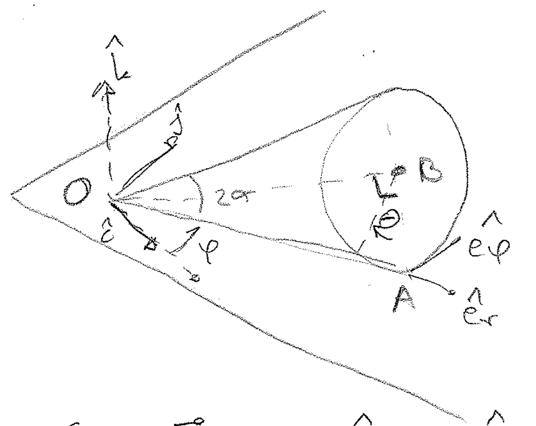
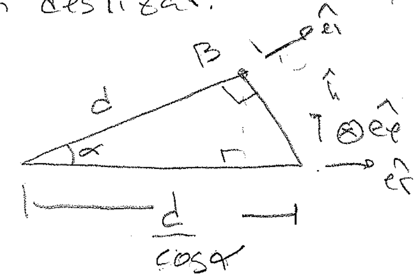
Problemas de Clase. Práctico 5

$\boxed{1}$



Rueda sin deslizar.



- Sea $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$

$\hat{e}_r = \cos\varphi\,\hat{i} + \sin\varphi\,\hat{j}$

$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$

velocidad de 3 puntos del rígido

$\vec{v}_O = 0$ por rodadura sin deslizar

$\vec{v}_A = 0$ " " "

$\vec{v}_B = d\cos\alpha\,\dot{\varphi}\,\hat{e}_\varphi$

aplico distribución de velocidades entre O y A

$$\vec{r}_A - \vec{r}_O = \underbrace{\frac{d}{\cos\alpha}}_{L}\,\hat{e}_r$$

$$\vec{v}_A = \vec{v}_O + \vec{\omega}\wedge(\vec{r}_A - \vec{r}_O)$$

$$= 0 + (\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})\wedge L\,\hat{e}_r$$

$$= \omega_1 L\sin\varphi\,\hat{k} + \omega_2 L\cos\varphi(-\hat{k}) + \omega_3 L\,\hat{e}_\varphi = 0$$

$$\Rightarrow \omega_3 = 0.$$

$$L(\omega_1\sin\varphi - \omega_2\cos\varphi) = 0$$

Entre O y B

$$\vec{v}_B = \vec{v}_O + \vec{\omega}\wedge(\vec{r}_B - \vec{r}_O) \qquad \vec{r}_B = d\cos\alpha\,\hat{e}_r + d\sin\alpha\,\hat{k}$$

$$d\dot{\varphi}\cos\alpha\,\hat{e}_\varphi = 0 + (\omega_1\hat{i} + \omega_2\hat{j})\wedge(d\cos\alpha\,\hat{e}_r + d\sin\alpha\,\hat{k})$$

$$= \omega_1 d\cos\alpha\sin\varphi\,\hat{k} - \omega_1 d\sin\alpha\,\hat{j} + \omega_2 d\cos\alpha\cos\varphi(-\hat{k}) + \omega_2 d\sin\alpha\,\hat{i}$$

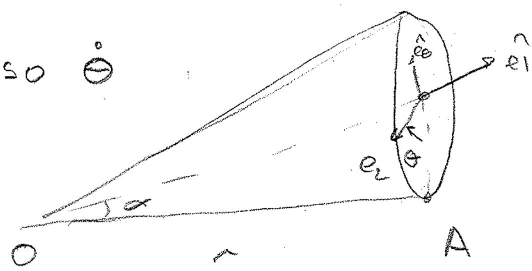
$$d\dot{\varphi}\cos\alpha(-\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}) = \omega_2 d\sin\alpha\,\hat{i} - \omega_1 d\sin\alpha\,\hat{j} + d\cos\alpha\underbrace{(\omega_1\sin\varphi - \omega_2\cos\varphi)}_{0}\hat{k}$$

$$-\dot{\varphi}\sin\varphi = \omega_2\frac{\sin\alpha}{\cos\alpha} \quad y \quad \dot{\varphi}\cos\varphi = -\omega_1\frac{\sin\alpha}{\cos\alpha}$$

$\Rightarrow \vec{\omega} = -\dot{\varphi}\dfrac{\cos\alpha}{\sin\alpha}\cos\varphi\,\hat{\imath} - \dot{\varphi}\dfrac{\cos\alpha}{\sin\alpha}\sin\varphi\,\hat{\jmath} = -\dot{\varphi}\dfrac{\cos\alpha}{\sin\alpha}\,\hat{e}_r$

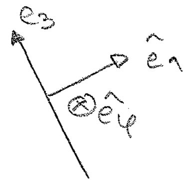
- Otra opción:

uso $\dot{\theta}$



base $\{\hat{e}_2, \hat{e}_\theta, \hat{e}_1\} = S_2$ solidaria al cono.

defino otro sistema



S_1: $\{\hat{e}_1, \hat{e}_\varphi, \hat{e}_3\}$ acompaña el eje del cono.

$$\vec{\omega}_1 = \dot{\varphi}\hat{k}$$

$$\vec{\omega}_{S_2 S_1} = -\dot{\theta}\,\hat{e}_1$$

por adición de velocidades angulares. $\vec{\omega}_{S_2} = \dot{\varphi}\hat{k} - \dot{\theta}\,\hat{e}_1$

Por rodadura sin deslizar los puntos del rígido entre O y A tiene velocidad nula. Hago distribución de velocidades entre O y A

$$\vec{v}_A = \vec{v}_0 = 0 \Rightarrow \vec{v}_A = \vec{\omega}\wedge\underbrace{(\vec{r}_A - \vec{r}_O)}_{L\hat{e}_r} = (\dot{\varphi}\hat{k} - \dot{\theta}\hat{e}_1)\wedge\hat{e}_r L$$

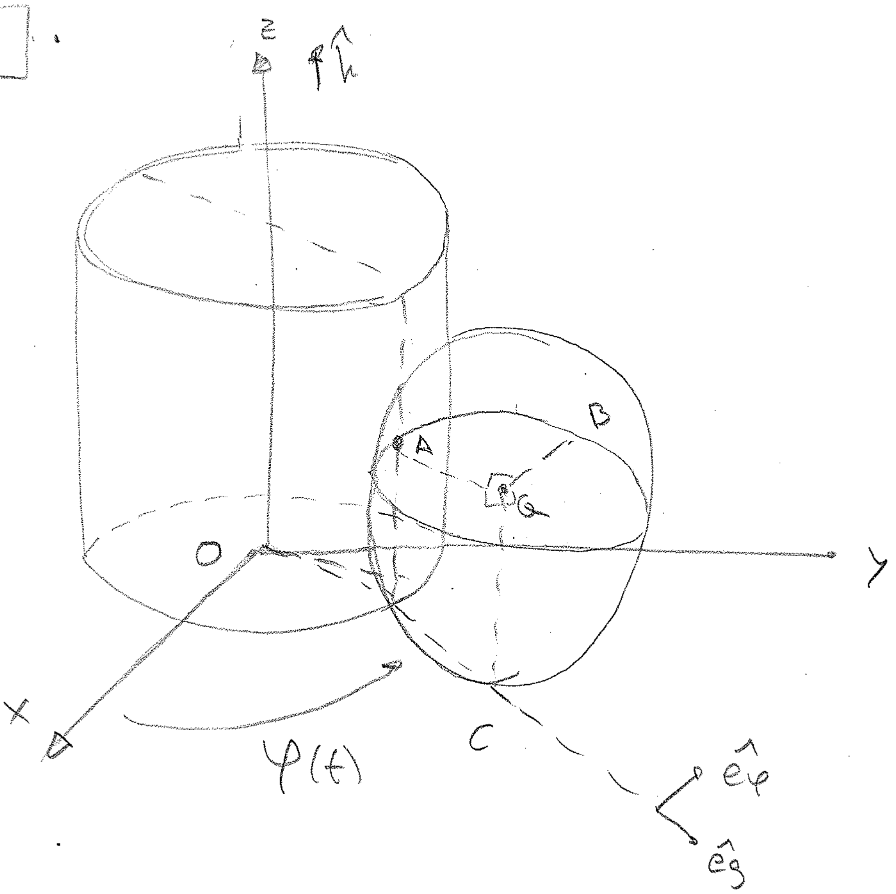
$$0 = L\dot{\varphi}\,\hat{e}_\varphi - L\dot{\theta}\,(\hat{e}_1\wedge\hat{e}_r) \qquad \hat{e}_1 = \cos\alpha\,\hat{e}_r + \sin\alpha\,\hat{k}$$

$$0 = L\dot{\varphi}\,\hat{e}_\varphi - L\dot{\theta}\sin\alpha\,\hat{e}_\varphi \rightarrow \dot{\theta} = \frac{\dot{\varphi}}{\sin\alpha}$$

$$\Rightarrow \vec{\omega}_{S_2} = \dot{\varphi}\hat{k} - \frac{\dot{\varphi}}{\sin\alpha}\left(\cos\alpha\,\hat{e}_r + \sin\alpha\,\hat{k}\right)$$

$$\boxed{\vec{\omega}_{S_2} = -\dot{\varphi}\,\frac{\cos\alpha}{\sin\alpha}\,\hat{e}_r}$$

2



$C = O + 2R\,\hat{e}_\rho$

$Q = C + R\hat{k}$

$B = Q + R\hat{e}_\varphi$

$A = Q - R\hat{e}_\rho$

$C = Q - R\hat{k}$

La esfera rueda sin deslizar a C y A

$\Rightarrow \vec{v}_C = 0 \quad y \quad \vec{v}_A = 0 \qquad \vec{v}_Q = 2R\dot{\varphi}\,\hat{e}_\varphi$ coincide con el punto geométrico

$\vec{\omega} = \omega_1 \hat{k} + \dot{\theta}\,\hat{e}_\rho + \dot{\psi}\,\hat{e}_\varphi$

$$\vec{v}_A = \underset{2R\dot{\varphi}\hat{e}_\varphi}{\underset{\shortparallel}{\vec{v}_Q}} + \vec{\omega}\wedge\underbrace{(\vec{r}_A - \vec{r}_Q)}_{Q - R\hat{e}_\rho - Q} = 2R\dot{\varphi}\,\hat{e}_\varphi + \vec{\omega}\wedge(-R\hat{e}_\rho)$$

$$\vec{\omega}\wedge(-R)\hat{e}_\rho = (\omega_1\hat{k} + \dot{\theta}\,\hat{e}_\rho + \dot{\psi}\,\hat{e}_\varphi)\wedge(-R)\,\hat{e}_\rho$$

$$= -R\omega_1\,\hat{e}_\varphi + R\dot{\psi}\,\hat{k} \rightarrow \dot{\psi} = 0$$

$$\vec{v}_A = 2R\dot{\varphi}\,\hat{e}_\varphi - R\omega_1\,\hat{e}_\varphi = 0 \rightarrow \boxed{\omega_1 = 2\dot{\varphi}}$$

$$\underset{0}{\underset{\shortparallel}{\vec{v}_C}} = \underset{0}{\underset{\shortparallel}{\vec{v}_A}} + \vec{\omega}\wedge\underbrace{(\vec{r}_C - \vec{r}_A)}_{-R\hat{k}+R\hat{e}_\rho} \rightarrow (2\dot{\varphi}\hat{k} + \dot{\theta}\,\hat{e}_\rho)\wedge(-R\hat{k} + R\hat{e}_\rho) =$$

$$2\dot{\varphi}R\,\hat{e}_\varphi + R\dot{\theta}\,\hat{e}_\varphi = 0 \Rightarrow \dot{\theta} = -2\dot{\varphi}$$

$$\boxed{\vec{\omega} = 2\dot{\varphi}(\hat{k} - \hat{e}_\rho)}$$

IV

$\vec{v}_B = \vec{v}_Q + \vec{\omega} \wedge (\vec{r}_B - \vec{r}_Q) = 2R\dot{\varphi}\hat{e}_\varphi + 2\dot{\varphi}(\hat{k} - \hat{e}_\xi) \wedge R\hat{e}_\varphi = 2R\dot{\varphi}\hat{e}_\varphi + 2R\dot{\varphi}(-\hat{e}_\xi - \hat{k})$

$\vec{v}_B = 2R\dot{\varphi}(\hat{e}_\varphi - \hat{e}_\xi - \hat{k})$

$\vec{a}_B = \vec{a}_Q + \dot{\vec{\omega}} \wedge (\vec{r}_B - \vec{r}_Q) + \vec{\omega} \wedge (\vec{\omega} \wedge (\vec{r}_B - \vec{r}_Q))$

$\vec{a}_Q = 2R\ddot{\varphi}\hat{e}_\varphi - 2R\dot{\varphi}^2\hat{e}_\xi$

$\dot{\vec{\omega}} = 2\ddot{\varphi}(\hat{k} - \hat{e}_\xi) - 2\dot{\varphi}\dot{\hat{e}}_\xi = 2\ddot{\varphi}(\hat{k} - \hat{e}_\xi) - 2\dot{\varphi}^2\hat{e}_\varphi$

$\dot{\vec{\omega}} \wedge \underbrace{(\vec{r}_B - \vec{r}_Q)}_{R\hat{e}_\varphi} = (2\ddot{\varphi}(\hat{k} - \hat{e}_\xi) - 2\dot{\varphi}^2\hat{e}_\varphi) \wedge R\hat{e}_\varphi = 2R\ddot{\varphi}(-\hat{e}_\xi - \hat{k})$

$\vec{\omega} \wedge (\vec{r}_B - \vec{r}_Q) = 2\dot{\varphi}(\hat{k} - \hat{e}_\xi) \wedge R\hat{e}_\varphi = 2R\dot{\varphi}(-\hat{e}_\xi - \hat{k})$

$\vec{\omega} \wedge (\vec{\omega} \wedge (\vec{r}_B - \vec{r}_Q)) = 2\dot{\varphi}(\hat{k} - \hat{e}_\xi) \wedge 2R\dot{\varphi}(-1)(\hat{e}_\xi + \hat{k}) = -4R\dot{\varphi}^2(\hat{e}_\varphi + \hat{e}_\varphi)$

$\vec{a}_B = 2R\ddot{\varphi}\hat{e}_\varphi - 8R\dot{\varphi}^2\hat{e}_\varphi - 2R\ddot{\varphi}\hat{e}_\xi - 2R\ddot{\varphi}\hat{k} - 2R\dot{\varphi}^2\hat{e}_\xi$

$\vec{a}_B = 2R(\ddot{\varphi} - 4\dot{\varphi}^2)\hat{e}_\varphi - 2R(\ddot{\varphi} + \dot{\varphi}^2)\hat{e}_\xi - 2R\ddot{\varphi}\hat{k}$

$\vec{a}_B \cdot \vec{v}_B = 0 + 2R\dot{\varphi}\,2R(\ddot{\varphi} - 4\dot{\varphi}^2) + 2R\dot{\varphi}\,2R(\ddot{\varphi} + \dot{\varphi}^2) + 2R\,2R\,\dot{\varphi}\ddot{\varphi} = 0$

$0 = \dot{\varphi}\,4R^2(3\ddot{\varphi} - 3\dot{\varphi}^2) \rightarrow \dot{\varphi} = 0$ solución trivial

$\rightarrow \ddot{\varphi} - \dot{\varphi}^2 = 0 \rightarrow \boxed{\ddot{\varphi} = \dot{\varphi}^2}$

c) $\vec{a}_B = 2R(-3\ddot{\varphi}\hat{e}_\varphi - 2\ddot{\varphi}\hat{e}_\xi - \ddot{\varphi}\hat{k}) = -2R\ddot{\varphi}(3\hat{e}_\varphi + 2\hat{e}_\xi + \hat{k})$

$= -2R\dot{\varphi}^2(3\hat{e}_\varphi + 2\hat{e}_\xi + \hat{k})$

Además

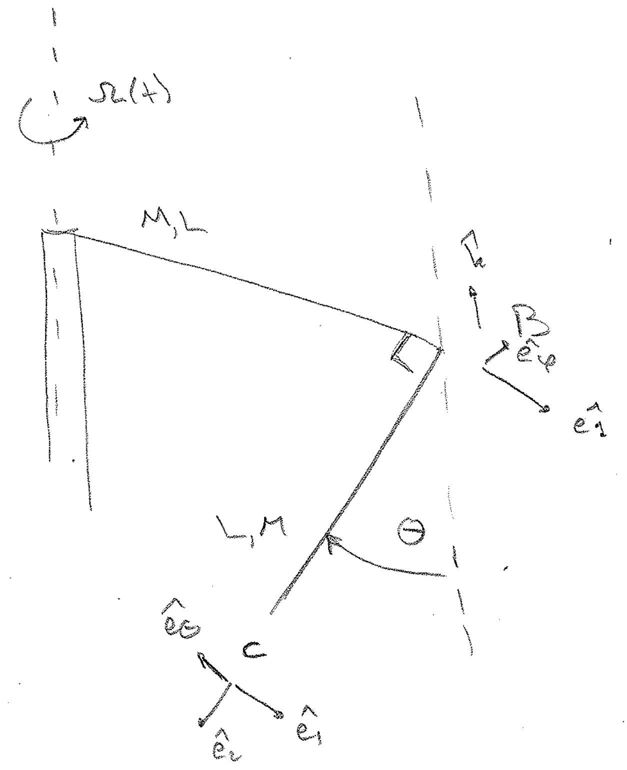
$\frac{d\dot{\varphi}}{dt} = \dot{\varphi}^2 \rightarrow \frac{d\dot{\varphi}}{\dot{\varphi}^2} = dt \rightarrow \int_{\dot{\varphi}(0)}^{\dot{\varphi}} \frac{du}{u^2} = t \rightarrow \frac{1}{\dot{\varphi}} - \frac{1}{\dot{\varphi}(0)} = -t$

$\frac{1}{\dot{\varphi}} = \frac{1}{\dot{\varphi}(0)} - t \rightarrow \boxed{\dot{\varphi}(t) = \frac{\dot{\varphi}(0)}{1 - t\dot{\varphi}(0)}}$

Si $\vec{v}_Q = v_0\hat{e}_\varphi$

$\rightarrow \boxed{\dot{\varphi}(0) = \frac{v_0}{2R}}$

3



a) $\mathbb{I}_A$ en la base $\{\hat{e}_1, \hat{e}_2, \hat{e}_\Theta\}$

$$\mathbb{I}_A^{\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}} = \begin{pmatrix} \int dV\rho(y^2+z^2) & -\int dV\rho\, xy & -\int dV\rho\, xz \\ -\int dV\rho\, yx & \int dV\rho(x^2+z^2) & -\int dV\rho\, yz \\ -\int dV\rho\, zx & -\int dV\rho\, zy & \int dV\rho(x^2+y^2) \end{pmatrix}$$

para cada barra $\rho = \frac{M}{L}$ $\quad \vec{r} = (x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3)$

tengo 2 regiones para el rígido 1 $x \in [0, L]$ $\left\{ \begin{array}{l} \rho(x,0,0) = \frac{M}{L} \\ \rho(x,y,z) = 0 \\ \quad \forall y, z \neq 0 \end{array} \right.$

2 $x = L$, $y \in [0, L]$ $\left\{ \begin{array}{ll} \rho(L,y,0) = \frac{M}{L} & \\ \rho(L,y,z) = 0 & \forall z \neq 0 \\ \rho(x,y,z) = 0 & \forall x \neq L \end{array} \right.$

$dV = dx\,dy\,dz$

$$\Rightarrow \int dV\rho(y^2+z^2) = \int_0^L dx \int_0^L dy \int_0^L dz\, \rho(y^2+z^2) = \int_0^L dx\, \rho(0) + \int_0^L dy\, \rho y^2 = \frac{1}{3}\rho L^3$$

$$\int dV\rho(xy) = \int_0^L dx \int_0^L dy \int_0^L dz\, \rho\, xy = \int_0^L dx\, \rho x(0) + \int_0^L dy\, \rho L y = \rho \frac{L^2}{2} L$$

$$\ldots \int dV \rho\, xz = \int_0^L dx \int_0^L dy \int_0^L dz\, \rho\, x z = \int_0^L dx\, \rho\, x \underset{0}{\underset{\shortparallel}{(0)}} + \int_0^L dz\, \rho\, \underset{0}{\underset{\shortparallel}{L z}} = 0$$

$$\int dV \rho\, yz = \int_0^L dx \int_0^L dy \int_0^L dz\, \rho\, y z = \int_0^L dx\, \rho\, (0)(0) + \int_0^L dy\, \rho\, 0 \cdot 0 = 0$$

$$\int dV \rho \cdot (x^2+z^2) = \int_0^L dx \int_0^L dy \int_0^L dz\, \rho (x^2+z^2) = \int_0^L \rho\, x^2 dx + \int_0^L dy\, \rho\, L^2 = \rho \frac{L^3}{3} + \rho L^2 L$$

$$\int \rho\, dV (y^2+x^2) = \int_0^L dx \int_0^L dy \int_0^L dz\, \rho (y^2+x^2) = \int_0^L \rho\, dx\, x^2 + \int_0^L dy\, \rho (y^2+L^2) = \frac{\rho}{3} L^3 + \rho \left(\frac{L^3}{3} + L^2 L\right)$$

$$\Rightarrow \mathbb{I}_A = \rho L^3 \begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & -0 \\ -1/2 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{5}{3} \end{pmatrix}$$

Otra forma: Pienso el rígido como la suma de 2 barras

barra 1: OB calculo $\mathbb{I}_A$ para c/barra en A y los sumo

barra 2: BC calculo $\mathbb{I}_G$ para la barra 2 y hago steiner.

barra 1 $\quad \mathbb{I}_A^{(1)\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}} = \rho \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{L^3}{3} & 0 \\ 0 & 0 & L^3/3 \end{pmatrix}$

barra 2 $\quad \mathbb{I}_G^{\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}} = \rho \begin{pmatrix} L^3/12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L^3/12 \end{pmatrix}$

Hago Steiner desde G a A $\quad \bar{r}_G = L\hat{e}_1 + \frac{L}{2}\hat{e}_2 \quad \bar{r}_A = 0$

$$\mathbb{I}_A = \mathbb{I}_G + \mathbb{I}_A^{(M,G)} \qquad \mathbb{I}_A^{(M,G)} = M\left(L^2 + \frac{L^2}{4}\right)\delta_{\alpha\beta} - M(\bar{r}_G - \bar{r}_A)_\alpha (\bar{r}_G - \bar{r}_A)_\beta$$

$$\mathbb{I}_A^{(M,G)} = M L^2 \begin{pmatrix} \frac{5}{4} - 1 & -1\, 1/2 & 0 \\ -1\, 1/2 & 5/4 - 1/4 & 0 \\ 0 & 0 & 5/4 \end{pmatrix}$$

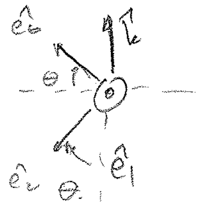
$$\therefore\ \mathbb{I}_A = ML^2 \begin{pmatrix} \frac{1}{12}+\frac{1}{4} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{12}+\frac{5}{4} \end{pmatrix} = ML^2 \begin{pmatrix} \frac{4}{12}=\frac{1}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{16}{12}=\frac{4}{3} \end{pmatrix}$$

$$\mathbb{I}_A = ML^2 \begin{pmatrix} \frac{1}{3} & -1/2 & 0 \\ -1/2 & 4/3 & 0 \\ 0 & 0 & 5/3 \end{pmatrix}$$

b) $\vec{L}_P = M(\vec{r}_G - \vec{r}_P) \wedge \vec{v}_P + \mathbb{I}_P \vec{\omega}$

tomo $\vec{r}_P = \vec{r}_A = 0 \implies \vec{L}_A = \mathbb{I}_A \vec{\omega}$

$\vec{\omega} = \Omega(t)\hat{k} - \dot{\theta}\,\hat{e}_1 \implies \vec{\omega} = -\Omega\cos\theta\,\hat{e}_2 + \Omega\sin\theta\,\hat{e}_\theta - \dot{\theta}\,\hat{e}_1$



$\hat{k} = -\cos\theta\,\hat{e}_2 + \sin\theta\,\hat{e}_\theta$

$$\mathbb{I}_A^{\{\hat{e}_1,\hat{e}_2,\hat{e}_\theta\}}\vec{\omega} = ML^2 \begin{pmatrix} 1/3 & -1/2 & 0 \\ -1/2 & 4/3 & 0 \\ 0 & 0 & 5/3 \end{pmatrix} \begin{pmatrix} -\dot{\theta} \\ -\Omega\cos\theta \\ \Omega\sin\theta \end{pmatrix}$$

$$\vec{L}_A = ML^2\left[\left(-\frac{1}{3}\dot{\theta} + \frac{1}{2}\Omega\cos\theta\right)\hat{e}_1 + \left(\frac{1}{2}\dot{\theta} - \frac{4}{3}\Omega\cos\theta\right)\hat{e}_2 + \frac{5}{3}\Omega\sin\theta\right]$$