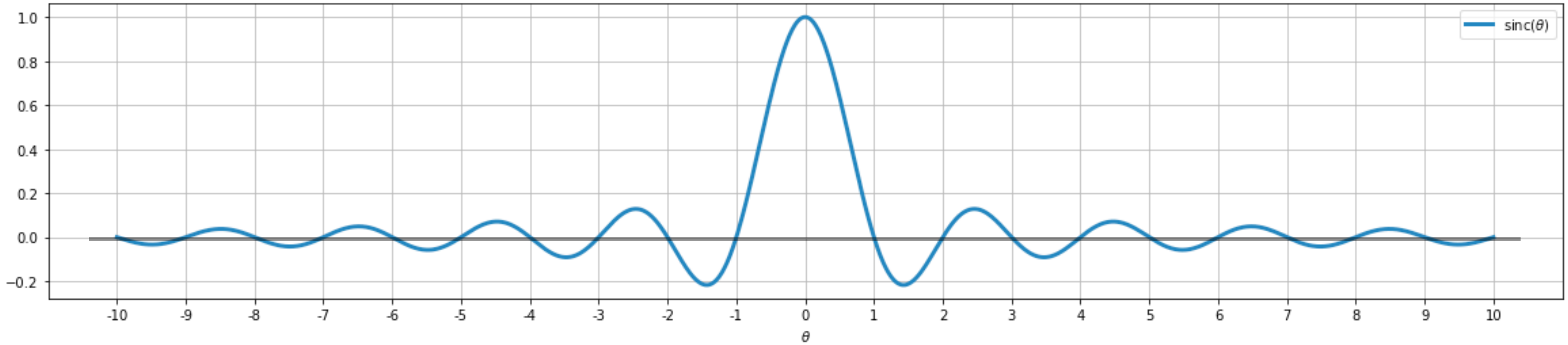


$\overset{\text{TF}}{\longleftrightarrow}$



Seno cardinal (sinc)

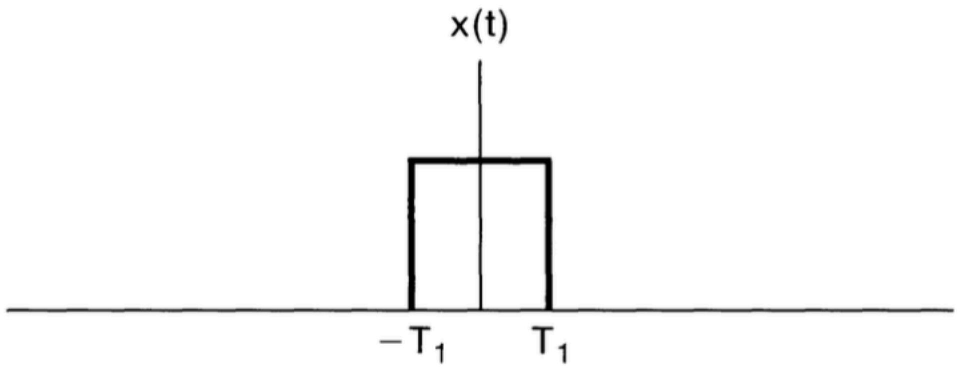
$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$



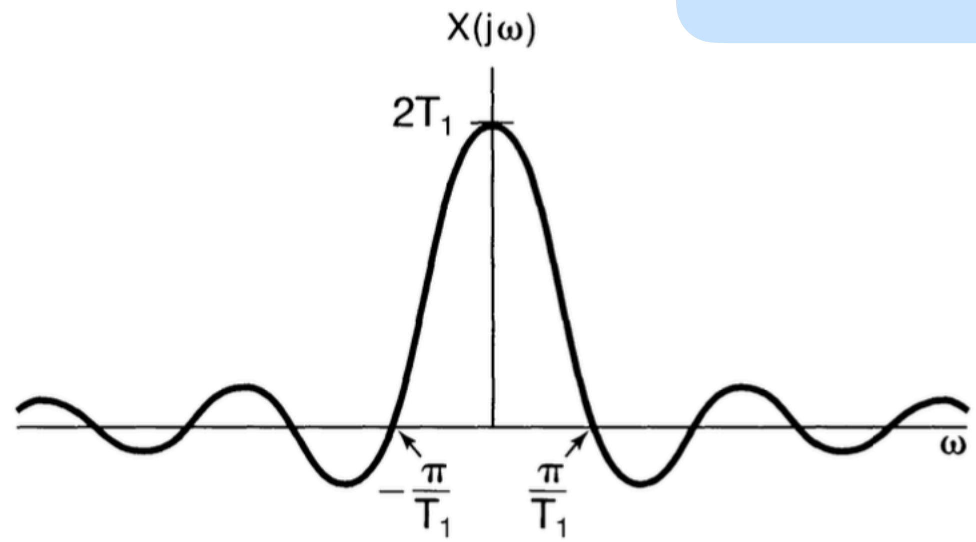
- Área unitaria.
- Vale 1 en $\theta = 0$: $\mathrm{sinc}(0) = 1$
- Cruces periódicos en $\theta \in Z$.
- Soporte infinito.

Ejemplos

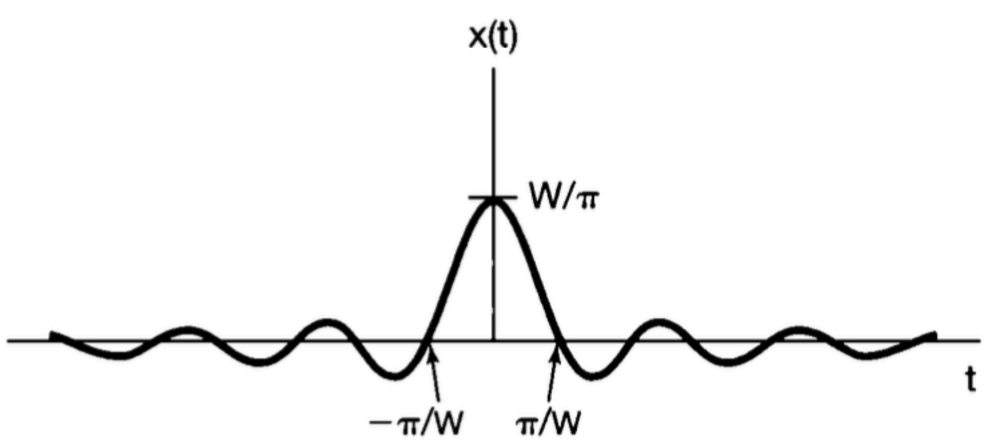
$$x(t) = \Pi\left(\frac{t}{2T_1}\right) \quad \overset{\text{TF}}{\longleftrightarrow} \quad X(j\omega) = 2\frac{\sin(\omega T_1)}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$



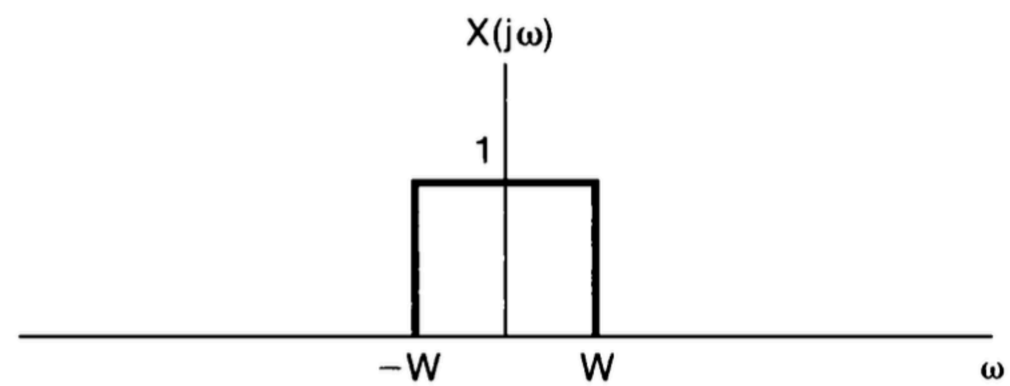
$\overset{\text{TF}}{\longleftrightarrow}$



$$X(j\omega) = \Pi\left(\frac{\omega}{2W}\right) \quad \overset{\text{TF}}{\longleftrightarrow} \quad x(t) = \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

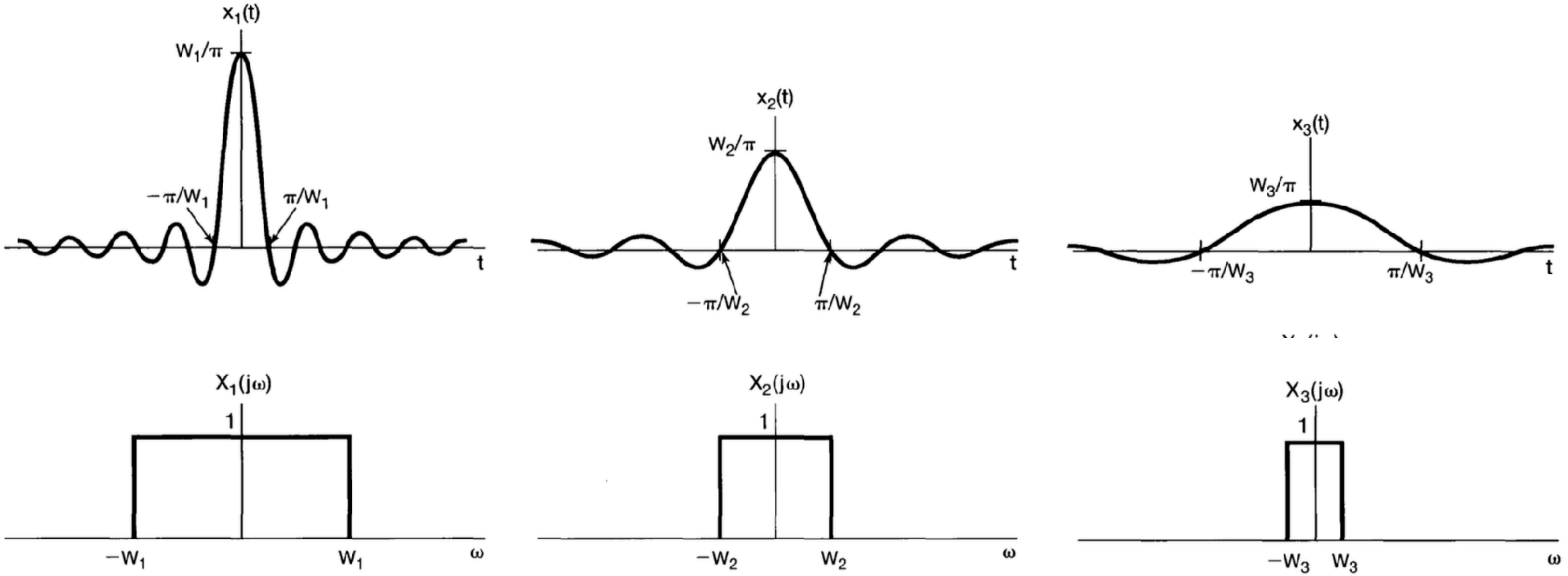


$\overset{\text{TF}}{\longleftrightarrow}$



Ejemplos

$$W_1 > W_2 > W_3$$

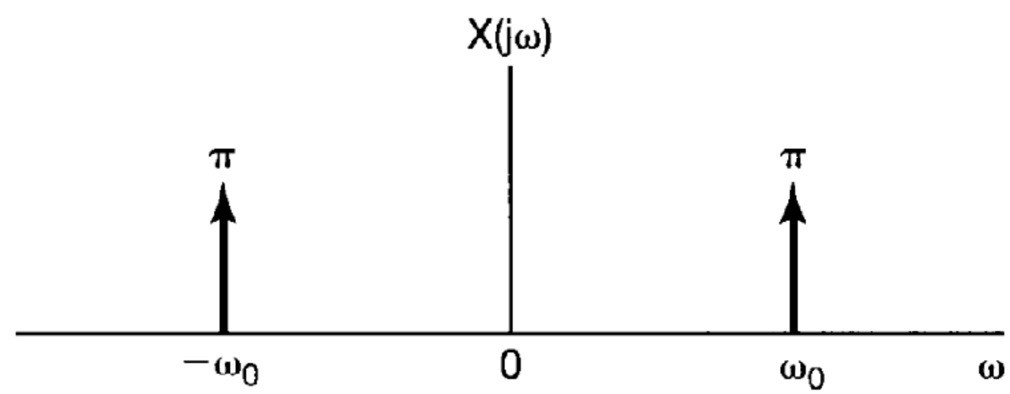


(a) (b) (c)

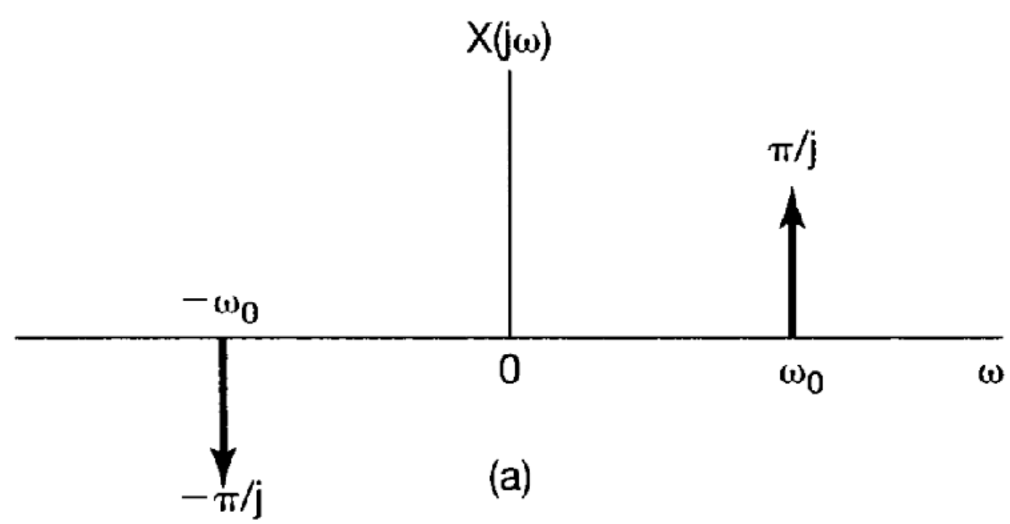
Ejemplos

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \quad \overset{\text{TF}}{\longleftrightarrow} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{-j\omega t}d\omega = e^{j\omega_0 t}$$

$$x(t) = \cos(\omega_0 t) \quad \overset{\text{TF}}{\longleftrightarrow} \quad X(j\omega) = \pi\left(\delta(\omega+\omega_0+\delta(\omega-\omega_0)\right)$$



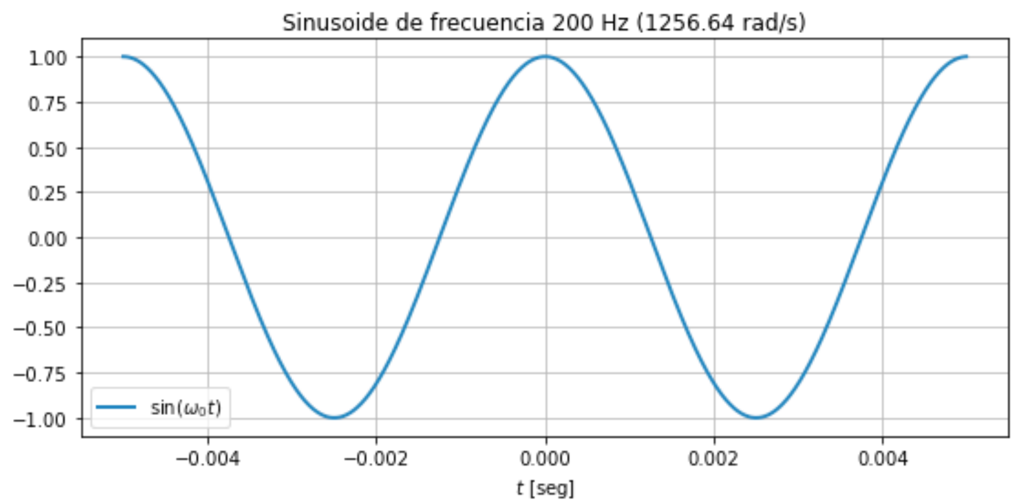
$$x(t) = \sin(\omega_0 t) \quad \overset{\text{TF}}{\longleftrightarrow} \quad X(j\omega) = \frac{\pi}{j}\left(-\delta(\omega+\omega_0+\delta(\omega-\omega_0)\right)$$



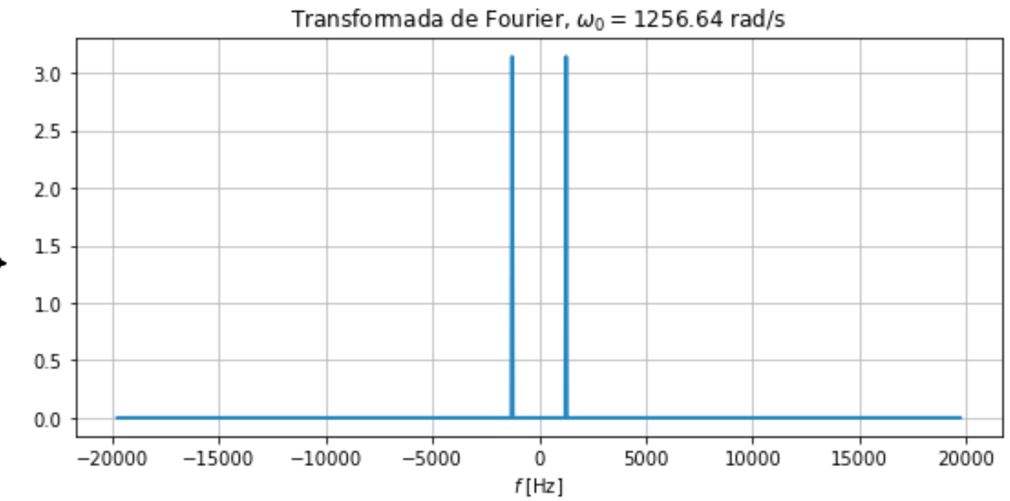
(a)

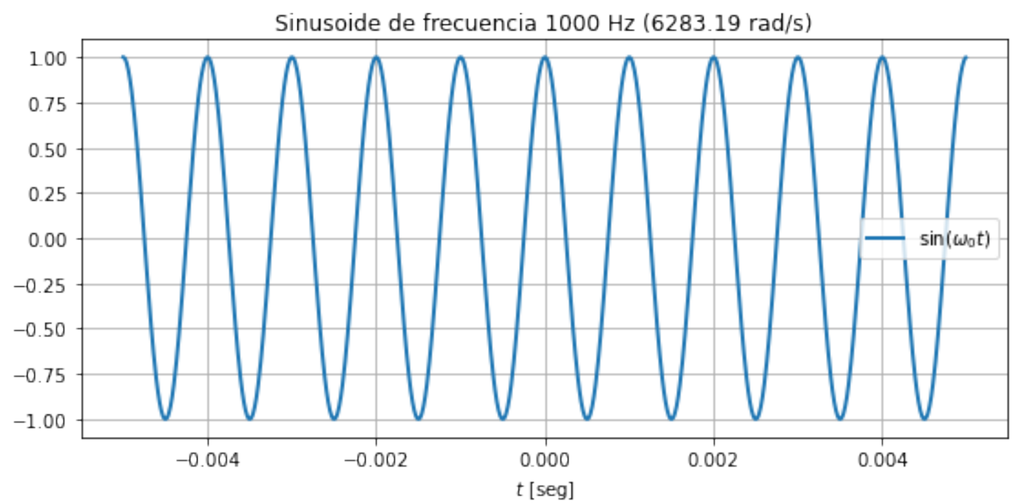
Ejemplos

$$x(t) = \cos(\omega_0 t) \quad \overset{\text{TF}}{\longleftrightarrow} \quad X(j\omega) = 2\pi\left(-\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right)$$

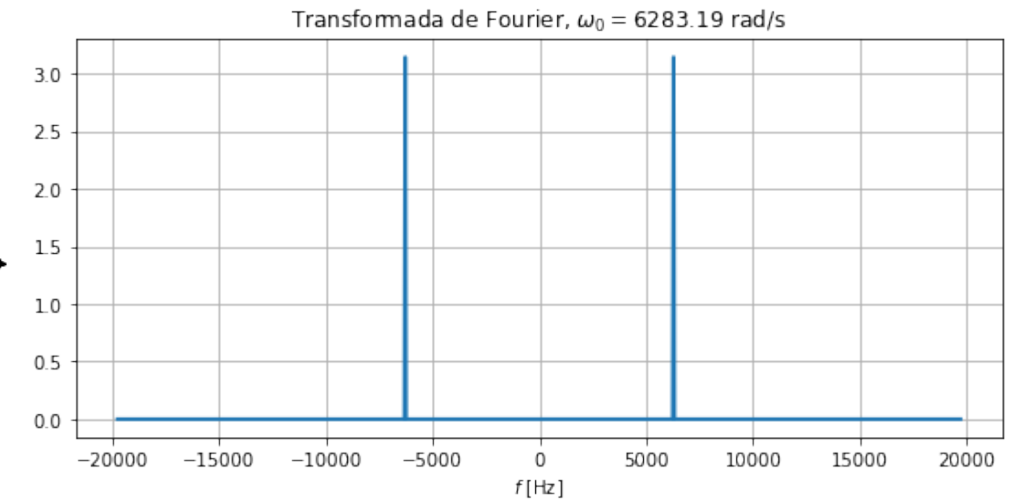


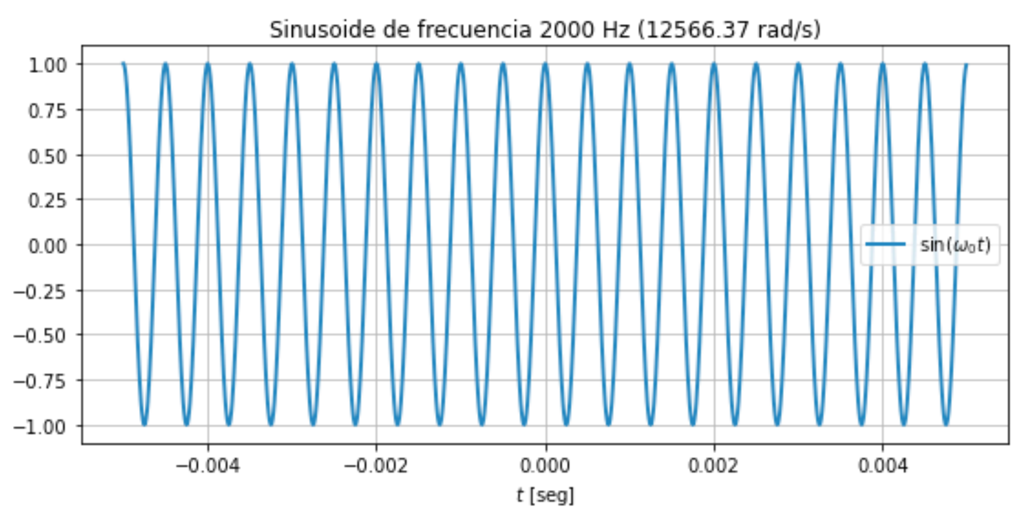
TF ⟷



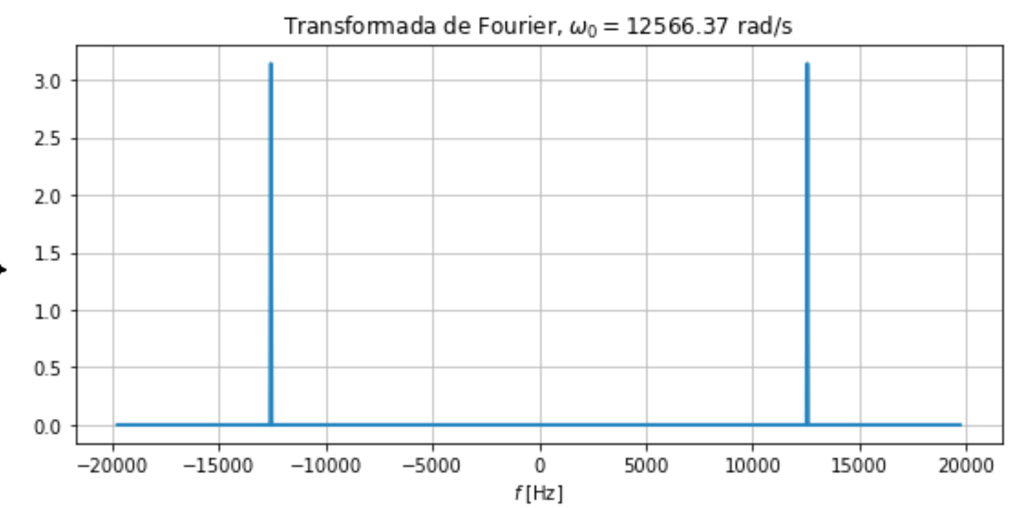


TF ⟷





TF ⟷



CTFT de una señal periódica

$$X(j\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_0) \quad \overset{\text{TF}}{\longleftrightarrow} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - k\omega_0) e^{j\omega t} d\omega$$

$$= \sum_k a_k e^{jk\omega_0 t}$$

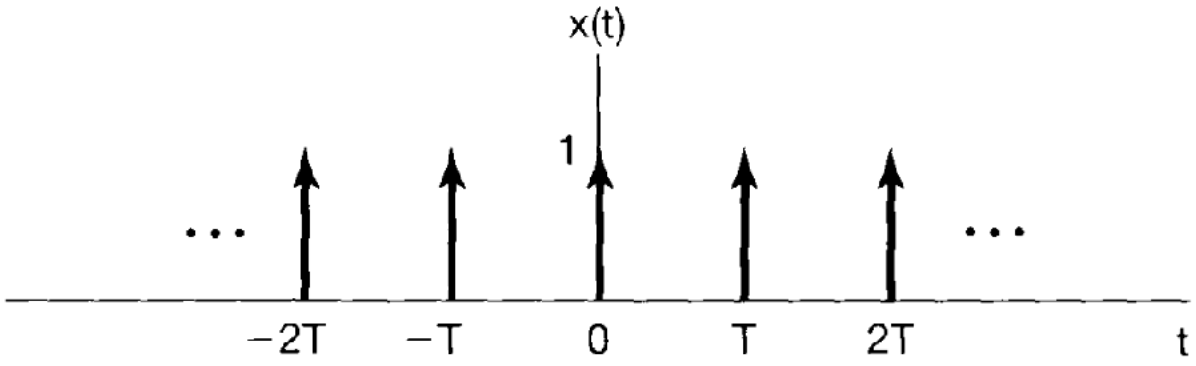
$x(t)$ admite una representación en Serie de Fourier, es una señal periódica de periodo $T_0 = 2\pi/\omega_0$.

La Transformada de Fourier de una señal periódica es un tren de impulsos en el dominio de la frecuencia, con las áreas de los impulsos proporcionales (2π) a los coeficientes de la Serie de Fourier.

Ejemplos

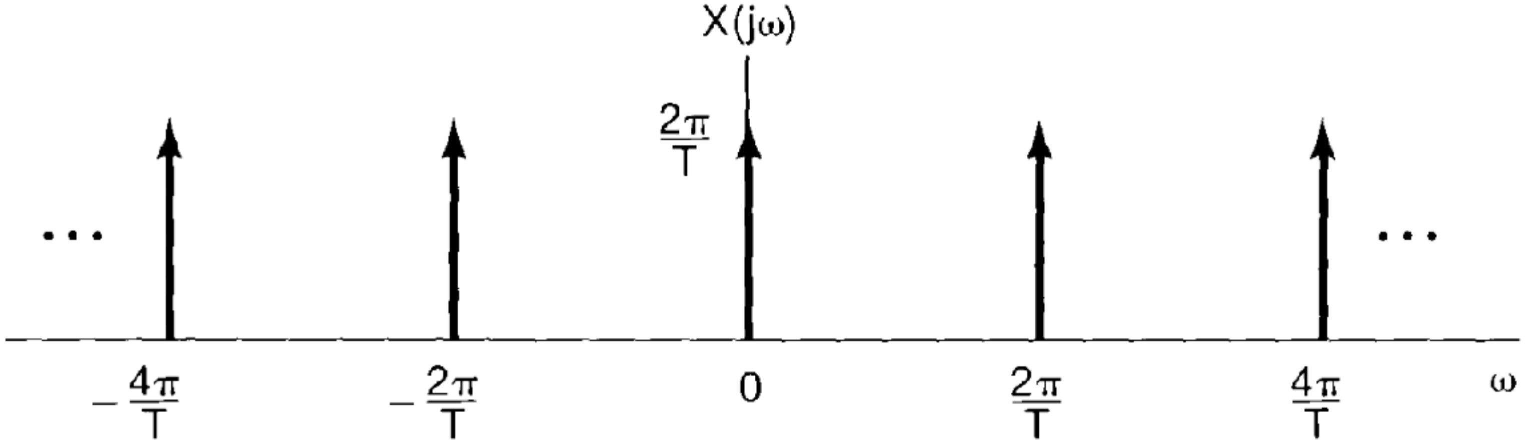
- Peine de Dirac o Tren de Impulsos

$$p_T(t) = \sum_k \delta(t - kT)$$



Serie de Fourier: $p_T[k] = \frac{1}{T} \forall k$

$$P_T(j\omega) = \frac{2\pi}{T} \sum_k \delta(\omega - k\omega_0)$$



CTFT: propiedades

$$x(t) \overset{\text{TF}}{\longleftrightarrow} X(j\omega) = \mathcal{F}\{x\}$$

$$y(t) \overset{\text{TF}}{\longleftrightarrow} Y(j\omega) = \mathcal{F}\{y\}$$

- Linealidad

$$a\,x(t) + b\,y(t) \overset{\text{TF}}{\longleftrightarrow} a\,X(j\omega) + b\,Y(j\omega)$$

- Desplazamiento temporal

$$x(t - t_0) \overset{\text{TF}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

- Conjugación

$$x^*(t) \overset{\text{TF}}{\longleftrightarrow} X^*(-j\omega)$$

$x(t)$ es real

$$X(-j\omega) = X^*(j\omega)$$

- Diferenciación

$$\frac{d\,x(t)}{dt} \overset{\text{TF}}{\longleftrightarrow} j\omega X(j\omega)$$

- Integración

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{TF}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

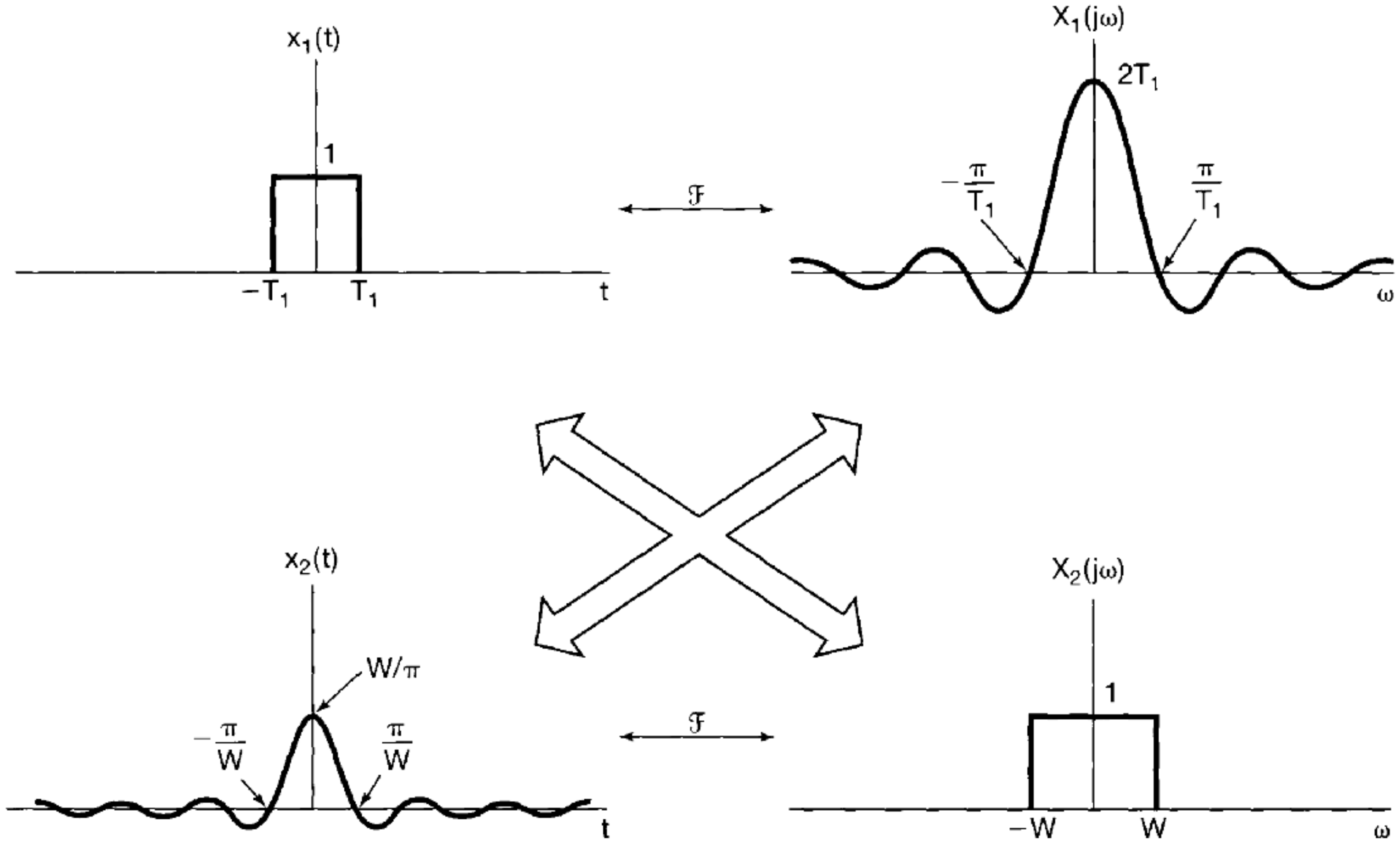
- Escalado tiempo y frecuencia

$$x(at) \overset{\text{TF}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

CTFT: propiedades

$$x(t) \stackrel{\text{TF}}{\longleftrightarrow} X(j\omega) = \mathcal{F}\{x\}$$

- Dualidad



$$-jtx(t) \stackrel{\text{TF}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t}x(t) \stackrel{\text{TF}}{\longleftrightarrow} X\left(j\left(\omega - \omega_0\right)\right)$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \stackrel{\text{TF}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)d\eta$$

CTFT: propiedades

$$x(t) \overset{\text{TF}}{\longleftrightarrow} X(j\omega) = \mathcal{F}\{x\}$$

$$y(t) \overset{\text{TF}}{\longleftrightarrow} Y(j\omega) = \mathcal{F}\{y\}$$

$$h(t) \overset{\text{TF}}{\longleftrightarrow} H(j\omega) = \mathcal{F}\{y\}$$

- Identidad de Parseval

$$\int_{-x}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

- **Convolución**

$$y(t) = h(t) * x(t) \overset{\text{TF}}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega)$$

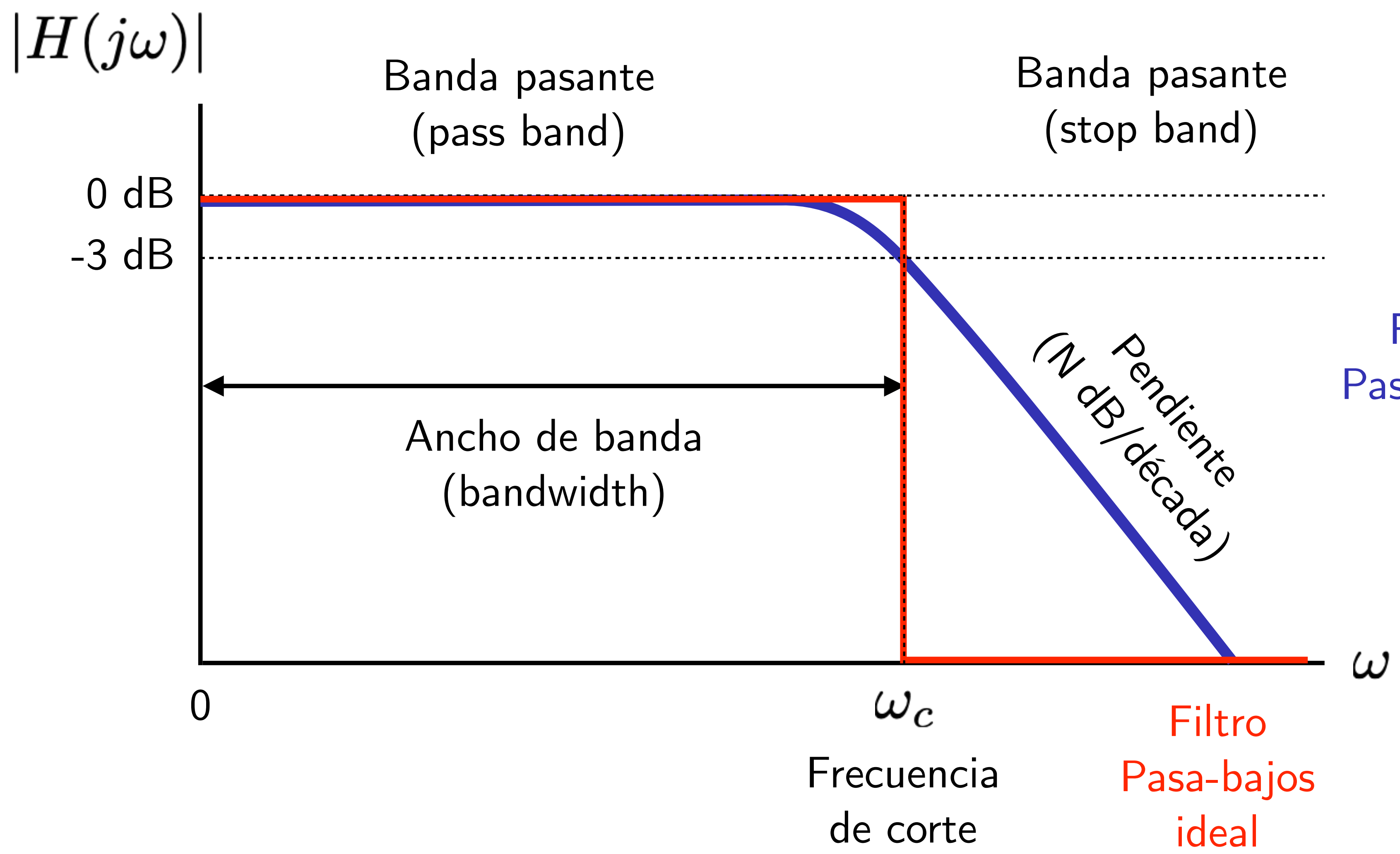
- $H(j\omega)$ es la respuesta frecuencial del sistema, existe si $h(t)$ es absolutamente integrable (para sistemas físicos)
  - un SLIT BIBO estable va a tener respuesta frecuencial $H(j\omega)$
- $h(t)$ y $H(j\omega)$ caracterizan el sistema,
- es usual llamar a este producto por $H(j\omega)$ el filtrado de la señal $X(j\omega)$.

Filtrado

Decibel es la décima parte de un Bel; es una medida relativa a un valor P_0

$$\frac{P}{P_0}\mathrm{dB} = 10\log_{10}\frac{P}{P_0}$$

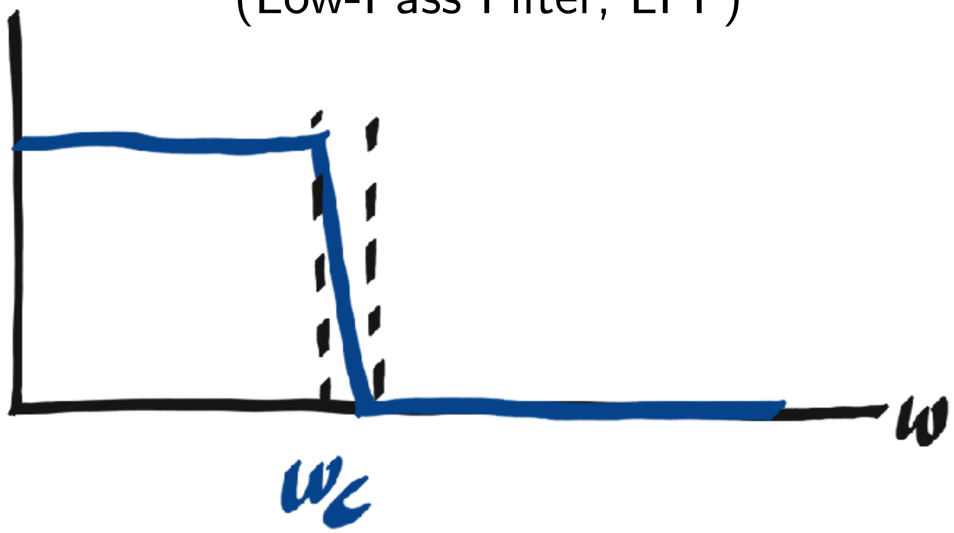
- Respuesta frecuencia de un filtro $H(j\omega)$:
  - Módulo



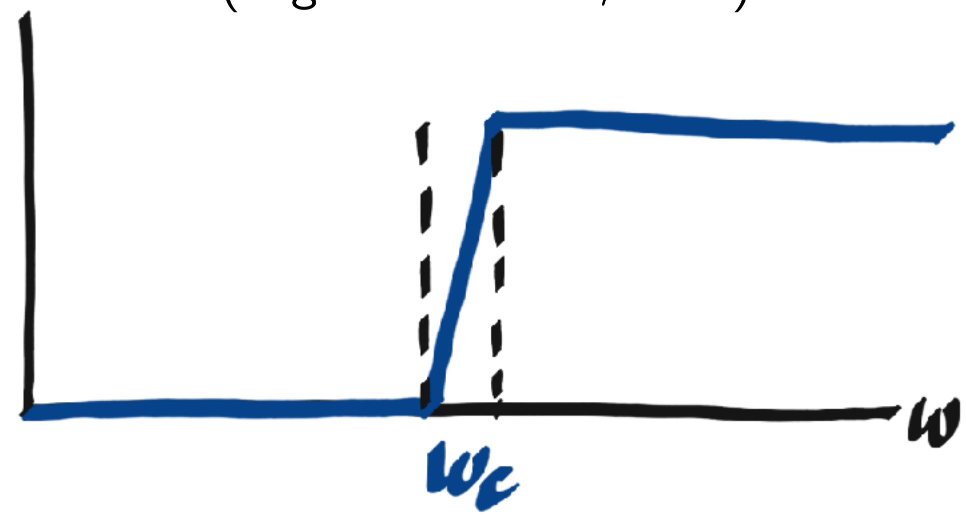
  - Igual con la fase $\sphericalangle H(j\omega)$

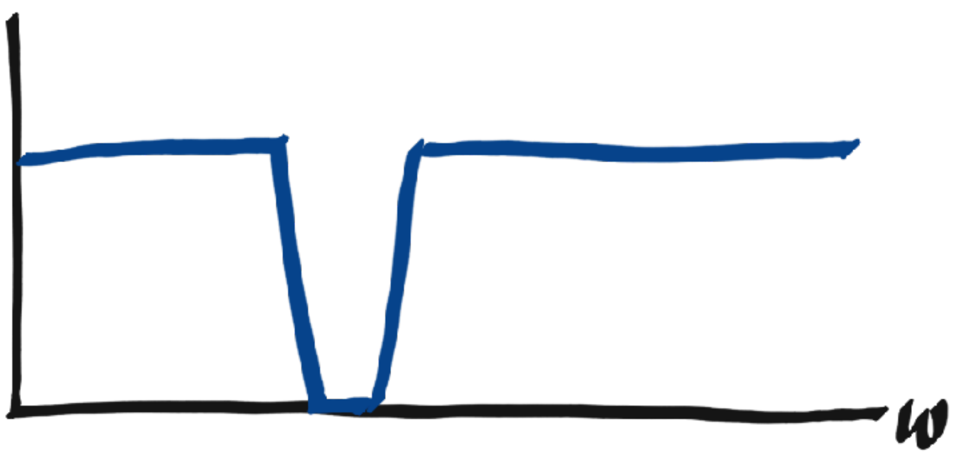
Filtrado

Filtro pasabajos
(Low-Pass Filter, LPF)

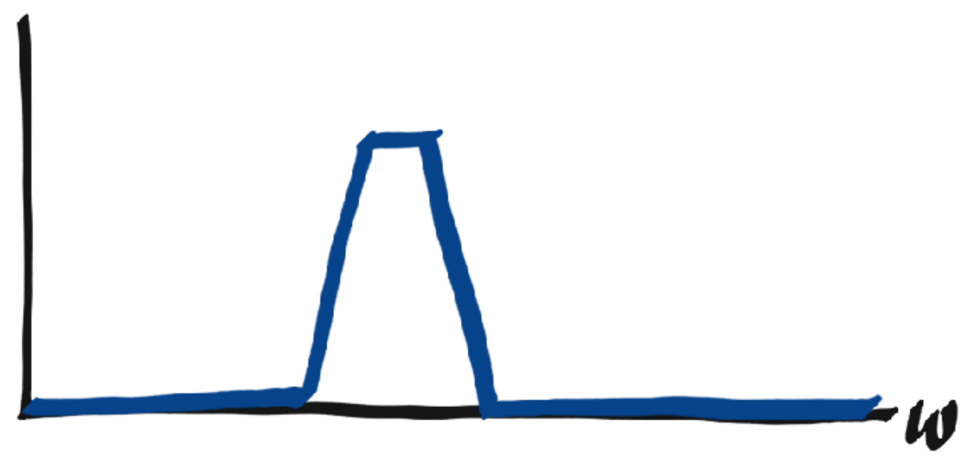


Filtro pasaaltos
(High-Pass Filter, HPF)





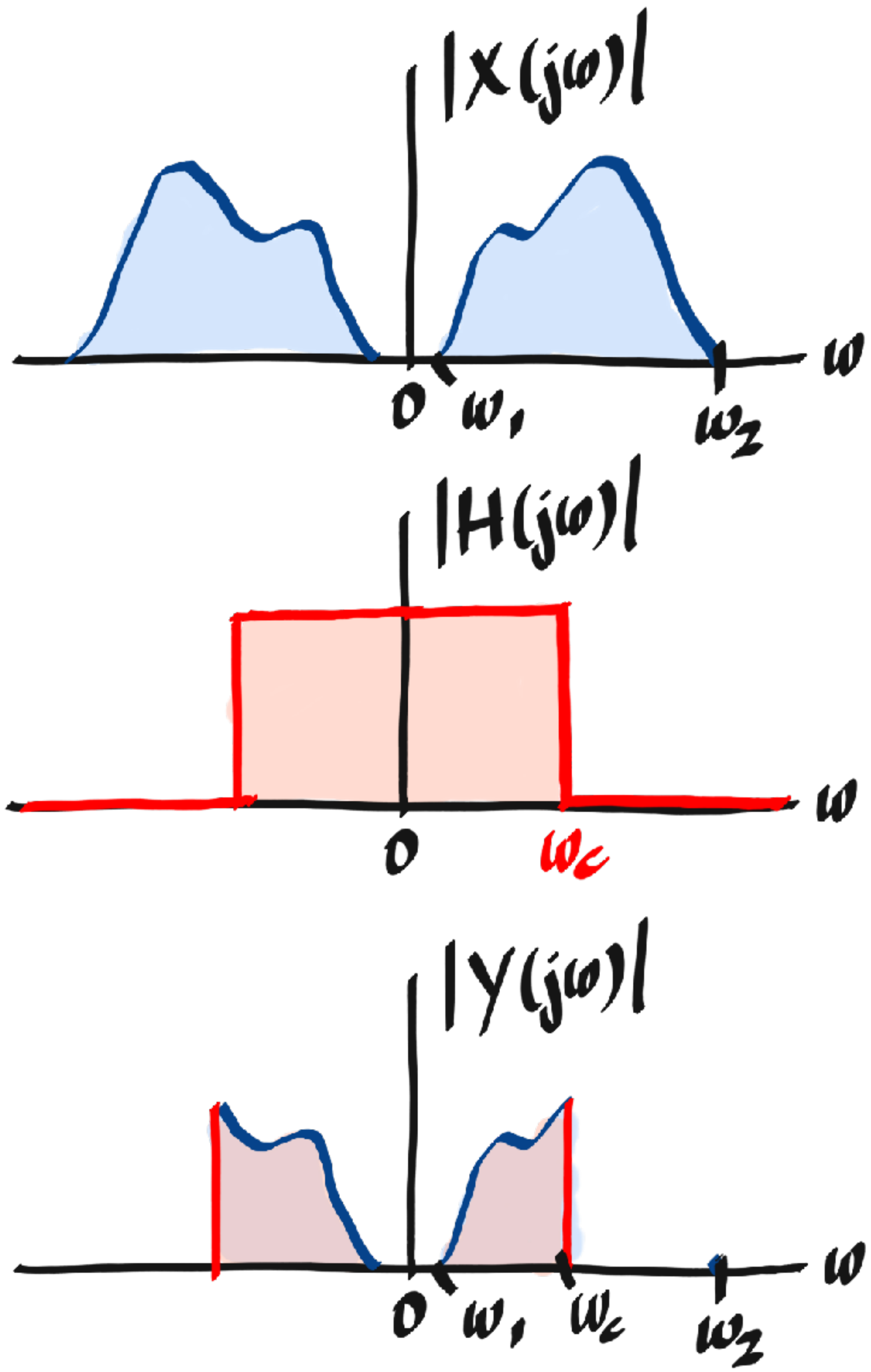
Filtro “matabanda” 😁,
filtro rechaza banda
filtro trampa, filtro notch
(Band-Stop Filter, BSF)



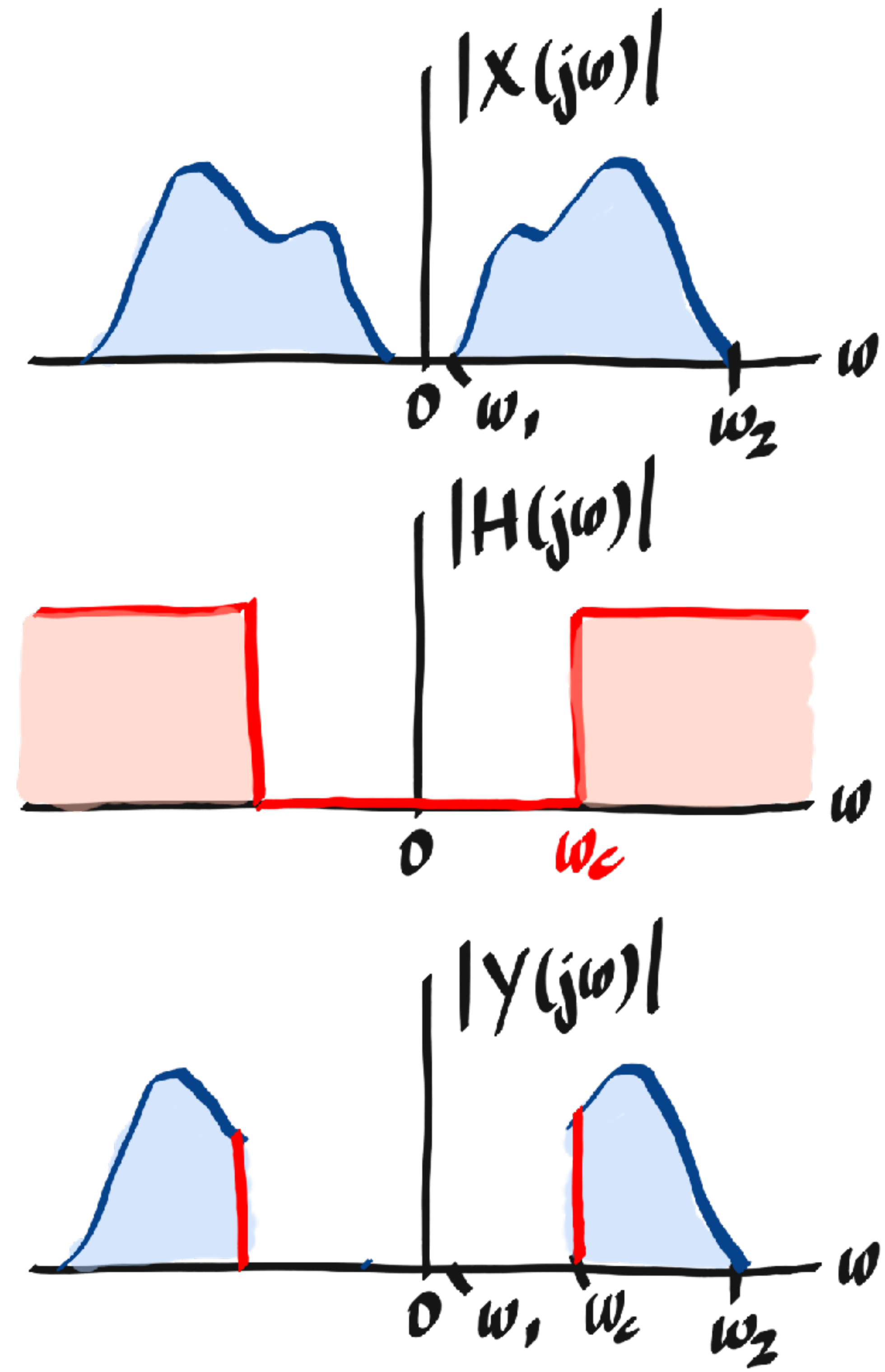
Filtro pasabanda
(Band-Pass Filter, BPF)

Filtrado

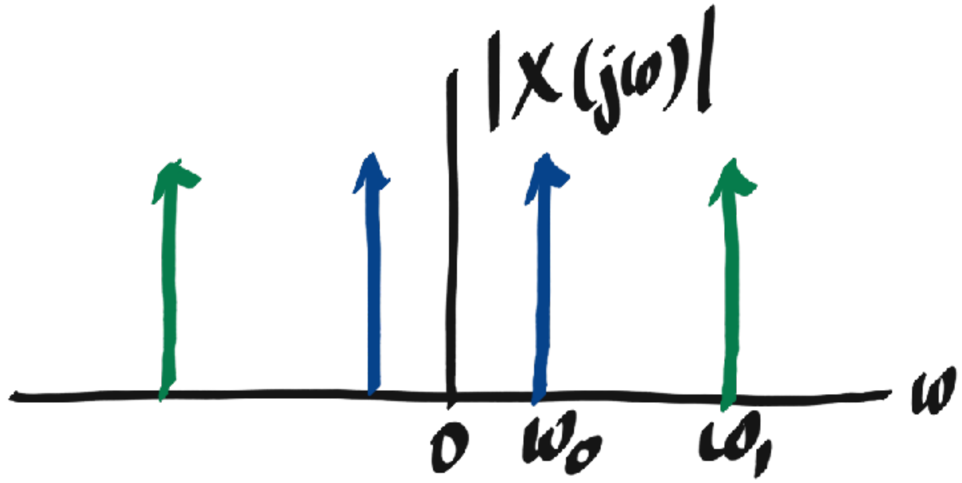
LPF



HPF

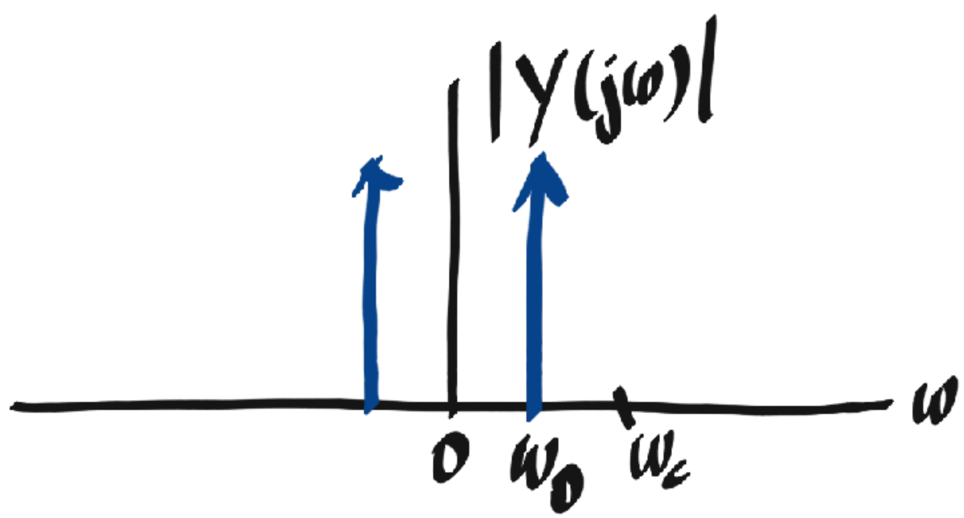


Filtrado



$$x(t) = \cos(\omega_0 t) + \cos(\omega_1 t)$$

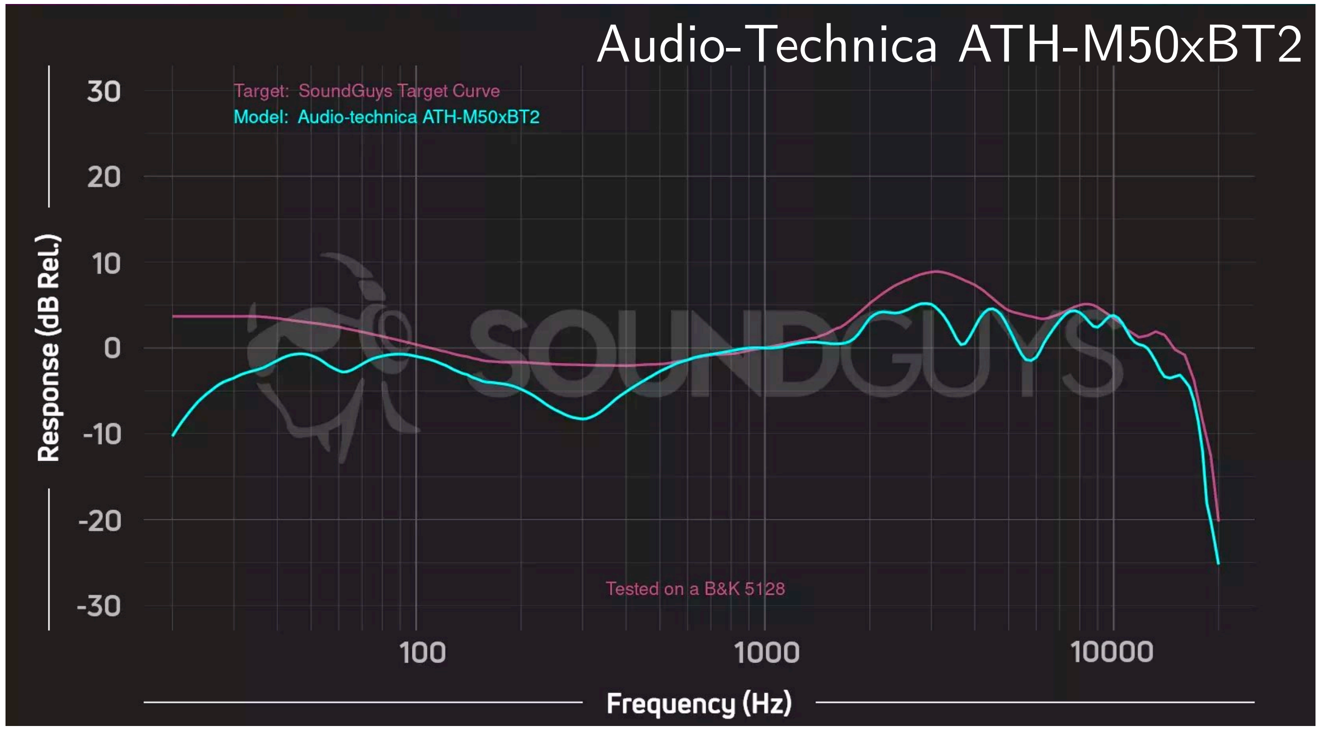
$$h(t) = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right) \qquad \omega_0 < \omega_c < \omega_1$$

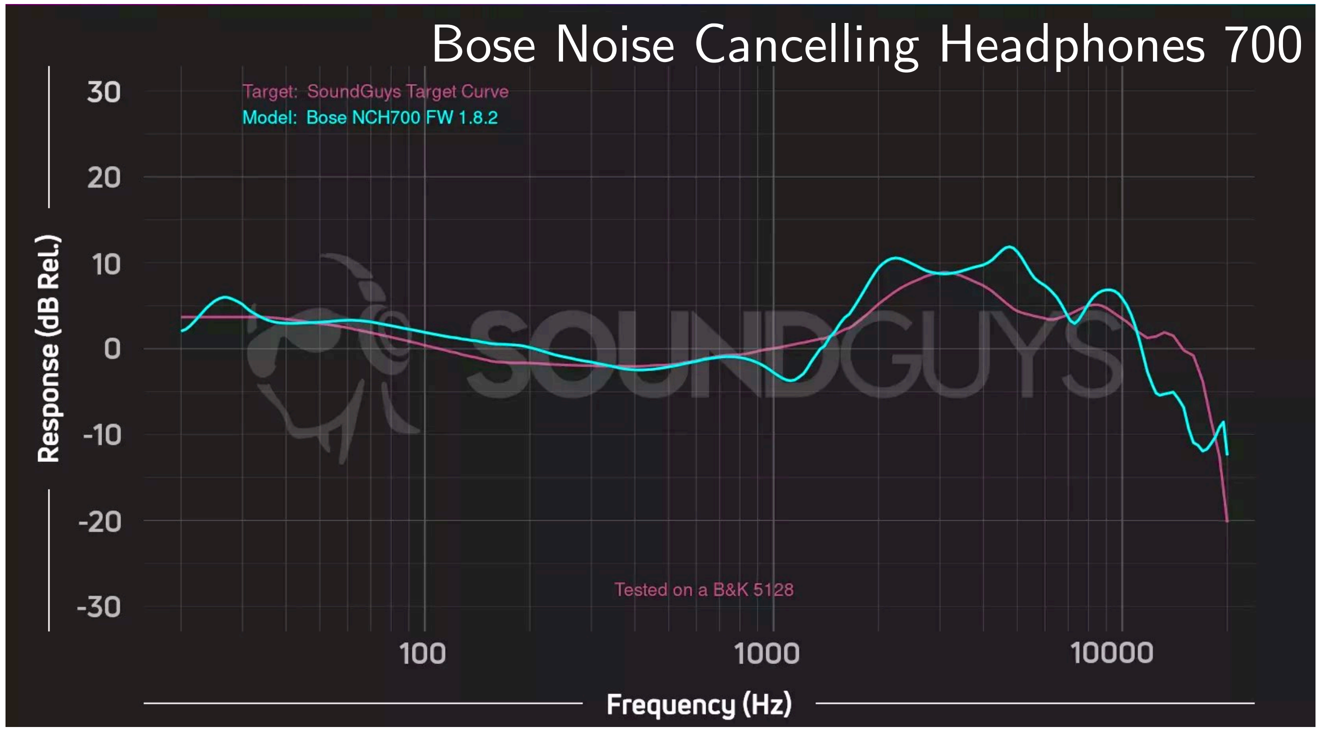


$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \Big(\cos(\omega_0 t) + \cos(\omega_1 t)\Big) * \left(\frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right)\right) \\ &= \cos(\omega_0 t) \end{aligned}$$

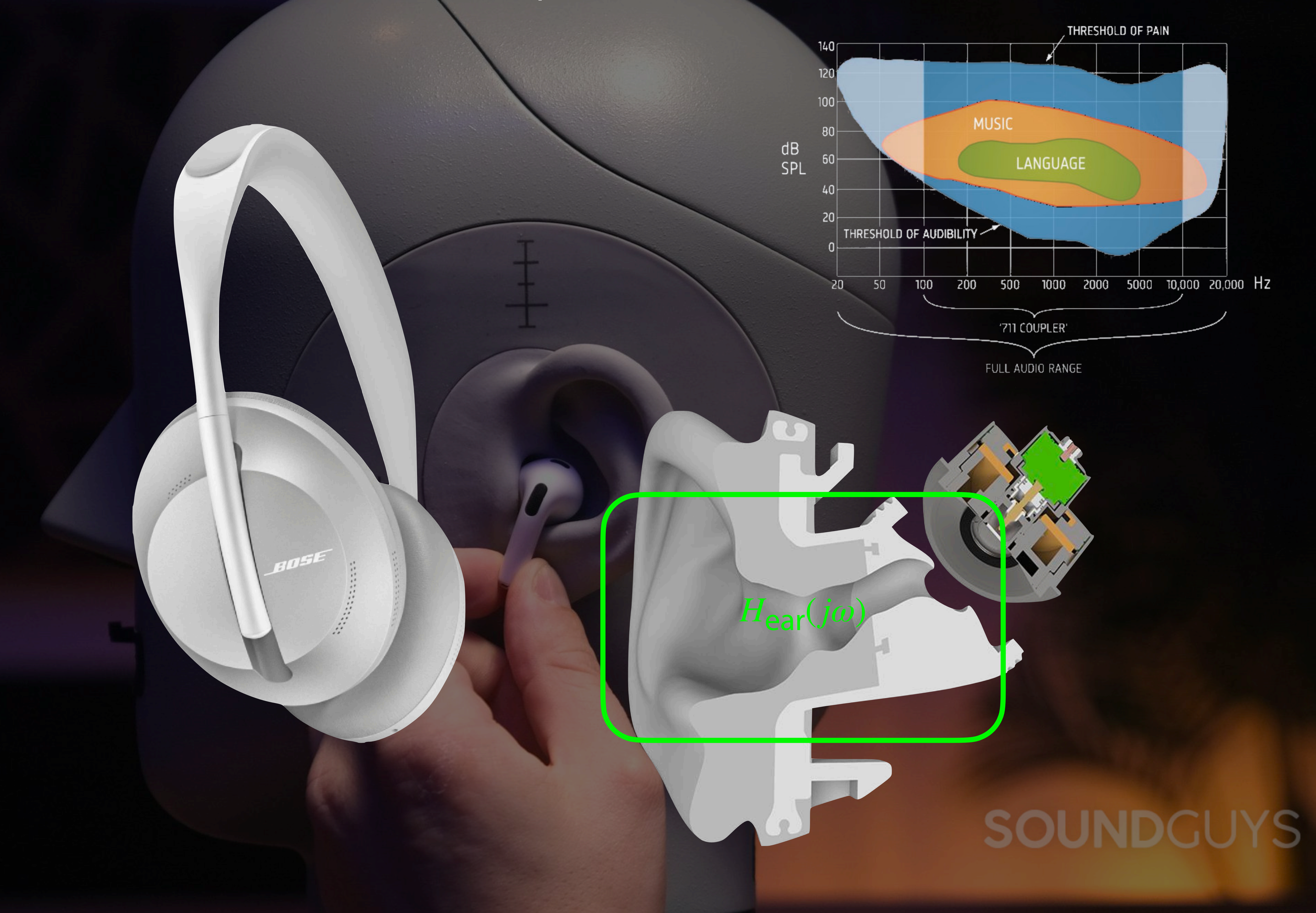
$$Y(j\omega) = X(j\omega) * H(j\omega)$$

Respuesta frecuencial

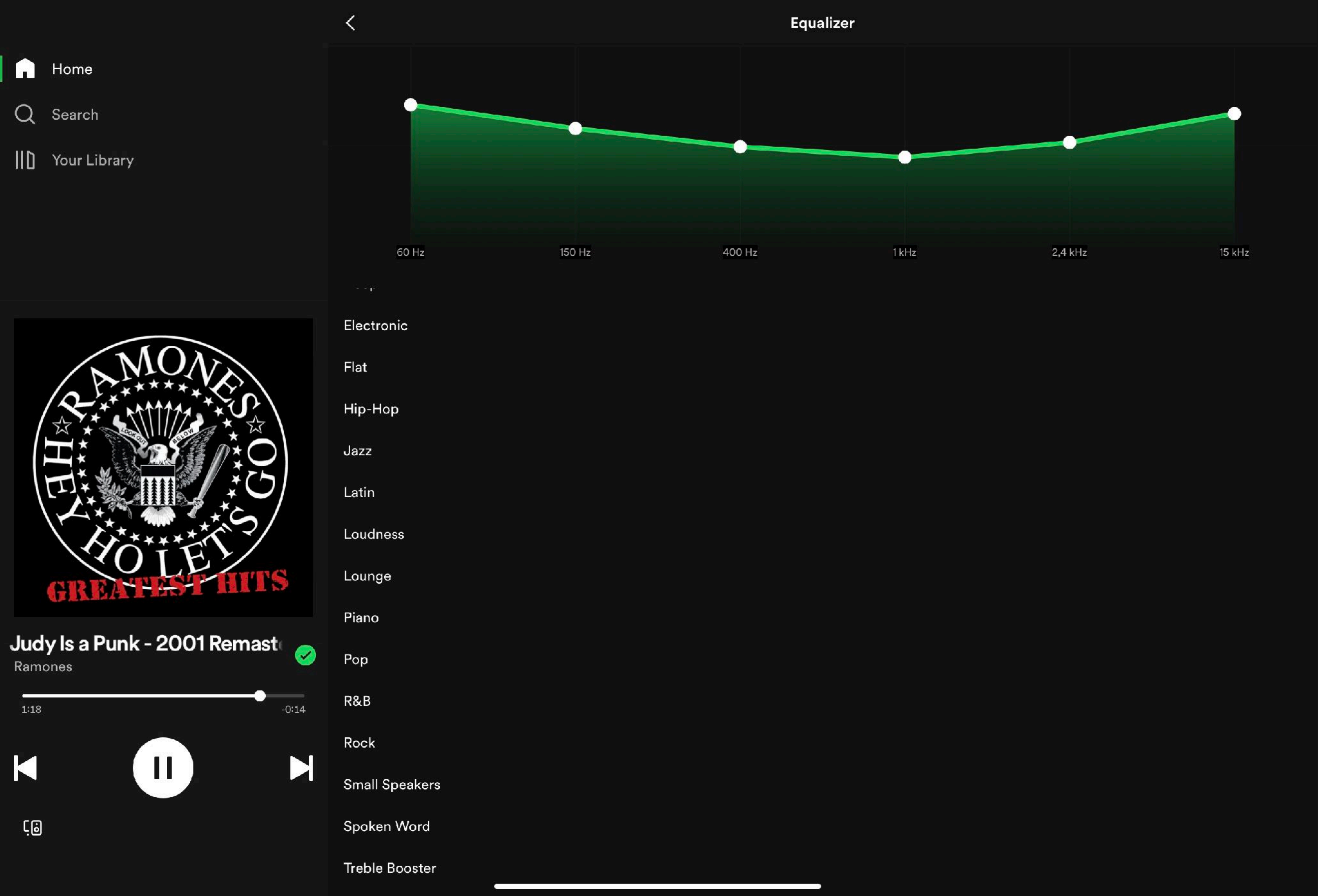




Respuesta frecuencial



Respuesta frecuencial



CTFT: propiedades

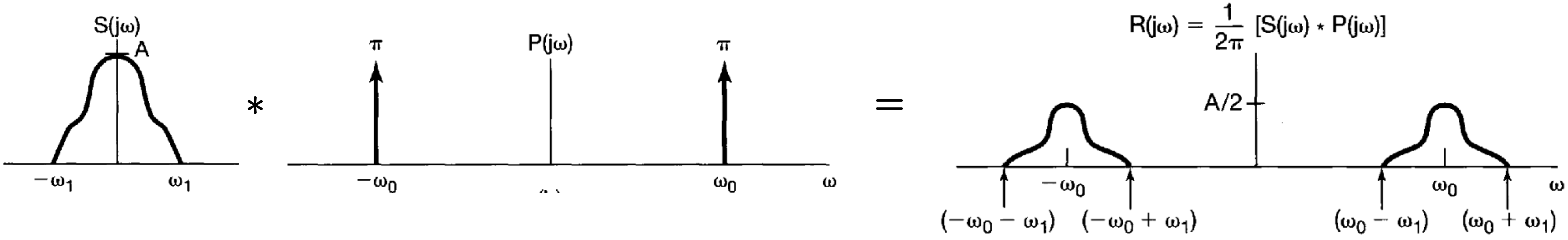
- Multiplicación

$$r(t) = s(t)p(t) \overset{TF}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(j\eta) P(j(\omega - \eta)) d\eta$$

- **Modulación**

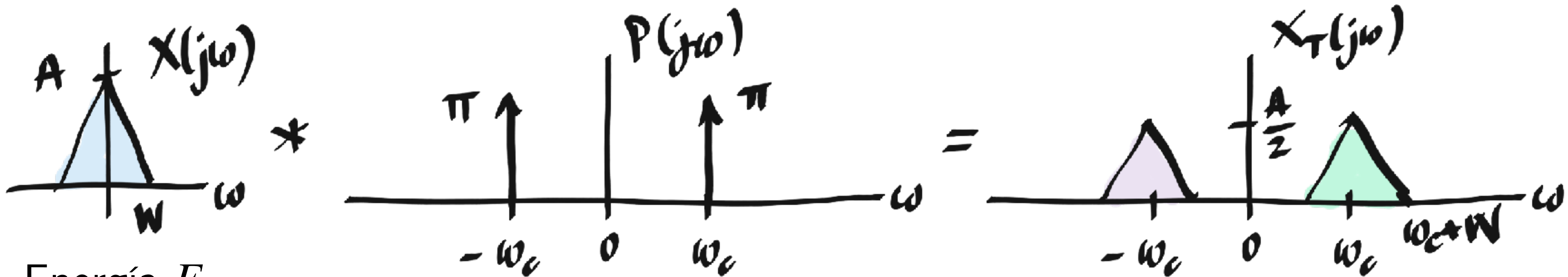
$$p(t) = \cos(\omega_0 t) \overset{TF}{\longleftrightarrow} P(j\omega) = \pi\delta\left(\omega - \omega_0\right) + \pi\delta\left(\omega + \omega_0\right)$$



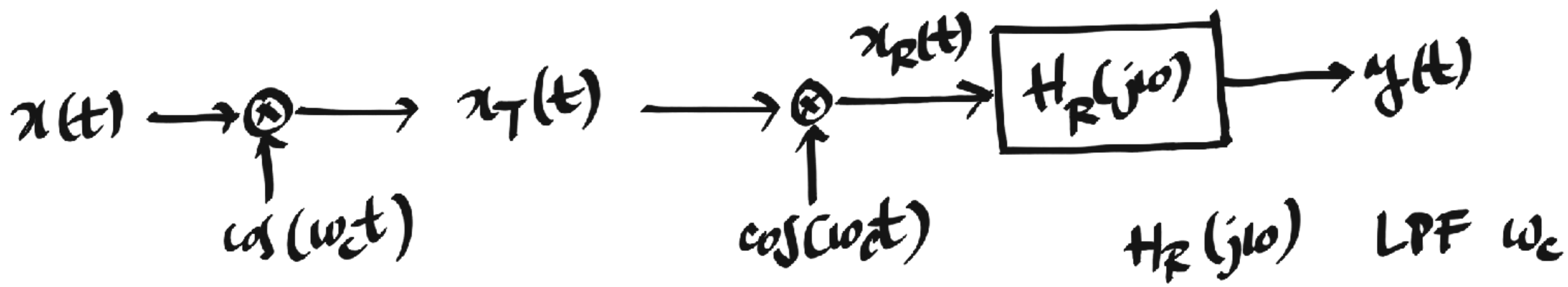
$$x_T(t) = x(t)\cos(\omega_c t) \overset{TF}{\longleftrightarrow} X_T(j\omega) = \frac{1}{2}\left(X(j(\omega + \omega_c)) + X(j(\omega - \omega_c))\right)$$

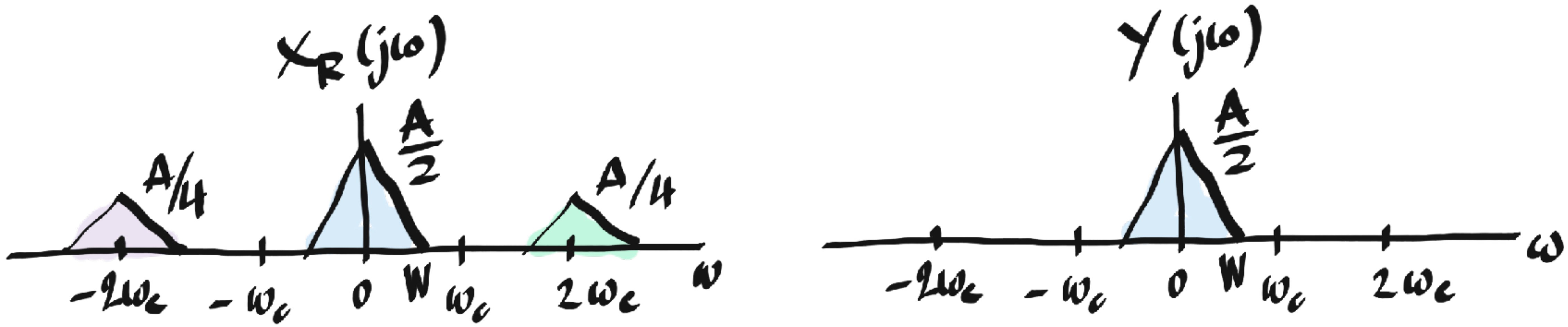
Modulación

$$x_T(t) = x(t)\cos(\omega_c t) \overset{\text{TF}}{\longleftrightarrow} X_T(j\omega) = \frac{1}{2}\left(X(j(\omega+\omega_c)) + X(j(\omega-\omega_c))\right)$$



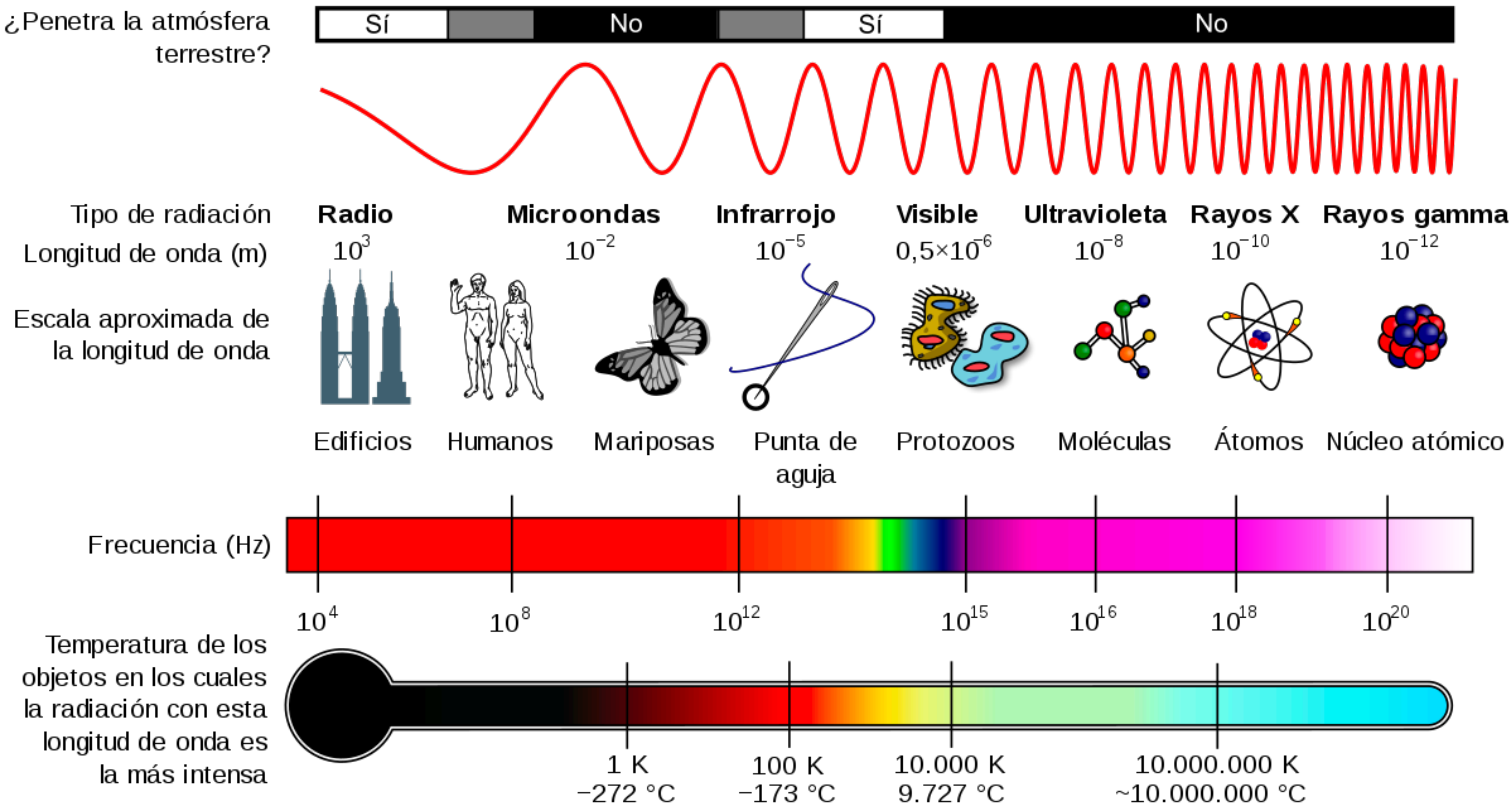
Energía E_x





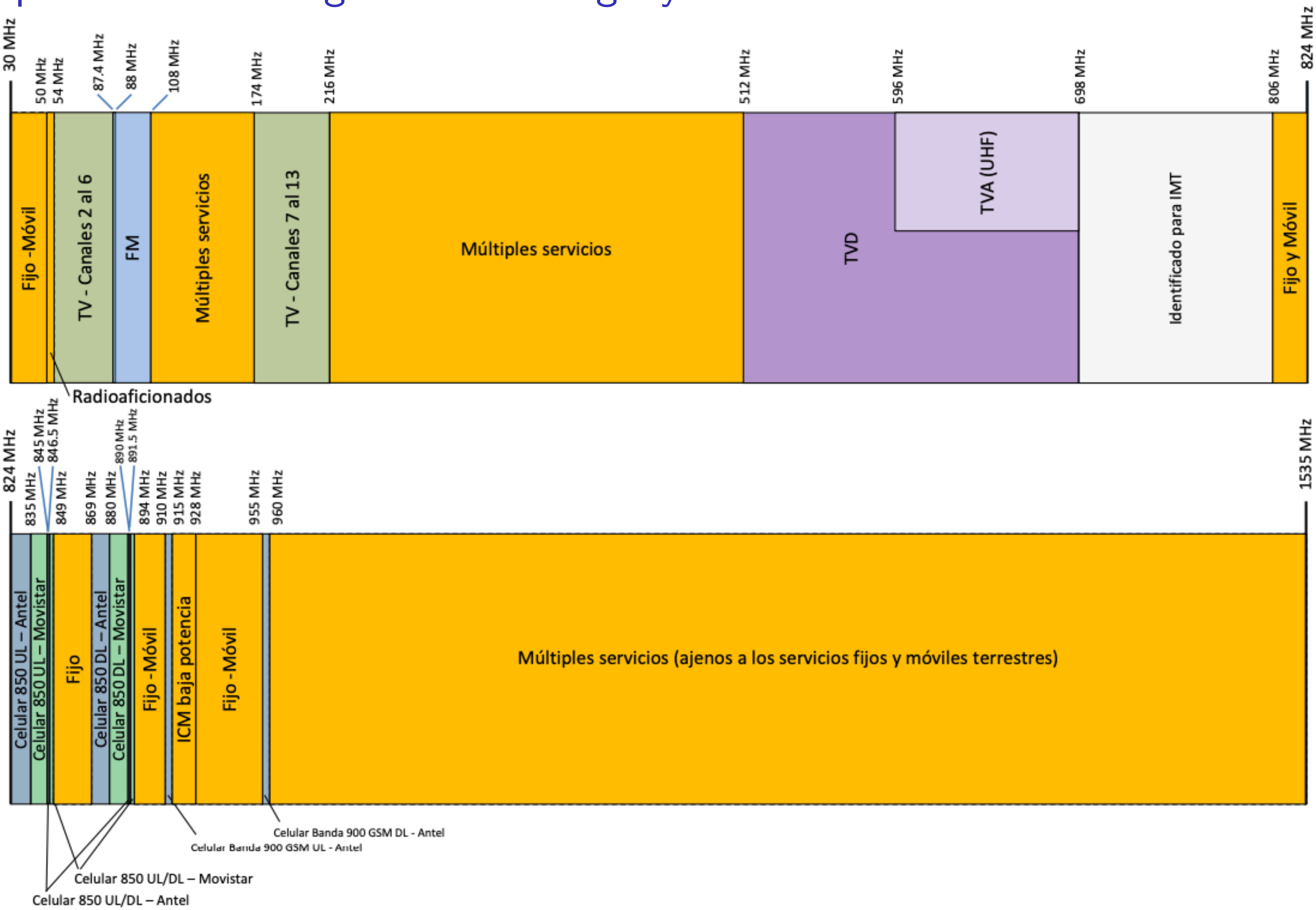
¿Energía de $y(t)$?

Espectro electromagnético



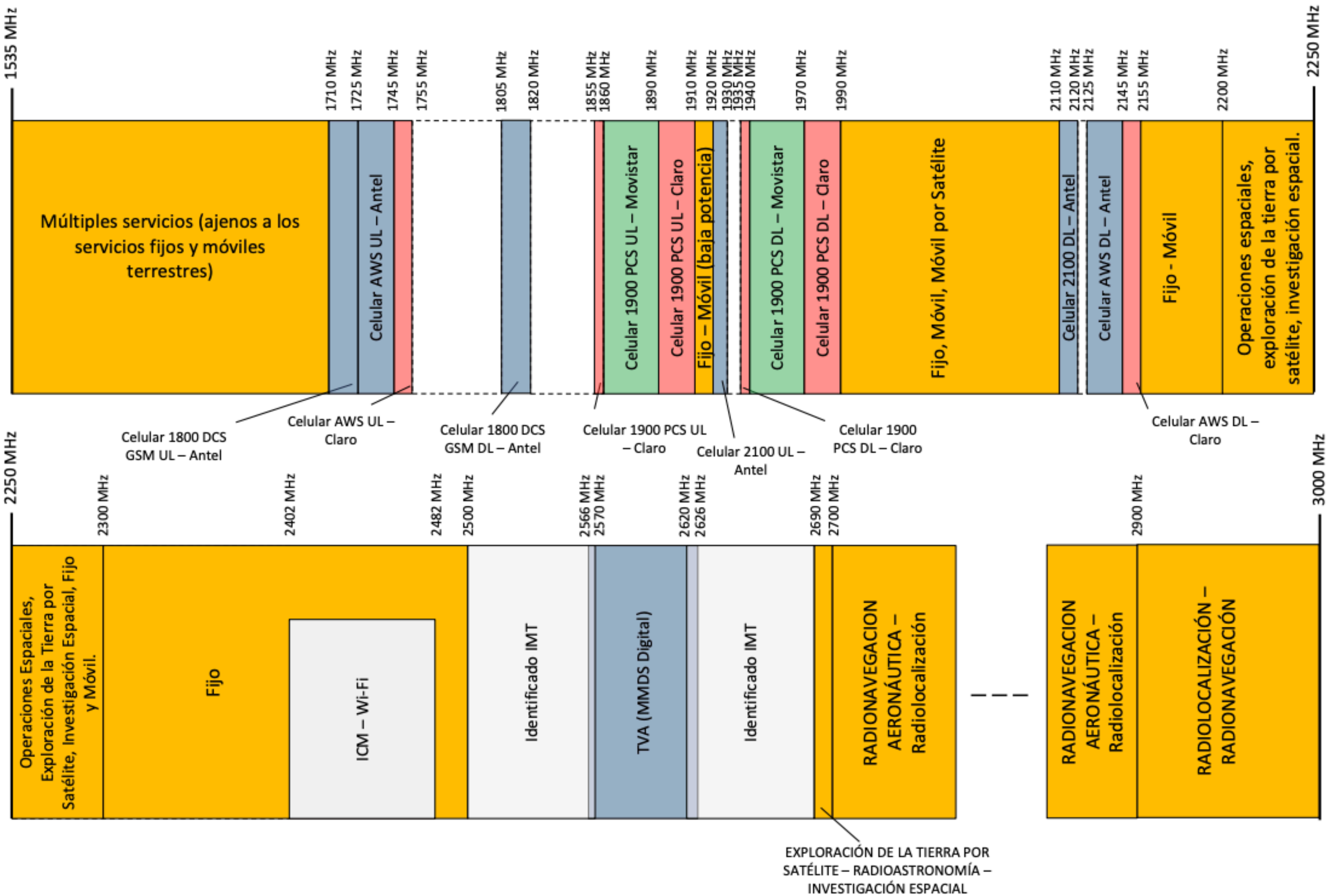
[Wikipedia]

Espectro electromagnético en Uruguay



Carro, G., Hernández, P. "Uso del espectro radioeléctrico en Uruguay y oportunidades para el uso de radio cognitiva". Tesis de grado, Universidad de la República (Uruguay). Facultad de Ingeniería, 2016.

Espectro electromagnético en Uruguay



Carro, G., Hernández, P. "Uso del espectro radioeléctrico en Uruguay y oportunidades para el uso de radio cognitiva". Tesis de grado, Universidad de la República (Uruguay). Facultad de Ingeniería, 2016.