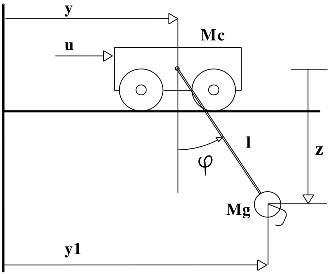
SyC - Hoja 5 - Ej. 3

3*) El dispositivo de la figura es una grúa móvil, consistente de un carro que se mueve sobre rieles horizontales, del que cuelga un gancho a través de una barra articulada de longitud **l**. Se despreciará la masa de esta barra.



a) Encuentre una representación en variables de estado para este sistema, lineal, para pequeños ángulos de apartamento de la vertical.
Considere como estado

$$x = [y, \quad \dot{y}, \quad \varphi, \quad \dot{\varphi}]^t$$

y la entrada es la fuerza **u**.

b) Dibuje un diagrama de bloques para esta representación.

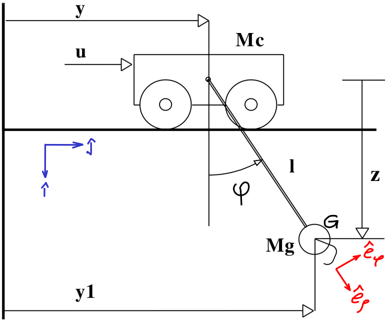
c) Considere el subsistema del gancho, es decir:

$$\begin{bmatrix} \dot{\varphi} \\ \ddot{\varphi} \end{bmatrix} = A_1 \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix} + B_1 u$$

Calcule la matriz de transición de este subsistema $\mathbf{e}^{\mathbf{A1t}}$.

Modelado

Cálculo de la aceleración del punto G (gancho)



$$\hat{e}_\rho = \cos\varphi\,\hat{\imath} + \operatorname{sen}\varphi\,\hat{\jmath}$$
$$\hat{e}_\varphi = -\operatorname{sen}\varphi\,\hat{\imath} + \cos\varphi\,\hat{\jmath}$$

$$\hat{\imath} = \cos\varphi\,\hat{e}_\rho - \operatorname{sen}\varphi\,\hat{e}_\varphi$$
$$\hat{\jmath} = \operatorname{sen}\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi$$

$$\begin{aligned}
\vec{r}_G &= y\hat{\jmath} + l\hat{e}_\rho \\
\vec{v}_G &= \dot{y}\hat{\jmath} + l\dot{\varphi}\hat{e}_\varphi \\
\vec{a}_G &= \ddot{y}\hat{\jmath} + l\ddot{\varphi}\hat{e}_\varphi - l\dot{\varphi}^2\hat{e}_\rho \\
&= \ddot{y}\left(\operatorname{sen}\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi\right) + l\ddot{\varphi}\hat{e}_\varphi - l\dot{\varphi}^2\hat{e}_\rho \\
&= \left(\ddot{y}\operatorname{sen}\varphi - l\dot{\varphi}^2\right)\hat{e}_\rho + \left(\ddot{y}\cos\varphi + l\ddot{\varphi}\right)\hat{e}_\varphi
\end{aligned}$$

Modelado

Aplicación de la 2da ley Newton al gancho y al carro

Segunda ley de Newton aplicada al gancho:

$$\text{Según } \hat{e}_\rho : \quad M_g g \cos\varphi - F = M_g \left(\ddot{y} \operatorname{sen}\varphi - l\dot{\varphi}^2\right) \qquad (1)$$

$$\text{Según } \hat{e}_\varphi : \quad -M_g g \operatorname{sen}\varphi = M_g \left(\ddot{y}\cos\varphi + l\ddot{\varphi}\right) \qquad (2)$$

Segunda ley de Newton aplicada al carro:

$$\text{Según } \hat{\jmath} : \quad u + F \operatorname{sen}\varphi = M_c \ddot{y} \qquad (3)$$

Despejando F de (1) para sustituirla en (3):

$$u + M_g g \operatorname{sen}\varphi\cos\varphi - M_g \ddot{y} \operatorname{sen}^2\varphi + M_g l\dot{\varphi}^2 \operatorname{sen}\varphi = M_c \ddot{y}$$

Despejando $\ddot{y}$ de esta última ecuación:

$$\ddot{y} = \frac{u + M_g g \operatorname{sen}\varphi\cos\varphi + M_g l\dot{\varphi}^2 \operatorname{sen}\varphi}{M_c + M_g \operatorname{sen}^2\varphi} \qquad (\text{I})$$

Representación en variables de estado (no lineal)

Sustituyendo $\ddot{y}$ en (2) y despejando $\ddot{\varphi}$:

$$\ddot{\varphi} = -\frac{g}{l}\operatorname{sen}\varphi - \frac{u\cos\varphi + M_g g \operatorname{sen}\varphi\cos^2\varphi + M_g l\dot{\varphi}^2\operatorname{sen}\varphi\cos\varphi}{l\left(M_c + M_g\operatorname{sen}^2\varphi\right)} \tag{II}$$

Eligiendo u como variable de entrada, $x = \begin{bmatrix} y & \dot{y} & \varphi & \dot{\varphi}\end{bmatrix}^{\mathsf{T}}$ como variable de estado y $w = x$ como variable de salida, se obtiene la siguiente representación en variables de estado:

$$\dot{x} = f(x,u) = \begin{bmatrix} f_{\dot{y}}(x,u) \\ f_{\ddot{y}}(x,u) \\ f_{\dot{\varphi}}(x,u) \\ f_{\ddot{\varphi}}(x,u) \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \frac{u + M_g g\operatorname{sen}\varphi\cos\varphi + M_g l\dot{\varphi}^2\operatorname{sen}\varphi}{M_c + M_g\operatorname{sen}^2\varphi} \\ \dot{\varphi} \\ -\frac{g}{l}\operatorname{sen}\varphi - \frac{u\cos\varphi + M_g g\operatorname{sen}\varphi\cos^2\varphi + M_g l\dot{\varphi}^2\operatorname{sen}\varphi\cos\varphi}{l(M_c + M_g\operatorname{sen}^2\varphi)} \end{bmatrix}$$

$$w = g(x,u) = \begin{bmatrix} g_y(x,u) \\ g_{\dot{y}}(x,u) \\ g_{\varphi}(x,u) \\ g_{\dot{\varphi}}(x,u) \end{bmatrix} = x$$

Puntos de equilibrio

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \frac{u + M_g g \operatorname{sen}\varphi \cos\varphi + M_g l \dot{\varphi}^2 \operatorname{sen}\varphi}{M_c + M_g \operatorname{sen}^2\varphi} \\ \dot{\varphi} \\ -\frac{g}{l}\operatorname{sen}\varphi - \frac{u\cos\varphi + M_g g \operatorname{sen}\varphi \cos^2\varphi + M_g l \dot{\varphi}^2 \operatorname{sen}\varphi\cos\varphi}{l(M_c + M_g \operatorname{sen}^2\varphi)} \end{bmatrix}$$

Son de la forma:

$$\left(\begin{bmatrix} y & \dot{y} & \varphi & \dot{\varphi} \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} y_0 & 0 & h\pi & 0 \end{bmatrix}^{\mathsf{T}}, \quad u = 0 \right)$$

donde $y_0 \in \mathbb{R}$ y $h \in \mathbb{Z}$.

Sean $\quad x_0 := \begin{bmatrix} y_0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}, \quad u_0 := 0, \quad w_0 := g\left(x_0, u_0\right) = x_0.$

Linealización

Definiciones

Linealizaremos en torno a (x_0, u_0).

Sean:

$$\tilde{x} = \begin{bmatrix} \tilde{y} \\ \dot{\tilde{y}} \\ \tilde{\varphi} \\ \dot{\tilde{\varphi}} \end{bmatrix} := x - x_0 = \begin{bmatrix} y \\ \dot{y} \\ \varphi \\ \dot{\varphi} \end{bmatrix} - \begin{bmatrix} y_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} y - y_0 \\ \dot{y} \\ \varphi \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \tilde{y} \\ \dot{y} \\ \varphi \\ \dot{\varphi} \end{bmatrix}$$

$$\tilde{u} := u - 0 = u$$

$$\tilde{w} := w - w_0$$

Linealización

Cálculo de las matrices jacobianas

$$\frac{\partial f}{\partial x}(x_0, u_0) = \left.\begin{bmatrix} \frac{\partial f_{\dot y}}{\partial y} & \frac{\partial f_{\dot y}}{\partial \dot y} & \frac{\partial f_{\dot y}}{\partial \varphi} & \frac{\partial f_{\dot y}}{\partial \dot\varphi} \\ \frac{\partial f_{\ddot y}}{\partial y} & \frac{\partial f_{\ddot y}}{\partial \dot y} & \frac{\partial f_{\ddot y}}{\partial \varphi} & \frac{\partial f_{\ddot y}}{\partial \dot\varphi} \\ \frac{\partial f_{\dot\varphi}}{\partial y} & \frac{\partial f_{\dot\varphi}}{\partial \dot y} & \frac{\partial f_{\dot\varphi}}{\partial \varphi} & \frac{\partial f_{\dot\varphi}}{\partial \dot\varphi} \\ \frac{\partial f_{\ddot\varphi}}{\partial y} & \frac{\partial f_{\ddot\varphi}}{\partial \dot y} & \frac{\partial f_{\ddot\varphi}}{\partial \varphi} & \frac{\partial f_{\ddot\varphi}}{\partial \dot\varphi} \end{bmatrix}\right|_{\substack{x = x_0 \\ u = u_0}} \qquad \frac{\partial f}{\partial u}(x_0, u_0) = \left.\begin{bmatrix} \frac{\partial f_{\dot y}}{\partial u} \\ \frac{\partial f_{\ddot y}}{\partial u} \\ \frac{\partial f_{\dot\varphi}}{\partial u} \\ \frac{\partial f_{\ddot\varphi}}{\partial u} \end{bmatrix}\right|_{\substack{x = x_0 \\ u = u_0}}$$

$$\frac{\partial g}{\partial x}(x_0, u_0) = \left.\begin{bmatrix} \frac{\partial g_{y}}{\partial y} & \frac{\partial g_{y}}{\partial \dot y} & \frac{\partial g_{y}}{\partial \varphi} & \frac{\partial g_{y}}{\partial \dot\varphi} \\ \frac{\partial g_{\dot y}}{\partial y} & \frac{\partial g_{\dot y}}{\partial \dot y} & \frac{\partial g_{\dot y}}{\partial \varphi} & \frac{\partial g_{\dot y}}{\partial \dot\varphi} \\ \frac{\partial g_{\varphi}}{\partial y} & \frac{\partial g_{\varphi}}{\partial \dot y} & \frac{\partial g_{\varphi}}{\partial \varphi} & \frac{\partial g_{\varphi}}{\partial \dot\varphi} \\ \frac{\partial g_{\dot\varphi}}{\partial y} & \frac{\partial g_{\dot\varphi}}{\partial \dot y} & \frac{\partial g_{\dot\varphi}}{\partial \varphi} & \frac{\partial g_{\dot\varphi}}{\partial \dot\varphi} \end{bmatrix}\right|_{\substack{x = x_0 \\ u = u_0}} \qquad \frac{\partial f}{\partial u}(x_0, u_0) = \left.\begin{bmatrix} \frac{\partial g_{y}}{\partial u} \\ \frac{\partial g_{\dot y}}{\partial u} \\ \frac{\partial g_{\varphi}}{\partial u} \\ \frac{\partial g_{\dot\varphi}}{\partial u} \end{bmatrix}\right|_{\substack{x = x_0 \\ u = u_0}}$$

Linealización

Cálculo de las matrices jacobianas

$$\dot{x} = f(x,u) = \begin{bmatrix} f_{\dot{y}}(x,u) \\ f_{\ddot{y}}(x,u) \\ f_{\dot{\varphi}}(x,u) \\ f_{\ddot{\varphi}}(x,u) \end{bmatrix} = \begin{bmatrix} \dot{y} \\ \frac{u + M_g g \operatorname{sen}\varphi \cos\varphi + M_g l \dot{\varphi}^2 \operatorname{sen}\varphi}{M_c + M_g \operatorname{sen}^2\varphi} \\ \dot{\varphi} \\ -\frac{g}{l}\operatorname{sen}\varphi - \frac{u\cos\varphi + M_g g \operatorname{sen}\varphi \cos^2\varphi + M_g l \dot{\varphi}^2 \operatorname{sen}\varphi \cos\varphi}{l(M_c + M_g \operatorname{sen}^2\varphi)} \end{bmatrix}$$

$$\frac{\partial f}{\partial x}(x_0,u_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\partial f_{\ddot{y}}}{\partial \varphi} & \frac{\partial f_{\ddot{y}}}{\partial \dot{\varphi}} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{\partial f_{\ddot{\varphi}}}{\partial \varphi} & \frac{\partial f_{\ddot{\varphi}}}{\partial \dot{\varphi}} \end{bmatrix}\Bigg|_{\substack{x = x_0 \\ u = u_0}} \qquad \frac{\partial f}{\partial u}(x_0,u_0) = \begin{bmatrix} 0 \\ \frac{\partial f_{\ddot{y}}}{\partial u} \\ 0 \\ \frac{\partial f_{\ddot{\varphi}}}{\partial u} \end{bmatrix}\Bigg|_{\substack{x = x_0 \\ u = u_0}}$$

$$\frac{\partial g}{\partial x}(x_0,u_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \frac{\partial g}{\partial u}(x_0,u_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Linealización

Cálculo de las matrices jacobianas

$$f_{\ddot{y}}(x,u) = \frac{u + M_g g \operatorname{sen}\varphi \cos\varphi + M_g l \dot{\varphi}^2 \operatorname{sen}\varphi}{M_c + M_g \operatorname{sen}^2\varphi}$$

$$\frac{\partial f_{\ddot{y}}}{\partial \varphi}(x,u) = \frac{1}{\left(M_c + M_g s^2\right)^2} \left\{ \left[M_g g \left(c^2 - s^2\right) + M_g l \dot{\varphi}^2 c \right] \left(M_c + M_g s^2\right) \right. \\ \left. - \left(u + M_g g s c + M_g l \dot{\varphi}^2 s\right)\left(s M_g s c\right) \right\}$$

donde $s = \operatorname{sen}\varphi$ y $c = \cos\varphi$.

$$\frac{\partial f_{\ddot{y}}}{\partial \varphi}(x_0,u_0) = \frac{M_g g}{M_c}$$

$$\frac{\partial f_{\ddot{y}}}{\partial \dot{\varphi}}(x_0,u_0) = 0$$

$$\frac{\partial f_{\ddot{y}}}{\partial u}(x_0,u_0) = \frac{1}{M_c}$$

Linealización

Cálculo de las matrices jacobianas

$$f_{\ddot{\varphi}}(x,u) = -\frac{g}{l}\operatorname{sen}\varphi - \frac{u\cos\varphi + M_g g \operatorname{sen}\varphi\cos^2\varphi + M_g l\dot{\varphi}^2\operatorname{sen}\varphi\cos\varphi}{l\left(M_c + M_g\operatorname{sen}^2\varphi\right)}$$

$$\begin{aligned}\frac{\partial f_{\ddot{\varphi}}}{\partial\varphi}(x,u) = -\frac{g}{l}c - \frac{1}{l\left(M_c + M_g s^2\right)^2}\Big\{ \\ \left[-us + M_g g\left(c^3 - 2s^2c\right) + M_g l\dot{\varphi}^2\left(c^2 - s^2\right)\right]\left(M_c + M_g s^2\right) \\ -\left(uc + M_g gsc^2 + M_g l\dot{\varphi}^2 sc\right)\left(2M_g sc\right)\Big\}\end{aligned}$$

$$\frac{\partial f_{\ddot{\varphi}}}{\partial\varphi}(x_0,u_0) = -\frac{g}{l} - \frac{M_g g M_c}{lM_c^2} = -\frac{g}{l}\left(1 + \frac{M_g}{M_c}\right)$$

$$\frac{\partial f_{\ddot{\varphi}}}{\partial\dot{\varphi}}(x_0,u_0) = 0$$

$$\frac{\partial f_{\ddot{\varphi}}}{\partial u}(x_0,u_0) = -\frac{1}{lM_c}$$

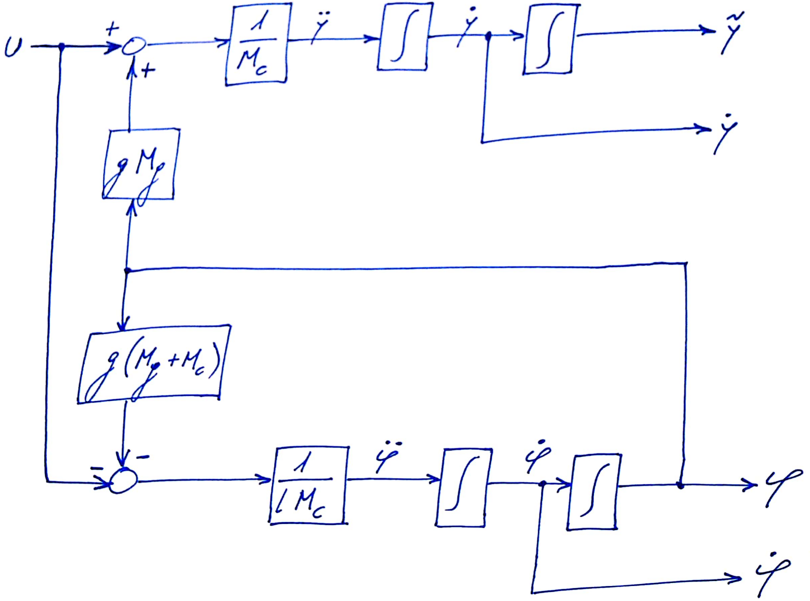
Representación en variables de estado del sistema linealizado

Resultado de la parte a:

$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\tilde{y}\\ \dot{y}\\ \varphi\\ \dot{\varphi}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & \frac{M_g}{M_c}g & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\frac{g}{l}\left(1+\frac{M_g}{M_c}\right) & 0\end{bmatrix}}_{A}\begin{bmatrix}\tilde{y}\\ \dot{y}\\ \varphi\\ \dot{\varphi}\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \frac{1}{M_c}\\ 0\\ -\frac{1}{lM_c}\end{bmatrix}}_{B}\begin{bmatrix}u\end{bmatrix}\\[2em]
\begin{bmatrix}\tilde{y}\\ \dot{y}\\ \varphi\\ \dot{\varphi}\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}_{C}\begin{bmatrix}\tilde{y}\\ \dot{y}\\ \varphi\\ \dot{\varphi}\end{bmatrix}
\end{cases}
$$

Diagrama de bloques del sistema linealizado

Resultado de la parte b:



Subsistema del gancho

Sea $\omega := \sqrt{\frac{g}{l}\left(1 + \frac{M_g}{M_c}\right)}$.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\varphi \\ \dot{\varphi}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 \\ -\omega^2 & 0\end{bmatrix}}_{A_1}\begin{bmatrix}\varphi \\ \dot{\varphi}\end{bmatrix} + \underbrace{\begin{bmatrix}0 \\ -\frac{1}{lM_c}\end{bmatrix}}_{B_1}\begin{bmatrix}u\end{bmatrix} \\ \\ \begin{bmatrix}\varphi \\ \dot{\varphi}\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}}_{C_1}\begin{bmatrix}\varphi \\ \dot{\varphi}\end{bmatrix} \end{cases}$$

Matriz de transición de estados del subsistema del gancho

Parte c

$$\mathrm{e}^{A_1 t} = \mathcal{L}^{-1}\left\{\left(s\mathrm{I} - A_1\right)^{-1}\right\}$$

$$\left(s\mathrm{I} - A_1\right) = \begin{bmatrix} s & -1 \\ \omega^2 & s \end{bmatrix}, \quad \det\left(s\mathrm{I} - A_1\right) = s^2 + \omega^2$$

$$\left(s\mathrm{I} - A_1\right)^{-1} = \frac{\mathrm{adj}\left(s\mathrm{I} - A_1\right)}{\det\left(s\mathrm{I} - A_1\right)} = \frac{\begin{bmatrix} s & -\omega^2 \\ 1 & s \end{bmatrix}^{\mathsf{T}}}{\det\left(s\mathrm{I} - A_1\right)} = \frac{\begin{bmatrix} s & 1 \\ -\omega^2 & s \end{bmatrix}}{\det\left(s\mathrm{I} - A_1\right)}$$

Entonces,

$$\left(s\mathrm{I} - A_1\right)^{-1} = \frac{1}{s^2 + \omega^2}\begin{bmatrix} s & 1 \\ -\omega^2 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{1}{s^2+\omega^2} \\ -\frac{\omega^2}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{1}{\omega}\frac{\omega}{s^2+\omega^2} \\ -\omega\frac{\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix}$$

$$\boxed{\mathrm{e}^{A_1 t} = \mathcal{L}^{-1}\left\{\left(s\mathrm{I} - A_1\right)^{-1}\right\} = \begin{bmatrix} \cos\left(\omega t\right) & \frac{1}{\omega}\,\mathrm{sen}\left(\omega t\right) \\ -\omega\,\mathrm{sen}\left(\omega t\right) & \cos\left(\omega t\right) \end{bmatrix}}$$

Matriz de transición de estados del subsistema del gancho

Otra forma de invertir $(s\mathrm{I} - A_1)$

$$(s\mathrm{I} - A_1)^{-1} = \frac{B(s)}{d(s)}, \qquad \begin{array}{l} B(s) = s^{n-1}B_0 + s^{n-2}B_1 + \ldots + sB_{n-2} + B_{n-1} \\ d(s) = \det(s\mathrm{I} - A_1) = s^n + d_1 s^{n-1} + \cdots + d_n \end{array}$$

$$\begin{cases} B_0 = \mathrm{I} \\ B_1 = B_0 A + d_1 \mathrm{I} \\ B_2 = B_1 A + d_2 \mathrm{I} \\ \vdots \\ B_{n-1} = B_{n-2} A + d_{n-1} \mathrm{I} \\ 0 = B_{n-1} A + d_n \mathrm{I} \end{cases}$$

Como $A_1 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$,

$$(s\mathrm{I} - A_1)^{-1} = \frac{sB_0 + B_1}{s^2 + d_1 s + d_2} = \frac{sB_0 + B_0 A + d_1 \mathrm{I}}{s^2 + d_1 s + d_2} = \frac{s\mathrm{I} + A}{s^2 + \omega^2} = \frac{\begin{bmatrix} s & 1 \\ -\omega^2 & s \end{bmatrix}}{s^2 + \omega^2}$$