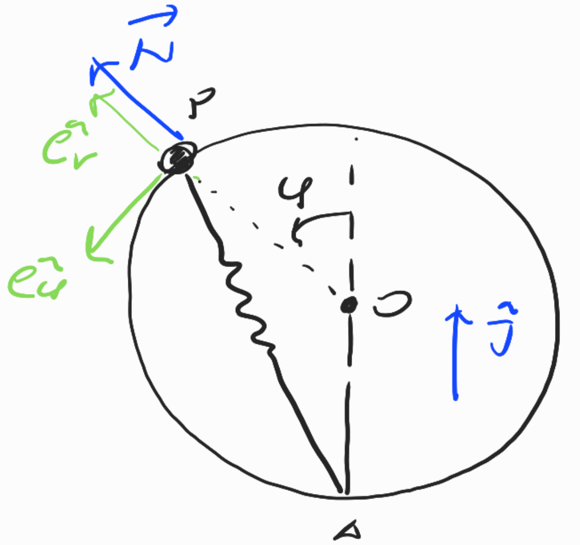
Mecánica Newtoniana, examen 13/02/25

Ejercicio 1)



Dado que $\vec{N}\cdot\vec{v}=0$, la energía de la partícula se conserva!

$$E=\frac{1}{2}m\vec{v}^2+U_{el}\ ,\quad \vec{v}=R\dot{\varphi}\hat{e_\varphi}\ ,\quad U_{el}=\frac{1}{2}K(P-A)^2$$

$$P-A=P-O+O-A=R\hat{e_r}+R\hat{j}\ ;$$

$$(P-A)^2=2R^2(1+\hat{j}\cdot\hat{e_r})=$$
$$=2R^2(1+\cos\varphi)$$

$$E=\frac{1}{2}mR^2\dot{\varphi}^2+\frac{1}{2}K\,2R^2(1+\cos\varphi)$$

$$E(0)=\frac{1}{2}mv_0^2+\frac{1}{2}K(2R)^2$$

$$\boxed{\frac{1}{2}mR^2\dot{\varphi}^2+KR^2(1+\cos\varphi)=\frac{1}{2}mv_0^2+2KR^2}$$

b) 2da ley de Newton proyectada según $\hat{e_r}$:

$$-mR\dot{\varphi}^2=N-K\underbrace{(P-A)\cdot\hat{e_r}}_{R(1+\hat{j}\cdot\hat{e_r})}\ :$$

$$N=KR(1+\cos\varphi)-mR\dot{\varphi}^2\ ;$$

Despejando $\dot{\varphi}^2$ de la ec. de movimiento y sustituyendo en la anterior:

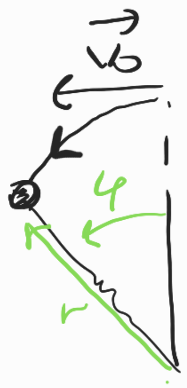
$$N(\varphi) = 3KR\cos\varphi - KR - mv_0^2/R$$

para $\varphi = \varphi_d$ $\quad N(\varphi_d) = 0$! $\quad \boxed{\cos\varphi_d = \frac{1}{3} + \frac{mv_0^2}{3KR^2}} \le 1$

si: $v_0^2 \le \frac{2K}{m} R^2$

si $v_0^2 \ge \frac{2K}{m} R^2$ la partícula se desprende en el instante inicial

c)



La partícula se mueve bajo la acción de una fuerza central conservativa $\begin{cases} l = cte. \\ E = cte. \end{cases}$

$$l = mr^2\dot{\varphi}$$

$$E = \frac{1}{2}m\vec{v}^2 + U_{el} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2 + \frac{1}{2}Kr^2 \rightarrow$$

$$\rightarrow E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\left(\frac{l}{mr^2}\right)^2 + \frac{1}{2}Kr^2 =$$

$$\frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + \frac{1}{2}Kr^2 :$$

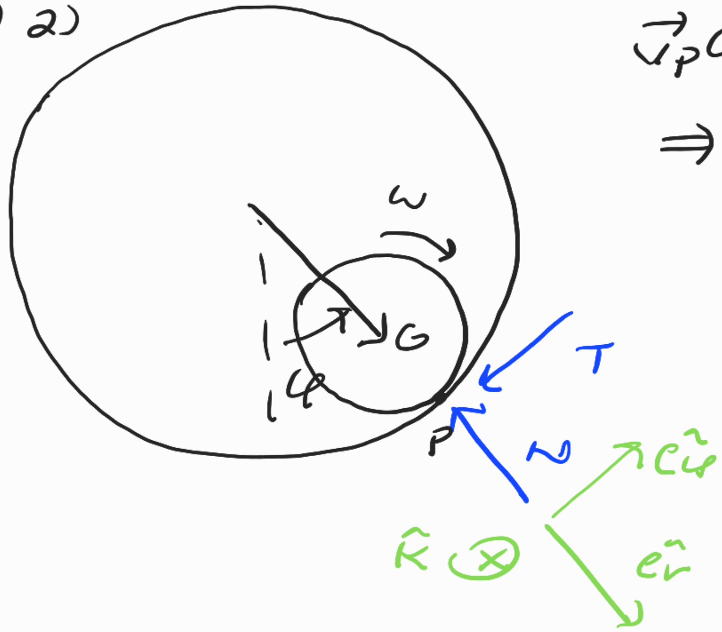
$$\boxed{\dot{r}^2 = \frac{2}{m}\left[E - \frac{l^2}{2mr^2} - \frac{1}{2}Kr^2\right]}$$

donde:

$$l = m2Rv_0$$

$$E = \frac{1}{2}mv_0^2 + 2KR^2$$

Ejercicio 2) a)



$\vec v_P(0) = v_0 \hat e_\varphi$

$\Rightarrow \vec T = -T \hat e_\varphi$,

$T = f_0 N$

1era Cardinal al disco: $\hat e_r)$ $-N = -m\,3r\,\dot\varphi^2$ (i)

$\hat e_\varphi)$ $-T = m\,3r\,\ddot\varphi$ (ii)

Usando $T = f_0 N$ y eliminando N entre las anteriores:

$$\boxed{\ddot\varphi + f_0 \dot\varphi^2 = 0} \quad (I)$$

2da Cardinal desde G: $\underbrace{I_{G,\hat k}}_{\frac{1}{2}mr^2}\dot\omega = rT$:

$\frac{1}{2} m r \dot\omega = T$ (iii)

Eliminando T entre (ii) y (iii):

$$\boxed{\dot\omega + 6\ddot\varphi = 0} \quad (II)$$

b.I) Integrando (II) en el tiempo:

$\omega - \omega(0) + 6(\dot\varphi - \dot\varphi(0)) = 0$

$$\Rightarrow \quad \omega = \underbrace{-6\dot\varphi}_{\alpha} + \underbrace{\frac{2v_0}{r}}_{\beta}$$

$$\begin{pmatrix} \omega(0) = 0 \\ \dot\varphi(0) = \frac{v_0}{3r} \end{pmatrix}$$

II) La ecuación (I) se puede escribir como:

$$\frac{d}{dt}\dot\varphi = -f_0\dot\varphi^2 \longrightarrow$$

integrando en variables separadas:

$$\int_{\underbrace{\dot\varphi(0)}_{v_0/3r}}^{\dot\varphi} \frac{d\dot\varphi}{\dot\varphi^2} = -f_0 t :$$

$$\frac{3r}{v_0} - \frac{1}{\dot\varphi} = -f_0 t : \quad \dot\varphi = \Big(\underbrace{f_0}_{\gamma} t + \underbrace{\frac{3r}{v_0}}_{\delta}\Big)^{-1}$$

c) La velocidad del pto. P es:

$$\vec{v}_P = \vec{v}_G + \underbrace{\vec\omega}_{\omega\hat{k}} \times \underbrace{(P-G)}_{r\hat{e}_r} = (3r\dot\varphi - \omega r)\hat{e}_\varphi$$

$$= \left[3r\dot\varphi - \left(-6\dot\varphi + \frac{2v_0}{r}\right)r\right]\hat{e}_\varphi =$$

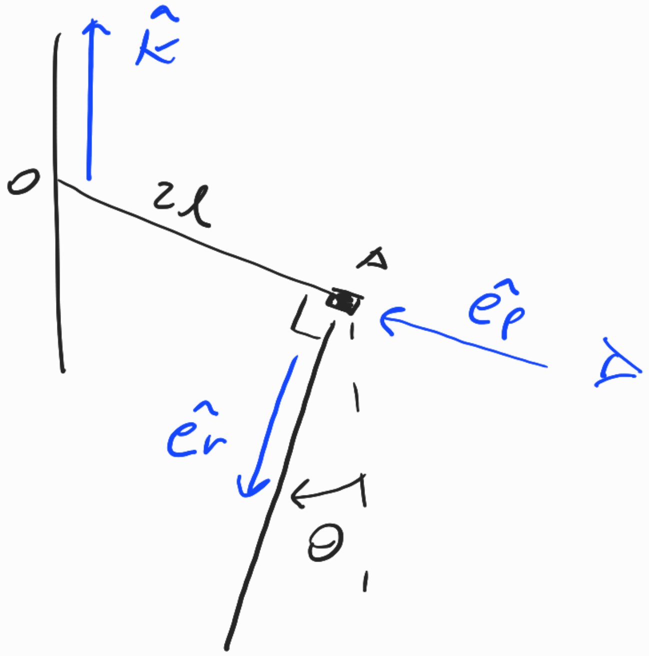
$$(9r\dot\varphi - 2v_0)\,\hat{e}_\varphi$$

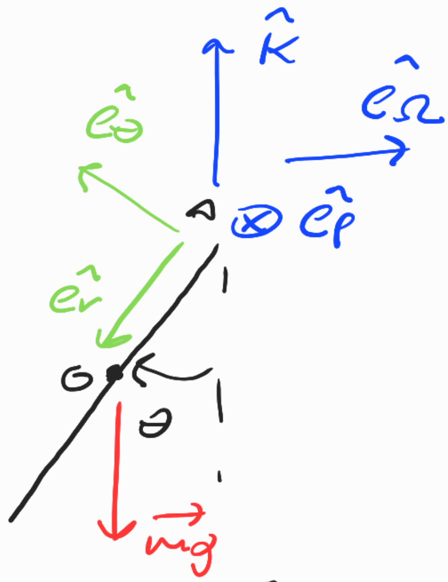
Para $t = t_d$ el disco comienza a rodar sin deslizarse:

$$9r\dot\varphi - 2v_0 = 0 : \quad \dot\varphi^{-1} = \frac{9}{2}\frac{r}{v_0} :$$

$$f_0 t_d + \frac{3r}{v_0} = \frac{9}{2}\frac{r}{v_0} : \quad \boxed{t_d = \frac{3}{2}\frac{r}{v_0 f_0}}$$

Ejercicio 3)





a) 2^{da} Cardinal en A a la barra proyectada según $\hat{e}_\rho$:

$$\left(m(G-A)\times\vec{a}_A + \frac{d}{dt}(\mathbb{I}_A\vec{\omega}) = \vec{M}_A^{(ext)}\right)\cdot\hat{e}_\rho ,$$

$$\vec{M}_A^{(ext)}\cdot\hat{e}_\rho = \vec{M}_A^{(peso)}\cdot\hat{e}_\rho \quad (\text{ya que } \vec{M}_A^{(aro)}\cdot\hat{e}_\rho = 0)$$

$$= -mgl\,\text{sen}\,\theta ;$$

$$\left.\begin{aligned} G-A &= l\,\hat{e}_r \\ \vec{a}_A &= 2l\,\Omega^2\,\hat{e}_\rho \end{aligned}\right] \; m(G-A)\times\vec{a}_A\cdot\hat{e}_\rho = 0$$

$$\mathbb{I}_A\vec{\omega}: \quad \mathbb{I}_A^{(\hat{e}_r,\hat{e}_\theta,\hat{e}_\rho)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{A,\hat{e}_\theta} & 0 \\ 0 & 0 & I_{A,\hat{e}_\rho} \end{pmatrix},$$

$$I_{A,\hat{e}_\theta} = I_{A,\hat{e}_\rho} = \int_0^{2l} \frac{m}{(2l)}\,dx\,x^2 = \frac{4}{3}ml^2 ;$$

$$\vec{\omega} = \Omega \hat{k} + \dot{\theta}\,\hat{e_\rho} = \Omega\left(-\cos\theta\,\hat{e_r} + \operatorname{sen}\theta\,\hat{e_\theta}\right) + \dot{\theta}\,\hat{e_\rho}$$

$$\Rightarrow \mathbb{I}_\Delta \vec{\omega} = \Omega\Big(-\cos\theta\,\underbrace{\mathbb{I}_\Delta \hat{e_r}}_{=0} + \operatorname{sen}\theta\,\underbrace{\mathbb{I}_\Delta \hat{e_\theta}}_{\frac{4}{3}ml^2\hat{e_\theta}}\Big) + \dot{\theta}\,\underbrace{\mathbb{I}_\Delta \hat{e_\rho}}_{\frac{4}{3}ml^2\hat{e_\rho}}$$

$$= \frac{4}{3}ml^2\left[\Omega \operatorname{sen}\theta\,\hat{e_\theta} + \dot{\theta}\,\hat{e_\rho}\right];$$

$$\frac{d}{dt}(\mathbb{I}_\Delta\vec{\omega})\cdot\hat{e_\rho} = \frac{4}{3}ml^2\Big[\Omega\operatorname{sen}\theta\,\underbrace{\dot{\hat{e_\theta}}}_{\Omega\hat{k}\times\hat{e_\theta}}\cdot\hat{e_\rho} + \ddot{\theta}\Big]$$

$$\Omega\hat{k}\times\hat{e_\theta}\cdot\hat{e_\rho} = -\Omega\cos\theta$$

$$\Rightarrow \boxed{\ddot{\theta} + \underbrace{\left(\frac{3}{4}g/l - \Omega^2\cos\theta\right)\operatorname{sen}\theta}_{\frac{dU_{eff}}{d\theta}} = 0}$$

b) Las posiciones de equilibrio corresponden a $\ddot{\theta} = 0$ $\left(\frac{dU_{eff}}{d\theta} = 0\right)$

$$\operatorname{sen}\theta\left[\frac{3}{4}g/l - \Omega^2\cos\theta\right] = 0:$$

$$\begin{cases} \operatorname{sen}\theta = 0: \quad \theta = 0, \pi \\ \cos\theta = \dfrac{\frac{3}{4}g/l}{\Omega^2} \end{cases}$$

c) $\theta_{eq} = 0, \pi$ existen siempre

$\cos\theta_{eq} = \left(\frac{3}{4}g/l\right)/\Omega^2 \leq 1$: $\exists$ si $\Omega^2 \geq \Omega_c^2 = \frac{3}{4}g/l$

Estabilidad: $\dfrac{d^2U_{eff}}{d\theta^2} = \Omega^2 + (\Omega_c^2 - 2\Omega^2\cos\theta)\cos\theta =$

$$\begin{cases} (\theta_{eq} = 0) \quad \Omega_c^2 - \Omega^2 \geq 0 \text{ si } \Omega^2 \leq \Omega_c^2 : \text{ estable si } \Omega^2 \leq \Omega_c^2 \\ (\theta_{eq} = \pi) \quad -(\Omega_c^2 + \Omega^2) < 0 \quad : \text{ siempre inestable} \\ \left(\cos\theta_{eq} = \left(\frac{\Omega}{\Omega_c}\right)^2\right) \quad \dfrac{\Omega^4 - \Omega_c^4}{\Omega^2} \geq 0 \text{ si } \Omega^2 \geq \Omega_c^2 \quad \text{estable mientras exista} \end{cases}$$