Funciones generatrices

sucesión $\longleftrightarrow$ función generatriz

$a_0, a_1, a_2, a_3, \ldots$

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ "polinomio infinito"

Ejercicio 1

a) función generatriz de $C_0^6, C_1^6, C_2^6, \ldots, C_6^6, \ldots$

$$\begin{array}{ccccccc} C_0^6 & C_1^6 & C_2^6 & \ldots & C_6^6 & 0 & \ldots \\ \shortparallel & \shortparallel & \shortparallel & & \shortparallel & \shortparallel & \\ a_0 & a_1 & a_2 & & a_6 & a_7 & \end{array}$$

$$f(x) = C_0^6 + C_1^6 x + C_2^6 x^2 + \ldots + C_6^6 x^6 + 0 + 0 \ldots$$

$$= \sum_{n=0}^{6} C_n^6 x^n$$

$$= \sum_{n=0}^{6} C_n^6 x^n 1^{6-n}$$

$$= (x+1)^6$$

$$(a+b)^k = \sum_{i=0}^{k} C_i^k a^i b^{k-i}$$

generalización: $(1+x)^n = \sum_{i=0}^{n} C_i^n x^i 1^{n-i}$

$$= \sum_{i=0}^{n} C_i^n x^i$$

$$= C_0^n + C_1^n x + C_2^n x^2 + C_3^n x^3 + \dots$$

$(1+x)^n$ genera la sucesión: $C_0^n, C_1^n, C_2^n, \dots, C_n^n, 0, 0, \dots$

Ejemplo:

función generatriz de $a_n = 1 \quad n \in \mathbb{N} \to a_0 = 1, \; a_1 = 1, \; a_2 = 1, \; \vdots$

$$f(x) = 1 + 1x + 1x^2 + 1x^3 + \dots$$

$$= 1 + x + x^2 + x^3 + \dots$$

$$= \sum_{n=0}^{\infty} x^n$$

$$(1-x)(1+x+x^2+x^3+\dots) = (1+\cancel{x}+\cancel{x^2}+\cancel{x^3}+\dots) + (-\cancel{x}-\cancel{x^2}-\cancel{x^3}-\dots)$$

$$= 1$$

entonces $\boxed{1+x+x^2+x^3+\dots = \frac{1}{1-x}}$

(d) $0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, \ldots$

$$f(x) = 0 + 0x + 0x^2 + 0x^3 + 1x^4 + 1x^5 + 1x^6 + \ldots$$

$$= x^4 + x^5 + x^6 + \ldots$$

$$= x^4 (1 + x + x^2 + x^3 + \ldots)$$

$$= x^4 \frac{1}{1-x}$$

$$f(x) = \frac{x^4}{1-x}$$

(c) $1, -1, 1, -1, \ldots$

función generatriz de $a_n = (-1)^n$, $n \in \mathbb{N}$

$$f(x) = 1 - x + x^2 - x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

Sabemos

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$$

$$= \sum_{n=0}^{\infty} (-x)^n$$

$y = -x$

$$= \sum_{n=0}^{\infty} y^n$$

$$= \frac{1}{1-y}$$

$$= \frac{1}{1-(-x)} = \frac{1}{1+x}$$

(h) $0, 0, 1, a, a^2, a^3, a^4, a^5, a^6, \ldots$

$$f(x) = 0 + 0x + 1x^2 + ax^3 + a^2x^4 + a^3x^5 + \ldots$$
$$= x^2 + ax^3 + a^2x^4 + a^3x^5 + \ldots$$
$$= x^2(1 + ax + a^2x^2 + a^3x^3 + \ldots)$$
$$= x^2(1 + ax + (ax)^2 + (ax)^3 + \ldots)$$

$y = ax$

$$= x^2(1 + y + y^2 + y^3 + \ldots)$$
$$= x^2 \frac{1}{1-y}$$
$$= x^2 \frac{1}{1-ax}$$

$$(2x^2)^3 = 8x^6$$
$$(2x^2)^2 = 4x^4$$

(i) $1, 0, 2, 0, 4, 0, 8, 0, 16, 0, 32, 0, 64, 0, 128, \ldots$

$$f(x) = 1 + 0x + 2x^2 + 0x^3 + 4x^4 + \ldots$$
$$= 1 + 2x^2 + 4x^4 + 8x^6 + \ldots$$
$$= 1 + 2x^2 + (2x^2)^2 + (2x^2)^3 + \ldots$$

$y = 2x^2$

$$= 1 + y + y^2 + y^3 + \ldots$$
$$= \frac{1}{1-y}$$
$$= \frac{1}{1-2x^2}$$

Ejercicio 2

$$(2x-3)(2x-3)(2x-3)$$

(a) $f(x) = (2x-3)^3$

sucesión generada por $f(x) = (2x-3)^3$

buscamos $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \underbrace{a_4 x^4 + \dots}_{=0}$

$$(a+b)^k = \sum_{i=0}^{k} C_i^k a^i b^{k-i}$$

$$f(x) = (2x-3)^3$$

$$= \sum_{i=0}^{3} C_i^3 (2x)^i (-3)^{3-i}$$

$$C_1^3 = \frac{3!}{1!(3-1)!} = 3$$

$$= \overbrace{C_0^3}^{=1} \overbrace{(2x)^0}^{1} \overbrace{(-3)^3}^{-27} + \overbrace{C_1^3}^{=3} \overbrace{(2x)^1}^{2x} \overbrace{(-3)^2}^{=9} + \overbrace{C_2^3}^{=3} \overbrace{(2x)^2}^{4x^2} \overbrace{(-3)^1}^{-3} + \overbrace{C_3^3}^{=1} \overbrace{(2x)^3}^{8x^3} \overbrace{(-3)^0}^{1}$$

$$= -27 + 54x - 36x^2 + 8x^3$$

sucesión generada por $f(x)$:

$a_0 = -27$

$a_1 = 54$

$a_2 = -36$

$a_3 = 8$

$a_n = 0$ si $n \geq 4$

(d) $f(x) = 1/(1+3x)$

sucesión generada por $f(x) = \dfrac{1}{1+3x}$

$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + \dots$$

$$f(x) = \frac{1}{1+3x}$$

$$= \frac{1}{1-(-3x)}$$

$y = -3x$

$$(ab)^n = a^n b^n$$

$$= \frac{1}{1-y}$$

$$(-3x)^n \rightsquigarrow \underbrace{(-3)^n}\, x^n$$

$$= \sum_{n=0}^{\infty} y^n$$

$$= \sum_{n=0}^{\infty} (-3x)^n \rightsquigarrow \sum_{n=0}^{\infty} a_n x^n = f(x)$$

$$= \sum_{n=0}^{\infty} (-3)^n \underbrace{x^n} = 1 - 3x + 9x^2 - 27x^3 + \dots$$

la sucesión generada es $a_n = (-3)^n$

$1, -3, 9, -27, \dots$

$p(\lambda) = \lambda - 1 \rightarrow$ 1 raiz simple

$$a_{n+1} - a_n = 3n = 1^n \underbrace{(3n)}_{g(n)}$$

$$a_n^{(p)} = n\, 1^n (an + b) = 1^n (an^2 + bn)$$

$$a_{n+1} - a_n = n+3 = 1^n (n+3)$$

$$a_n^{(p)} = n\, 1^n (an + b)$$