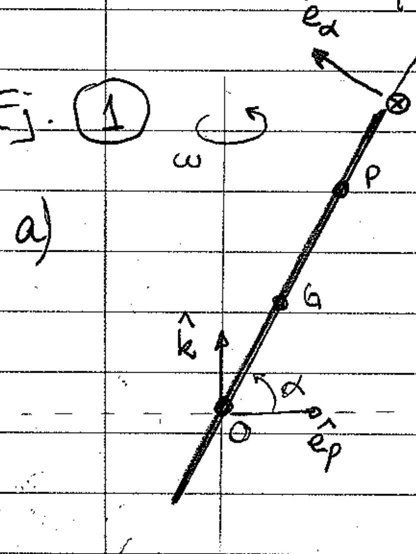
Problemas De Clase - Práctico 8

Ej. 1



Vel. angular del rígido: $\vec{\Omega} = \omega \hat{k} - \dot{\alpha}\hat{e}_\varphi$

a)

$$\vec{r}_G = r\hat{e}_r$$
$$\vec{V}_G = r\dot{\hat{e}}_r = r\omega\cos\alpha\,\hat{e}_\varphi + r\dot{\alpha}\hat{e}_\alpha$$
$$\vec{a}_G = -r\omega\dot{\alpha}\,\text{sen}\,\alpha\,\hat{e}_\varphi + r\omega\cos\alpha\,\dot{\hat{e}}_\varphi + r\ddot{\alpha}\hat{e}_\alpha + r\dot{\alpha}\dot{\hat{e}}_\alpha =$$
$$= -r\omega\dot{\alpha}\,\text{sen}\,\alpha\,\hat{e}_\varphi - r\omega^2\cos^2\alpha\,\hat{e}_r + r\omega^2\,\text{sen}\,\alpha\cos\alpha\,\hat{e}_\alpha + r\ddot{\alpha}\hat{e}_\alpha - r\omega\dot{\alpha}\,\text{sen}\,\alpha\,\hat{e}_\varphi - r\dot{\alpha}^2\hat{e}_r \implies$$

$$\left(\begin{aligned}
\dot{\hat{e}}_r &= \vec{\Omega}\times\hat{e}_r = (\omega\hat{k} - \dot{\alpha}\hat{e}_\varphi)\times\hat{e}_r = \omega\cos\alpha\,\hat{e}_\varphi + \dot{\alpha}\hat{e}_\alpha \\
\dot{\hat{e}}_\varphi &= \vec{\Omega}\times\hat{e}_\varphi = (\omega\hat{k} - \dot{\alpha}\hat{e}_\varphi)\times\hat{e}_\varphi = -\omega\hat{e}_\rho = -\omega\cos\alpha\,\hat{e}_r + \omega\,\text{sen}\,\alpha\,\hat{e}_\alpha \\
\dot{\hat{e}}_\alpha &= \vec{\Omega}\times\hat{e}_\alpha = (\omega\hat{k} - \dot{\alpha}\hat{e}_\varphi)\times\hat{e}_\alpha = -\omega\,\text{sen}\,\alpha\,\hat{e}_\varphi - \dot{\alpha}\hat{e}_r
\end{aligned}\right)$$

$$\implies \boxed{\vec{a}_G = -(r\dot{\alpha}^2 + r\omega^2\cos^2\alpha)\,\hat{e}_r - 2r\omega\dot{\alpha}\,\text{sen}\,\alpha\,\hat{e}_\varphi + (r\ddot{\alpha} + r\omega^2\,\text{sen}\,\alpha\cos\alpha)\,\hat{e}_\alpha}$$

b) Aplico 2ª cardinal desde O: $\dot{\vec{L}}_O = \vec{M}_O^{(ext)}$

$$\mathbb{I}_G^{\{e_r, e_\varphi, e_\alpha\}} = \begin{bmatrix} 0 & & \\ & \frac{1}{12}m(2l)^2 & \\ & & \frac{1}{12}m(2l)^2 \end{bmatrix} = \frac{ml^2}{3}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

Steiner: $$\mathbb{I}_O = \mathbb{I}_G + \begin{bmatrix} 0 & & \\ & mr^2 & \\ & & mr^2 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & \frac{ml^2}{3} + mr^2 & \\ & & \frac{ml^2}{3} + mr^2 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & I_O & \\ & & I_O \end{bmatrix}$$

$$\vec{w} = \omega\hat{k} - \dot{\alpha}\hat{e}_\varphi = \omega(\text{sen}\,\alpha\,\hat{e}_r + \cos\alpha\,\hat{e}_\alpha) - \dot{\alpha}\hat{e}_\varphi = \begin{bmatrix} \omega\,\text{sen}\,\alpha \\ -\dot{\alpha} \\ \omega\cos\alpha \end{bmatrix} \quad \text{En } \{\hat{e}_r, \hat{e}_\varphi, \hat{e}_\alpha\}$$

$$\implies \vec{L}_O = \mathbb{I}_O\vec{w} = -I_O\dot{\alpha}\hat{e}_\varphi + I_O\omega\cos\alpha\,\hat{e}_\alpha$$

$$\implies \dot{\vec{L}}_O = -I_O\ddot{\alpha}\hat{e}_\varphi - I_O\dot{\alpha}\dot{\hat{e}}_\varphi - I_O\omega\dot{\alpha}\,\text{sen}\,\alpha\,\hat{e}_\alpha + I_O\omega\cos\alpha\,\dot{\hat{e}}_\alpha$$

$\longrightarrow$

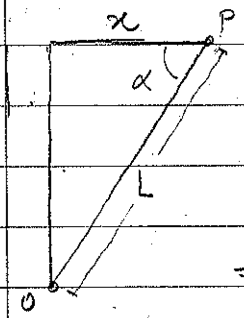
$$\Rightarrow \vec{L}_O = -I_O\ddot{\alpha}\,\hat{e}_\varphi - I_O\dot{\alpha}\left(-\omega\cancel{\cos\alpha\,\hat{e}_r} + \omega\,\mathrm{sen}\,\alpha\,\hat{e}_\alpha\right) - I_O\omega\dot{\alpha}\,\mathrm{sen}\,\alpha\,\hat{e}_\alpha +$$
$$+ I_O\omega\cos\alpha\left(-\omega\,\mathrm{sen}\,\alpha\,\hat{e}_\varphi - \cancel{\dot{\alpha}\,\hat{e}_r}\right) =$$

$$\Rightarrow \vec{L}_O = -I_O\left(\ddot{\alpha} + \omega^2\,\mathrm{sen}\,\alpha\cos\alpha\right)\hat{e}_\varphi - 2I_O\omega\dot{\alpha}\,\mathrm{sen}\,\alpha\,\hat{e}_\alpha \qquad (1).$$

MOMENTOS:

$$\vec{M}_O^{(ext)} = r\hat{e}_r \times (-mg\hat{k}) + L\hat{e}_r \times (-kx\,\hat{e}_\rho) + \underbrace{M_1^{react}\,\hat{e}_r + M_2^{react}\,\hat{e}_\alpha}$$

Por ser art. cilíndrica, no tiene componente según $\hat{e}_\varphi$



$x = L\cos\alpha$

$$\Rightarrow \vec{M}_O^{(ext)} = mgr\cos\alpha\,\hat{e}_\varphi - kL^2\cos\alpha\,\mathrm{sen}\,\alpha\,\hat{e}_\varphi + M_1\hat{e}_r + M_2\hat{e}_\alpha \qquad (2)$$

De (1) y (2) $\Rightarrow \begin{cases} -I_O(\ddot{\alpha} + \omega^2\,\mathrm{sen}\,\alpha\cos\alpha) = mgr\cos\alpha - kL^2\cos\alpha\,\mathrm{sen}\,\alpha \\ 0 = M_1^{react} \\ -2I_O\omega\dot{\alpha}\,\mathrm{sen}\,\alpha = M_2^{react} \end{cases}$

Ec. de MOV: $\boxed{\ddot{\alpha} + \left(\omega^2 - \dfrac{kL^2}{I_O}\right)\mathrm{sen}\,\alpha\cos\alpha + \dfrac{mgr}{I_O}\cos\alpha = 0}$

c) De la parte anterior se desprende

$$\boxed{\vec{M}_O^{react} = -2I_O\omega\dot{\alpha}\,\mathrm{sen}\,\alpha\,\hat{e}_\alpha}$$

Fuerzas:

$$\vec{F} = -mg\hat{k} - kx\,\hat{e}_\rho + \vec{F}^{reac} = m\vec{a}_G$$

$$\Rightarrow \boxed{\vec{F}^{reac} = m\vec{a}_G + mg\hat{k} - kL\cos\alpha\,\hat{e}_\rho}$$

Hallada en la parte (a)

d) $Pot = \vec{\mathcal{M}}_o^{\,react} \cdot \vec{\Omega} = -2 I_o \omega \dot{\alpha} \operatorname{sen}\alpha \, \hat{e}_\alpha \cdot (\omega \hat{k} - \dot{\alpha} \hat{e}_\varphi) =$

$$\Rightarrow \boxed{Pot = -2 I_o \omega^2 \dot{\alpha} \operatorname{sen}\alpha \cos\alpha}$$

e) Eq. Relativo se da cuando $\ddot{\alpha} = 0 \Rightarrow$ De la Ec. de mov. resulta:

$$\left(\omega^2 - \frac{kL^2}{I_o}\right) \operatorname{sen}\alpha_o \cos\alpha_o + \frac{mgr}{I_o} \cos\alpha_o = 0$$

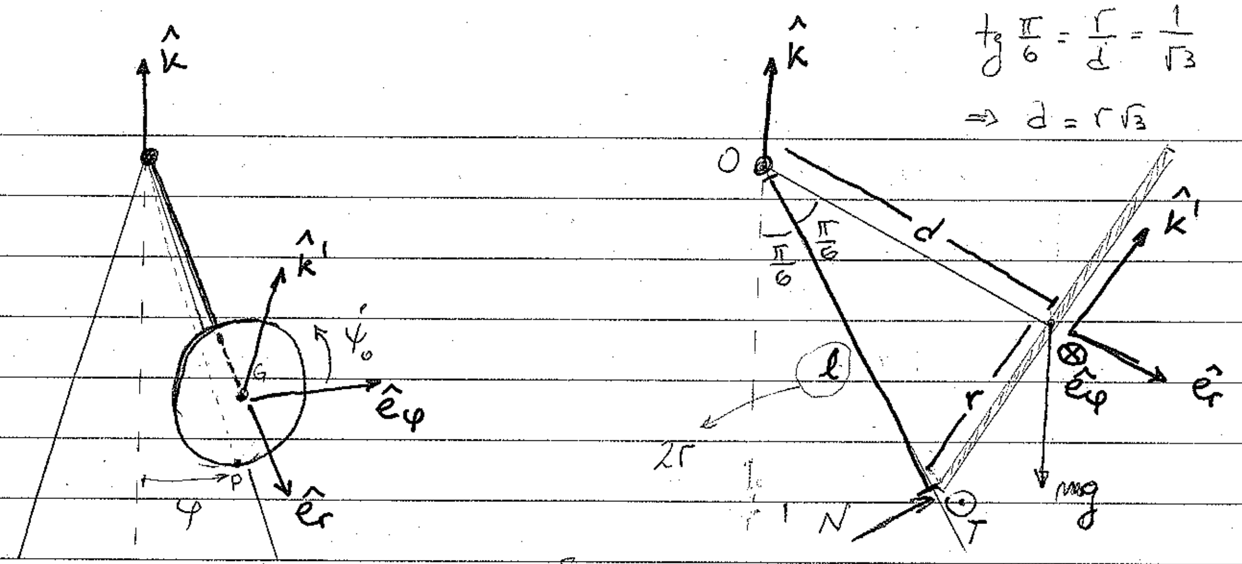
$$\Rightarrow \begin{cases} \cos\alpha_o = 0 \Rightarrow \boxed{\alpha_o = \pm \frac{\pi}{2}} \\ \operatorname{sen}\alpha_o = \dfrac{mgr}{kL^2 - I_o\omega^2} \Rightarrow \boxed{\alpha_o = \operatorname{sen}^{-1}\left(\dfrac{mgr}{kL^2 - I_o\omega^2}\right)} \end{cases}$$

$$\exists \text{ sii } \left|\frac{mgr}{kL^2 - I_o\omega^2}\right| < 1$$

Ej (2)

$\operatorname{tg}\frac{\pi}{6} = \frac{r}{d} = \frac{1}{\sqrt{3}}$

$\Rightarrow d = r\sqrt{3}$



$$\vec{\omega} = \dot{\varphi}\hat{k} + \dot{\psi}\hat{e}_r = \dot{\varphi}\Big(\underbrace{\operatorname{sen}(\pi/3)}_{=\frac{\sqrt{3}}{2}}\hat{k}' - \underbrace{\cos(\pi/3)}_{=\frac{1}{2}}\hat{e}_r\Big) + \dot{\psi}\hat{e}_r =$$

$$= \left(\dot{\psi} - \frac{\dot{\varphi}}{2}\right)\hat{e}_r + \frac{\sqrt{3}}{2}\dot{\varphi}\hat{k}'$$

OBS:
- La base $\{\hat{e}_r, \hat{e}_\varphi, \hat{k}'\}$ es semisolidaria al disco, de forma tal que su vel. angular es $\dot{\varphi}\hat{k}$
- Esta base es de vectores ppales del disco

$$\Rightarrow \mathbb{I}_G^{\{\hat{e}_r, \hat{e}_\varphi, \hat{k}'\}} = \begin{bmatrix} \frac{mr^2}{2} & & \\ & \frac{mr^2}{4} & \\ & & \frac{mr^2}{4} \end{bmatrix}$$

Steiner:

$$\mathbb{I}_O = \mathbb{I}_G + \mathbb{I}_O^{(m,G)}, \quad \text{con} \quad \left(\mathbb{I}_O^{(m,G)}\right)_{\alpha\beta} = m(\vec{r}_G - \vec{r}_O)^2\delta_{\alpha\beta} - m(\vec{r}_G - \vec{r}_O)_\alpha(\vec{r}_G - \vec{r}_O)_\beta$$

$$\vec{r}_G - \vec{r}_O = d\,\hat{e}_r = r\sqrt{3}\,\hat{e}_r$$

$$\Rightarrow \mathbb{I}_O^{(m,G)} = \begin{bmatrix} md^2 - md^2 & & \\ & md^2 & \\ & & md^2 \end{bmatrix} = \begin{bmatrix} 0 & & \\ & 3mr^2 & \\ & & 3mr^2 \end{bmatrix}$$

$$\Rightarrow \mathbb{I}_O = \begin{bmatrix} \frac{mr^2}{2} & & \\ & \frac{13mr^2}{4} & \\ & & \frac{13mr^2}{4} \end{bmatrix}. \quad \text{Calculo } \vec{L}_O = \mathbb{I}_O\vec{\omega}$$

$$\Rightarrow \vec{L}_O = \frac{mr^2}{2}\left(\dot{\psi} - \frac{\dot{\varphi}}{2}\right)\hat{e}_r + \frac{13\sqrt{3}}{8}mr^2\dot{\varphi}\hat{k}'$$

$$\Rightarrow \dot{\vec{L}}_O = \frac{mr^2}{2}\left[\left(\ddot{\psi} - \frac{\ddot{\varphi}}{2}\right)\hat{e}_r + \left(\dot{\psi} - \frac{\dot{\varphi}}{2}\right)\dot{\hat{e}}_r\right] + \frac{13\sqrt{3}}{8}mr^2\left(\ddot{\varphi}\hat{k}' + \dot{\varphi}\dot{\hat{k}}'\right)$$

$\rightarrow$

$$\dot{\hat{e}}_r = \dot{\varphi}\hat{k}\times\hat{e}_r = \cos\frac{\pi}{6}\,\dot{\varphi}\,\hat{e}_\varphi = \frac{\sqrt{3}}{2}\dot{\varphi}\,\hat{e}_\varphi$$

$$\dot{\hat{k}}' = \dot{\varphi}\hat{k}\times\hat{k}' = \operatorname{sen}\frac{\pi}{6}\,\dot{\varphi}\,\hat{e}_\varphi = \frac{\dot{\varphi}}{2}\hat{e}_\varphi$$

$$\Rightarrow \dot{\vec{L}}_O = \frac{mr^2}{2}\left(\ddot{\psi}-\frac{\ddot{\varphi}}{2}\right)\hat{e}_r + \left[\frac{mr^2}{2}\left(\dot{\psi}-\frac{\dot{\varphi}}{2}\right)\frac{\sqrt{3}}{2}\dot{\varphi} + \frac{13\sqrt{3}}{8}mr^2\frac{\dot{\varphi}^2}{2}\right]\hat{e}_\varphi + \frac{13\sqrt{3}}{8}mr^2\ddot{\varphi}\hat{k}'$$

$$\boxed{\dot{\vec{L}}_O = \frac{mr^2}{2}\left(\ddot{\psi}-\frac{\ddot{\varphi}}{2}\right)\hat{e}_r + \frac{\sqrt{3}}{4}mr^2\left(\dot{\psi}+\frac{15}{4}\dot{\varphi}\right)\dot{\varphi}\,\hat{e}_\varphi + \frac{13\sqrt{3}}{8}mr^2\ddot{\varphi}\,\hat{k}'}$$

MOMENTOS: Observar que definir el giro inicial $\dot{\psi}_0$ como antihorario determina el sentido de la Froz (pues es roz. dinámico).

$$\vec{M}_O^{(ext)} = \vec{M}_O^{peso} + \vec{M}_O^{N} + \vec{M}_O^{T} = d\hat{e}_r\times(-mg\hat{k}) - lN\hat{e}_\varphi + (d\hat{e}_r - r\hat{k}')\times(-T\hat{e}_\varphi)$$

$$= \sqrt{3}\,r\,mg\underbrace{\cos\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}\hat{e}_\varphi - 2Nr\hat{e}_\varphi - T\sqrt{3}r\hat{k}' - Tr\hat{e}_r =$$

$$= -Tr\hat{e}_r + \left(\frac{3}{2}mgr - 2Nr\right)\hat{e}_\varphi - \sqrt{3}Tr\hat{k}'$$

2ª cardinal: $\dot{\vec{L}}_O = \vec{M}_O^{(ext)}$

Además, $T = \mu_d N$

$$\Rightarrow \begin{cases} \frac{mr^2}{2}\left(\ddot{\psi}-\frac{\ddot{\varphi}}{2}\right) = -Tr & (1) \\ \frac{\sqrt{3}}{4}mr^2\dot{\varphi}\left(\dot{\psi}+\frac{11}{4}\dot{\varphi}\right) = \frac{3}{2}mgr - 2Nr & (2) \\ \frac{13\sqrt{3}}{8}mr^2\ddot{\varphi} = -\sqrt{3}Tr & (3) \end{cases}$$

$N = \frac{T}{\mu_d}$

De (1) y (3) $\Rightarrow \frac{mr^2}{2}\left(\ddot{\psi}-\frac{\ddot{\varphi}}{2}\right) = \frac{13mr^2}{8}\ddot{\varphi} \Rightarrow \boxed{\ddot{\psi} = \frac{15}{4}\ddot{\varphi}}$

De (3) y (2)

$$\Rightarrow \frac{\sqrt{3}}{4}mr^2\left(\dot{\psi}\dot{\varphi} + \frac{11}{4}\dot{\varphi}^2\right) = \frac{3}{2}mgr + 2\cdot\frac{13mr^2}{8\mu_d}\ddot{\varphi}$$

$$\Rightarrow \boxed{\frac{13}{4\mu_d}\ddot{\varphi} - \frac{11\sqrt{3}}{16}\dot{\varphi}^2 - \frac{\sqrt{3}}{4}\dot{\psi}\dot{\varphi} + \frac{3g}{2r} = 0}$$

Parte b) La vel. angular del disco en $t \to +\infty$ es la alcanzada en rodadura s/deslizar. $\Rightarrow \vec{V_P} = 0$ y aplico distr. de velocidad.

$$\vec{V_G} = \cancel{\vec{V_P}}^{\,0} + \vec{\omega} \times (\vec{r_G} - \vec{r_P}) \quad // \quad \vec{r_G} = d\hat{e_r} \Rightarrow \vec{V_G} = d\,\mathring{\hat{e_r}} = \sqrt{3}r\,\frac{\sqrt{3}}{2}\dot{\varphi}\,\hat{e_\varphi}$$

$$\Rightarrow \frac{3}{2}r\dot{\varphi}\,\hat{e_\varphi} = \left[\left(\dot{\psi} - \frac{\dot{\varphi}}{2}\right)\hat{e_r} - \frac{\sqrt{3}}{2}\dot{\varphi}\,\hat{k}'\right] \times r\hat{k}' \stackrel{!}{=} -\left(\dot{\psi} - \frac{\dot{\varphi}}{2}\right) r\,\hat{e_\varphi}$$

$$\Rightarrow \boxed{\dot{\psi} = -\dot{\varphi}}$$

De la 1ª Ec. de mov: $$\dot{\psi}(t) - \overbrace{\dot{\psi}_0}^{\text{Dato}} = \frac{15}{4}\left(\overbrace{\dot{\varphi}(t)}^{-\dot{\psi}(t)} - \overbrace{\dot{\varphi}_0}^{0 \text{ (cond. inicial)}}\right)$$

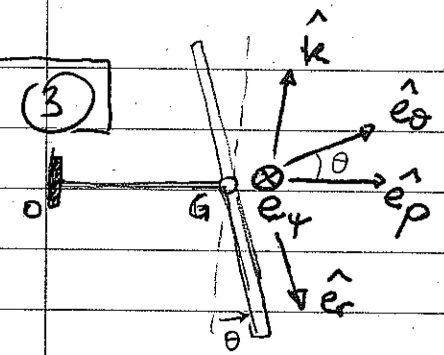
$$\Rightarrow \dot{\psi}(t_{rsd}) + \frac{15}{4}\dot{\psi}(t_{rsd}) = \dot{\psi}_0 \qquad \Rightarrow \dot{\psi}_{rsd} = \frac{4}{19}\dot{\psi}_0$$

como $\vec{\omega} = \dot{\varphi}\hat{k} + \dot{\psi}\hat{e_r} \Rightarrow \vec{\omega}_{rsd} = -\dot{\psi}\hat{k} + \dot{\psi}\hat{e_r}$

$$\Rightarrow \boxed{\vec{\omega}_{rsd} = \frac{4}{19}\dot{\psi}_0\left(\hat{e_r} - \hat{k}\right)}$$

Ej (3)

a)



$$\vec{L}_G = \mathbb{I}_G \vec{\omega}$$

$$\mathbb{I}_G^{\{\hat{e}_r, \hat{e}_\varphi, \hat{e}_\theta\}} = \begin{bmatrix} 0 & & \\ & \frac{m(2l)^2}{12} & \\ & & \frac{m(2l)^2}{12} \end{bmatrix}$$

$$\vec{\omega} = \dot{\varphi}\hat{k} - \dot{\theta}\hat{e}_\varphi = \dot{\varphi}(\operatorname{sen}\theta\,\hat{e}_\theta - \cos\theta\,\hat{e}_r) - \dot{\theta}\hat{e}_\varphi$$

$$\Rightarrow \vec{L}_G = \frac{ml^2}{3}\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} -\dot{\varphi}\cos\theta \\ -\dot{\theta} \\ \dot{\varphi}\operatorname{sen}\theta \end{bmatrix} = \frac{ml^2}{3}\left(-\dot{\theta}\,\hat{e}_\varphi + \dot{\varphi}\operatorname{sen}\theta\,\hat{e}_\theta\right)$$

Aplico $\vec{L}_O = \vec{L}_G + m\vec{v}_G \times (\vec{r}_O - \vec{r}_G)$

$\vec{r}_G = l\hat{e}_\rho \Rightarrow \vec{v}_G = l\dot{\hat{e}}_\rho = l\dot{\varphi}\hat{e}_\varphi$ $\Bigg\} \Rightarrow$

$$\Rightarrow \vec{L}_O = \vec{L}_G + ml\dot{\varphi}\,\hat{e}_\varphi \times (-l\hat{e}_\rho) = \vec{L}_G + ml^2\dot{\varphi}\hat{k}$$

$$\Rightarrow \boxed{\vec{L}_O = \frac{ml^2}{3}\left(3\dot{\varphi}\,\hat{k} - \dot{\theta}\,\hat{e}_\varphi + \dot{\varphi}\operatorname{sen}\theta\,\hat{e}_\theta\right)}$$

b) 2^a card. en O : $\dot{\vec{L}}_O = m\vec{v}_G \times \underbrace{\dot{\vec{r}}_O}_{=0} + \vec{M}_O^{(EXT)} = \vec{M}_O^{(react)} + \underbrace{\vec{r}_G}_{=l\hat{e}_\rho} \times (-mg\hat{k})$

$$\dot{\vec{L}}_O = \vec{M}_O^{(react)} + mgl\,\hat{e}_\varphi\ .$$

Observar que: $\dot{\vec{L}}_O \cdot \hat{k} = \vec{M}_O^{(react)} \cdot \hat{k} = 0$ pues O es art. cilíndrica

$\Rightarrow \vec{L}_O \cdot \hat{k} = cte \equiv L$. / $\vec{L}_O \cdot \hat{k} = ml^2\dot{\varphi} + \frac{ml^2}{3}\dot{\varphi}\operatorname{sen}^2\theta = L$

$$\Rightarrow \boxed{\dot{\varphi}(\operatorname{sen}^2\theta + 3) = \frac{3L}{ml^2}} \longrightarrow \text{cantidad conservada!}$$

Derivando obtengo

Ec. de mov.: $\boxed{(3 + \operatorname{sen}^2\theta)\,\ddot{\varphi} + 2\operatorname{sen}\theta\cos\theta\,.\,\dot{\theta}\dot{\varphi} = 0}$

Se conserva la E (pues Peso es conservativo y las react. en O son de pot. nula)

$$\Rightarrow E = T + U \quad , \quad \text{con } T = \frac{1}{2} m \vec{v}_G^{\,2} + \vec{\omega} \circ \mathbb{I}_G \vec{\omega} \quad \text{y } U = 0 \text{ (cte)}$$

$$T = \frac{1}{2} m l^2 \dot{\varphi}^2 + \frac{m l^2}{6} \left(\dot{\varphi}^2 \operatorname{sen}^2\theta + \dot{\theta}^2 \right)$$

$$\Rightarrow E = \frac{m l^2}{6} \left[\dot{\varphi}^2 (\operatorname{sen}^2\theta + 3) + \dot{\theta}^2 \right]$$

$$\Rightarrow \boxed{\dot{\varphi}^2 (\operatorname{sen}^2\theta + 3) + \dot{\theta}^2 = \frac{6E}{m l^2}}$$ cant. conservada

Derivando obtengo la otra Ec. de mov:

$$2\dot{\theta}\ddot{\theta} + \underbrace{2\dot{\varphi}\ddot{\varphi}(\operatorname{sen}^2\theta + 3)}_{=\,-2\operatorname{sen}\theta\cos\theta\,\dot{\theta}\dot{\varphi}} + \dot{\varphi}^2 (2 \operatorname{sen}\theta \cos\theta\, \dot{\theta}) = 0$$

(por cons. de $\vec{L}_O \cdot \hat{k}$)

$$\Rightarrow \boxed{\ddot{\theta} = \operatorname{sen}\theta \cos\theta\, \dot{\varphi}^2}$$